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ESTIMATED INCAPABILITY INDEX: RELIABILITY AND DECISION MAKING WITH SAMPLE INFORMATION

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SUMMARY

The process incapability index C_{pp}, which provides an uncontaminated separation between information concerning the process precision and process accuracy, has been proposed to measure process performance for industry applications. In this paper, we investigate the reliability of the natural estimator computationally, based on the α-level confidence relative error for various sample sizes. We also develop a decision-making procedure for judging if the process satisfies the preset quality requirement. The investigation is useful to the practitioners in determining the sample sizes required in their applications for the decisions reliable to the desired level. Copyright © 2002 John Wiley & Sons, Ltd.

KEY WORDS: process incapability index; non-central chi-square distribution; α-level confidence relative error

1. THE INCAPABILITY INDEX C_{pp}

A process incapability index C_{pp}, providing numerical measures on process performance, has been proposed by Greenwich and Jahr-Schaffrath [1]. The index C_{pp} is a simple transformation of C^*_{pm}, a general form of the capability index C_{pm} considered by Chan *et al.* [2], which provides an uncontaminated separation between information concerning the process precision and the process accuracy. The index C_{pp} is defined as the following:

$$C_{pp} = \left(\frac{\sigma}{D}\right)^2 + \left(\frac{\mu - T}{D}\right)^2$$

where μ is the process mean, σ is the process standard deviation, $D = \min\{(USL - T)/3, (T - LSL)/3\}$, USL and LSL are the upper and the lower specification limits, and T is the target value. If we define $C_{ip} = (\sigma/D)^2$ and $C_{ia} = [(\mu - T)/D]^2$, then C_{pp} can be expressed as $C_{pp} = C_{ip} + C_{ia}$. The index C_{ip} measures the process variation relative to the specification tolerance, which reflects process precision. The index C_{ia} measures the relative process departure, which reflects process accuracy. We note that the mathematical relationships $C_{ip} = 1/(C_p)^2$ and $C_{ia} = 9(1 - C_a)^2$ can be established, where C_p and C_a are two basic process capability indices considered by Kane [3] and Pearn *et al.* [4]. Thus, $C_{pp} = C_{ip} + C_{ia} = 1/(C_p)^2 + 9(1 - C_a)^2$.

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2. ESTIMATION OF C_{pp}

To estimate the process incapability (a combined measure of process imprecision and process inaccuracy), we consider the natural estimator $\hat{C}_{pp}$ defined as the following, where $\bar{X} = \sum_{i=1}^{n} X_i/n$, which also can be written as a function of C_{ip}

$$\hat{C}_{pp} = \frac{1}{n}\sum_{i=1}^{n}\frac{(X_i - \bar{X})^2}{D^2} + \frac{(\bar{X} - T)^2}{D^2} = \frac{C_{ip}}{n}\frac{n\hat{C}_{pp}}{C_{ip}} = \frac{C_{ip}}{n}\sum_{i=1}^{n}\frac{(X_i - T)^2}{\sigma^2}$$

If the process characteristic is normally distributed, then the estimator $\hat{C}_{pp}$ is distributed as $[C_{ip}/n]\chi^2_n(\delta)$, where $\chi^2_n(\delta)$ is a non-central chi-square distribution with n degrees of freedom and non-centrality parameter $\delta = n(\mu - T)^2/\sigma^2 = nC_{ia}/C_{ip}$. Therefore, the probability density function (PDF) of $\hat{C}_{pp}$ can be easily derived and expressed as the following, for $y > 0$, which can also be rewritten as a function of C_{ip} and C_{ia}

$$h(y) = \sum_{k=0}^{\infty}\left\{\frac{[(ny)/(2C_{ip})]^{k+n/2}\exp[-(ny)/(2C_{ip})]}{y\Gamma(k+n/2)} \times \frac{(\delta/2)^k\exp(-\delta/2)}{\Gamma(k+1)}\right\}$$

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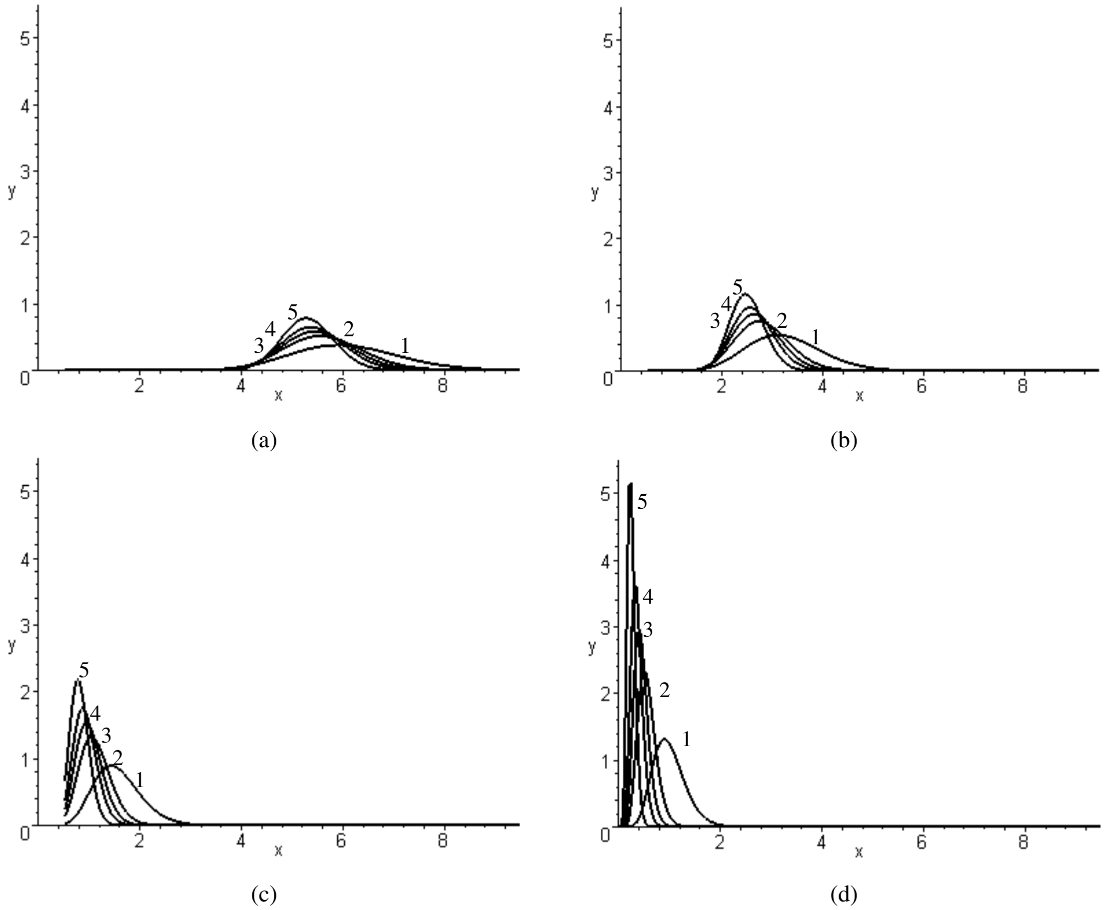

Figure 1. PDF plot of $\hat{C}_{pp}$ for (a) $C_{ia} = 5.06$, $C_{ip} = 1.00, 0.56, 0.44, 0.36, 0.25$, (b) $C_{ia} = 2.25$, $C_{ip} = 1.00, 0.56, 0.44, 0.36, 0.25$, (c) $C_{ia} = 0.56$, $C_{ip} = 1.00, 0.56, 0.44, 0.36, 0.25$ and (d) $C_{ia} = 0.00$, $C_{ip} = 1.00, 0.56, 0.44, 0.36, 0.25$

$$= \sum_{k=0}^{\infty} \left\{ \frac{[(ny)/(2C_{ip})]^{k+n/2} \exp[-(ny)/(2C_{ip})]}{y\Gamma(k+n/2)} \times \frac{[nC_{ia}/(2C_{ip})]^k \exp[-nC_{ia}/(2C_{ip})]}{\Gamma(k+1)} \right\}$$

In Figures 1(a)–(d), we plot the PDF of $\hat{C}_{pp}$ for $n = 20$, $C_{ia} = 5.06, 2.25, 0.56$ and 0.00, respectively, with commonly used values of $C_{ip} = 1.00$ (curve 1), 0.56 (curve 2), 0.44 (curve 3), 0.36 (curve 4), and 0.25 (curve 5). As can be seen from the figures, for a fixed value of C_{ia} the variance of $\hat{C}_{pp}$ decreases as the value of C_{ip} increases and for a fixed value of C_{ip} the variance of $\hat{C}_{pp}$ decreases as the value of C_{ia} decreases.

We note that those C_{ip} values are equivalent to the widely used capability requirements, $C_p = 1.00$, 1.33, 1.50, 1.67 and 2.00. For industry applications, a process is called 'incapable' if $C_{ip} > 1.00$ (equivalent to $C_p < 1.00$) and is called 'capable' if $0.56 < C_{ip} \leq 1.00$ (equivalent to $1.00 \leq C_p < 1.33$). A process is called 'satisfactory' if $0.44 < C_{ip} \leq 0.56$ (equivalent to $1.33 \leq C_p < 1.50$), called 'good' if $0.36 < C_{ip} \leq 0.44$ (equivalent to $1.50 \leq C_p < 1.67$), called 'excellent' if $0.25 < C_{ip} \leq 0.36$ (equivalent to $1.67 \leq C_p < 2.00$) and is called 'super' if $C_{ip} \leq 0.25$ (equivalent to $C_p \geqslant 2.00$). On the other hand, the values of C_{ia} are equivalent to $C_a = 0.25, 0.50, 0.75$ and 1.00 respectively. We note that if the process is perfectly centered, then $C_{ia} = 0.00$ (or equivalently $C_a = 1.00$) (see Pearn *et al.* [4]).

If the process characteristic follows the normal distribution, Pearn and Lin [5] showed that $\hat{C}_{pp}$ is the MLE, and the UMVUE of C_{pp}. They also showed that $\hat{C}_{pp}$ is consistent, $\sqrt{n}(\hat{C}_{pp} - C_{pp})$ converges to $N(0, 2C_{ip}C_{ia} + 2C_{ip}C_{pp})$ in distribution and $\hat{C}_{pp}$ is asymptotically efficient. Thus, the estimator $\hat{C}_{pp}$ has all the desired statistical properties and using $\hat{C}_{pp}$ to estimate process incapability would be reasonable.

```
OPTIONS REPLACE PAGESIZE = 58 LINESIZE = 78 NODATA;
DATA CRE;
 CIA = ;
 CIP1 = ; CIP2 = ; CIP3 = ; CIP4 = ; CIP5 = ;
DO N = 10 TO 200 BY 10;
 D1 = N*CIA/CIP1; D2 = N*CIA/CIP2;
 D3 = N*CIA/CIP3; D4 = N*CIA/CIP4; D5 = N*CIA/CIP5;
 A1 = CIP/(CIA + CIP1)*(1/N)*CINV(0.025, N, D1);
 B1 = CIP/(CIA + CIP1)*(1/N)*CINV(0.975, N, D1);
 C1 = MAX(ABS(A1 - 1), ABS(B1 - 1));
 A2 = CIP/(CIA + CIP2)*(1/N)*CINV(0.025, N, D2);
 B2 = CIP/(CIA + CIP2)*(1/N)*CINV(0.975, N, D2);
 C2 = MAX(ABS(A2 - 1), ABS(B2 - 1));
 A3 = CIP/(CIA + CIP3)*(1/N)*CINV(0.025, N, D3);
 B3 = CIP/(CIA + CIP3)*(1/N)*CINV(0.975, N, D3);
 C3 = MAX(ABS(A3 - 1), ABS(B3 - 1));
 A4 = CIP/(CIA + CIP4)*(1/N)*CINV(0.025, N, D4);
 B4 = CIP/(CIA + CIP4)*(1/N)*CINV(0.975, N, D4);
 C4 = MAX(ABS(A4 - 1), ABS(B4 - 1));
 A5 = CIP/(CIA + CIP5)*(1/N)*CINV(0.025, N, D5);
 B5 = CIP/(CIA + CIP5)*(1/N)*CINV(0.975, N, D5);
 C5 = MAX(ABS(A5 - 1), ABS(B5 - 1));
OUTPUT;
END;
FORMAT C1 C2 C3 C4 C5 6.4;
PROC PRINT DATA = CRE;
VAR N C1 C2 C3 C4 C5;
RUN;
```

Figure 2.

3. RELIABILITY ANALYSIS

To evaluate the reliability of the estimator $\hat{C}_{pp}$, we consider the measurement criteria called the α-level confidence relative error, which is defined as $\mathrm{CRE}_\alpha(\hat{C}_{pp}) = \max_\alpha\{|\hat{C}_{pp} - C_{pp}|/C_{pp}\} = \max_\alpha |(\hat{C}_{pp}/C_{pp}) - 1| = \max_\alpha\{|L_{\alpha/2} - 1|, |U_{1-\alpha/2} - 1|\}$, where $L_{\alpha/2}$ and $U_{1-\alpha/2}$ satisfy the probability equation $\mathrm{P}\{L_{\alpha/2} \le \hat{C}_{pp}/C_{pp} \le U_{1-\alpha/2}\} = 1 - \alpha$, which can be obtained as follows

$$\mathrm{P}\{L_{\alpha/2} \le \hat{C}_{pp}/C_{pp} \le U_{1-\alpha/2}\} = 1 - \alpha$$
$$= \mathrm{P}\left\{L_{\alpha/2} \le \left[\frac{C_{ip}}{n}\right]\left[\frac{n\hat{C}_{pp}}{C_{ip}}\right](C_{ip} + C_{ia})^{-1} \le U_{1-\alpha/2}\right\}$$
$$= \mathrm{P}\left\{L_{\alpha/2} \le \left[\frac{C_{ip}}{n(C_{ip} + C_{ia})}\right]\chi_n^2(\delta) \le U_{1-\alpha/2}\right\}$$

where once again $\chi_n^2(\delta)$ is a non-central chi-square distribution with n degrees of freedom and non-centrality parameter $\delta = n(\mu - T)^2/\sigma^2 = nC_{ia}/C_{ip}$. Therefore, the bounds $U_{1-\alpha/2}$, and $L_{\alpha/2}$, may be obtained as:

$$U_{1-\alpha/2} = \left[\frac{C_{ip}}{n(C_{ip} + C_{ia})}\right]\chi^2_{n,1-\alpha/2}(\delta)$$
$$L_{\alpha/2} = \left[\frac{C_{ip}}{n(C_{ip} + C_{ia})}\right]\chi^2_{n,\alpha/2}(\delta)$$

Thus, $\mathrm{CRE}_\alpha(\hat{C}_{pp}) = c$ presents that with at least $(1-\alpha)$ confidence the relative deviation (relative error) of $\hat{C}_{pp}$, $\max_\alpha\{|\hat{C}_{pp} - C_{pp}|/C_{pp}\} = \max_\alpha\{|(\hat{C}_{pp}/C_{pp}) - 1|\} = \max_\alpha\{|U_{1-\alpha/2} - 1|, |L_{\alpha/2} - 1|\}$, will be no greater than c. The $(1 - \alpha)\%$ confidence relative error of $\hat{C}_{pp}$ can be calculated using the SAS computer program (see Figure 2). Tables 1–3 display $\mathrm{CRE}_\alpha(\hat{C}_{pp})$ values with $\alpha = 0.05$, 0.025 and 0.01, for $C_{ip} = 1.00$, 0.56, 0.44, 0.36 and 0.25, and $C_{ia} = 5.06$, 2.25, 0.56 and 0.00, with $n = 10(10)200$.

For example, for $C_{ip} = 0.25$ and $C_{ia} = 5.06$, with $\alpha = 0.05$ and $n = 150$, we have $\mathrm{CRE}_\alpha(\hat{C}_{pp}) = 0.0695$, which indicates that with at least 95% confidence the obtained $\hat{C}_{pp}$ value will be within 6.95% of the true C_{pp} value. Thus, for the described

Table 1. $\mathrm{CRE}_\alpha(\hat{C}_{pp})$ for (a) $C_{ia} = 5.06$, $\alpha = 0.05$, various C_{ip}, $n = 10(10)200$; (b) $C_{ia} = 2.25$; (c) $C_{ia} = 0.56$; and (d) $C_{ia} = 0.00$

	C_{ip}						C_{ip}				
n	1.00	0.56	0.44	0.36	0.25	n	1.00	0.56	0.44	0.36	0.25
(a)						(c)					
10	0.5258	0.4091	0.3669	0.3322	0.2787	10	0.9591	0.8765	0.8331	0.7903	0.7100
20	0.3630	0.2838	0.2550	0.2312	0.1944	20	0.6507	0.5968	0.5682	0.5399	0.4867
30	0.2932	0.2297	0.2066	0.1874	0.1578	30	0.5209	0.4786	0.4561	0.4338	0.3917
40	0.2522	0.1979	0.1781	0.1616	0.1361	40	0.4457	0.4100	0.3909	0.3720	0.3362
50	0.2246	0.1764	0.1587	0.1441	0.1215	50	0.3953	0.3639	0.3472	0.3305	0.2989
60	0.2043	0.1606	0.1446	0.1313	0.1107	60	0.3586	0.3303	0.3152	0.3002	0.2716
70	0.1887	0.1483	0.1336	0.1213	0.1023	70	0.3303	0.3045	0.2906	0.2768	0.2505
80	0.1761	0.1385	0.1248	0.1133	0.0956	80	0.3078	0.2838	0.2709	0.2581	0.2337
90	0.1657	0.1304	0.1175	0.1067	0.0900	90	0.2892	0.2667	0.2547	0.2427	0.2198
100	0.1570	0.1236	0.1113	0.1012	0.0853	100	0.2736	0.2524	0.2410	0.2297	0.2081
110	0.1495	0.1177	0.1061	0.0964	0.0813	110	0.2602	0.2401	0.2293	0.2186	0.1980
120	0.1430	0.1126	0.1015	0.0922	0.0778	120	0.2486	0.2294	0.2191	0.2089	0.1893
130	0.1372	0.1081	0.0974	0.0885	0.0747	130	0.2383	0.2201	0.2102	0.2004	0.1816
140	0.1321	0.1041	0.0938	0.0853	0.0720	140	0.2293	0.2117	0.2023	0.1928	0.1748
150	0.1275	0.1005	0.0906	0.0823	0.0695	150	0.2212	0.2043	0.1952	0.1861	0.1687
160	0.1234	0.0972	0.0876	0.0797	0.0673	160	0.2138	0.1975	0.1887	0.1799	0.1632
170	0.1196	0.0943	0.0850	0.0773	0.0652	170	0.2072	0.1914	0.1829	0.1744	0.1582
180	0.1162	0.0916	0.0826	0.0751	0.0634	180	0.2011	0.1858	0.1776	0.1693	0.1536
190	0.1130	0.0891	0.0803	0.0730	0.0617	190	0.1955	0.1807	0.1727	0.1647	0.1494
200	0.1101	0.0868	0.0783	0.0712	0.0601	200	0.1904	0.1760	0.1682	0.1604	0.1455
(b)						(d)					
10	0.7100	0.5781	0.5258	0.4810	0.4091	10	1.0483	1.0483	1.0483	1.0483	1.0483
20	0.4867	0.3983	0.3630	0.3327	0.2838	20	0.7085	0.7085	0.7085	0.7085	0.7085
30	0.3917	0.3214	0.2932	0.2689	0.2297	30	0.5660	0.5660	0.5660	0.5660	0.5660
40	0.3362	0.2763	0.2522	0.2315	0.1979	40	0.4836	0.4836	0.4836	0.4836	0.4836
50	0.2989	0.2459	0.2246	0.2062	0.1764	50	0.4284	0.4284	0.4284	0.4284	0.4284
60	0.2716	0.2236	0.2043	0.1876	0.1606	60	0.3883	0.3883	0.3883	0.3883	0.3883
70	0.2505	0.2065	0.1887	0.1733	0.1483	70	0.3575	0.3575	0.3575	0.3575	0.3575
80	0.2337	0.1927	0.1761	0.1618	0.1385	80	0.3329	0.3329	0.3329	0.3329	0.3329
90	0.2198	0.1813	0.1657	0.1523	0.1304	90	0.3127	0.3127	0.3127	0.3127	0.3127
100	0.2081	0.1717	0.1570	0.1442	0.1236	100	0.2956	0.2956	0.2956	0.2956	0.2956
110	0.1980	0.1635	0.1495	0.1374	0.1177	110	0.2811	0.2811	0.2811	0.2811	0.2811
120	0.1893	0.1563	0.1430	0.1314	0.1126	120	0.2684	0.2684	0.2684	0.2684	0.2684
130	0.1816	0.1500	0.1372	0.1261	0.1081	130	0.2573	0.2573	0.2573	0.2573	0.2573
140	0.1748	0.1444	0.1321	0.1214	0.1041	140	0.2475	0.2475	0.2475	0.2475	0.2475
150	0.1687	0.1394	0.1275	0.1172	0.1005	150	0.2387	0.2387	0.2387	0.2387	0.2387
160	0.1632	0.1349	0.1234	0.1134	0.0972	160	0.2308	0.2308	0.2308	0.2308	0.2308
170	0.1582	0.1307	0.1196	0.1100	0.0943	170	0.2235	0.2235	0.2235	0.2235	0.2235
180	0.1536	0.1270	0.1162	0.1068	0.0916	180	0.2169	0.2169	0.2169	0.2169	0.2169
190	0.1494	0.1235	0.1130	0.1039	0.0891	190	0.2108	0.2108	0.2108	0.2108	0.2108
200	0.1455	0.1203	0.1101	0.1012	0.0868	200	0.2053	0.2053	0.2053	0.2053	0.2053

condition, if the calculated value $\hat{C}_{pp} = 1.27$, then the true value of C_{pp} would be between $1.27/(1 + 6.95\%)$ and $1.27/(1 - 6.95\%)$ (with at least 95% confidence). We note that the α-level confidence relative error $\mathrm{CRE}_\alpha(\hat{C}_{pp})$, which is obtained from the same approach as used for finding the confidence interval, provides the practitioners with more direct and easily understood information than the confidence interval approach regarding the accuracy of their estimations and suggests a clear range on the true value of the process performance measure using the index C_{pp}.

4. A DECISION MAKING PROCEDURE

Under the usual normality assumption, $n\hat{C}_{pp}/(C_{pp} - C_{ia})$ is distributed as $\chi^2_n(\delta)$, a non-central chi-square distribution with n degrees of freedom and non-centrality parameter $\delta = n(\mu - T)^2/\sigma^2 =$

Table 2. $\text{CRE}_\alpha(\hat{C}_{pp})$ for (a) $C_{ia} = 5.06$, $\alpha = 0.025$, various C_{ip}, $n = 10(10)200$; (b) $C_{ia} = 2.25$; (c) $C_{ia} = 0.56$; and (d) $C_{ia} = 0.00$

	C_{ip}						C_{ip}				
n	1.00	0.56	0.44	0.36	0.25	n	1.00	0.56	0.44	0.36	0.25
(a)						(c)					
10	0.6138	0.4754	0.4258	0.3850	0.3223	10	1.1410	1.0382	0.9846	0.9323	0.8348
20	0.4214	0.3284	0.2947	0.2670	0.2241	20	0.7659	0.7001	0.6656	0.6316	0.5679
30	0.3395	0.2652	0.2383	0.2160	0.1816	30	0.6102	0.5591	0.5321	0.5055	0.4554
40	0.2916	0.2282	0.2052	0.1861	0.1566	40	0.5205	0.4777	0.4549	0.4325	0.3901
50	0.2593	0.2032	0.1828	0.1658	0.1396	50	0.4607	0.4232	0.4033	0.3836	0.3463
60	0.2357	0.1849	0.1663	0.1510	0.1272	60	0.4172	0.3836	0.3657	0.3479	0.3143
70	0.2175	0.1707	0.1536	0.1395	0.1175	70	0.3839	0.3532	0.3368	0.3205	0.2897
80	0.2029	0.1594	0.1434	0.1303	0.1098	80	0.3573	0.3289	0.3137	0.2986	0.2701
90	0.1909	0.1500	0.1350	0.1226	0.1034	90	0.3354	0.3089	0.2947	0.2806	0.2538
100	0.1808	0.1421	0.1279	0.1162	0.0980	100	0.3171	0.2921	0.2788	0.2655	0.2402
110	0.1721	0.1353	0.1218	0.1107	0.0933	110	0.3014	0.2778	0.2651	0.2525	0.2285
120	0.1645	0.1294	0.1165	0.1059	0.0893	120	0.2878	0.2653	0.2532	0.2412	0.2184
130	0.1579	0.1242	0.1119	0.1016	0.0857	130	0.2758	0.2543	0.2428	0.2313	0.2095
140	0.1519	0.1196	0.1077	0.0979	0.0825	140	0.2652	0.2446	0.2335	0.2225	0.2015
150	0.1466	0.1154	0.1040	0.0945	0.0797	150	0.2557	0.2359	0.2253	0.2146	0.1944
160	0.1494	0.1117	0.1006	0.0914	0.0771	160	0.2472	0.2281	0.2178	0.2075	0.1880
170	0.1375	0.1083	0.0976	0.0887	0.0748	170	0.2394	0.2209	0.2110	0.2011	0.1822
180	0.1335	0.1052	0.0948	0.0861	0.0727	180	0.2323	0.2144	0.2048	0.1952	0.1769
190	0.1299	0.1023	0.0922	0.0838	0.0707	190	0.2258	0.2085	0.1991	0.1898	0.1720
200	0.1265	0.0997	0.0898	0.0816	0.0689	200	0.2198	0.2030	0.1939	0.1848	0.1675
(b)						(d)					
10	0.8348	0.6762	0.6138	0.5605	0.4754	10	1.2558	1.2558	1.2558	1.2558	1.2558
20	0.5679	0.4631	0.4214	0.3857	0.3284	20	0.8381	0.8381	0.8381	0.8381	0.8381
30	0.4554	0.3725	0.3395	0.3110	0.2652	30	0.6656	0.6656	0.6656	0.6656	0.6656
40	0.3901	0.3197	0.2916	0.2673	0.2282	40	0.5666	0.5666	0.5666	0.5666	0.5666
50	0.3463	0.2842	0.2593	0.2379	0.2032	50	0.5008	0.5008	0.5008	0.5008	0.5008
60	0.3143	0.2583	0.2357	0.2163	0.1849	60	0.4531	0.4531	0.4531	0.4531	0.4531
70	0.2897	0.2382	0.2175	0.1996	0.1707	70	0.4165	0.4165	0.4165	0.4165	0.4165
80	0.2701	0.2222	0.2029	0.1863	0.1594	80	0.3874	0.3874	0.3874	0.3874	0.3874
90	0.2538	0.2090	0.1909	0.1753	0.1500	90	0.3634	0.3634	0.3634	0.3634	0.3634
100	0.2402	0.1979	0.1808	0.1660	0.1421	100	0.3434	0.3434	0.3434	0.3434	0.3434
110	0.2285	0.1883	0.1721	0.1580	0.1353	110	0.3263	0.3263	0.3263	0.3263	0.3263
120	0.2184	0.1800	0.1645	0.1511	0.1294	120	0.3114	0.3114	0.3114	0.3114	0.3114
130	0.2095	0.1727	0.1579	0.1450	0.1242	130	0.2984	0.2984	0.2984	0.2984	0.2984
140	0.2015	0.1662	0.1519	0.1396	0.1196	140	0.2869	0.2869	0.2869	0.2869	0.2869
150	0.1944	0.1604	0.1466	0.1347	0.1154	150	0.2765	0.2765	0.2765	0.2765	0.2765
160	0.1880	0.1552	0.1419	0.1304	0.1117	160	0.2672	0.2672	0.2672	0.2672	0.2672
170	0.1822	0.1504	0.1375	0.1264	0.1083	170	0.2587	0.2587	0.2587	0.2587	0.2587
180	0.1769	0.1460	0.1335	0.1227	0.1052	180	0.2510	0.2510	0.2510	0.2510	0.2510
190	0.1720	0.1420	0.1299	0.1194	0.1023	190	0.2439	0.2439	0.2439	0.2439	0.2439
200	0.1675	0.1383	0.1265	0.1163	0.0997	200	0.2374	0.2374	0.2374	0.2374	0.2374

$nC_{ia}/(C_{pp} - C_{ia})$. Let c_o be a statistic calculated from the sample data satisfying $\text{P}\{C_{pp} \le c_o\} = 1 - \alpha$. Then, c_o is a $100(1-\alpha)\%$ upper confidence limit for C_{pp}. We note that

$$\begin{aligned}
\text{P}\{C_{pp} \le c_o\} &= \text{P}\{C_{pp} - C_{ia} \le c_o - C_{ia}\} \\
&= \text{P}\{1/(C_{pp} - C_{ia}) \ge 1/(c_o - C_{ia})\} \\
&= \text{P}\{n\hat{C}_{pp}/(C_{pp} - C_{ia}) \ge n\hat{C}_{pp}/(c_o - C_{ia})\} \\
&= \text{P}\{\chi_n^2(\delta) \ge n\hat{C}_{pp}/(c_o - C_{ia})\} = 1 - \alpha
\end{aligned}$$

Therefore, $n\hat{C}_{pp}/(c_o - C_{ia}) = \chi^2_{n,\alpha}(\delta)$, where $\chi^2_{n,\alpha}(\delta)$ is the (lower) αth percentile of the $\chi_n^2(\delta)$ distribution. A $100(1 - \alpha)\%$ upper confidence limit on C_{pp} can be written in terms of $\hat{C}_{pp}$ as $c_o = C_{ia} + [n\hat{C}_{pp}/\chi^2_{n,\alpha}(\delta)]$. The value C_{ia} is

Table 3. $\text{CRE}_\alpha(\hat{C}_{pp})$ for (a) $C_{ia} = 5.06$, $\alpha = 0.01$, various C_{ip}, $n = 10(10)200$; (b) $C_{ia} = 2.25$; (c) $C_{ia} = 0.56$; and (d) $C_{ia} = 0.00$

	C_{ip}						C_{ip}				
n	1.00	0.56	0.44	0.36	0.25	n	1.00	0.56	0.44	0.36	0.25
(a)						(c)					
10	0.7217	0.5564	0.4974	0.4491	0.3751	10	1.3693	1.2399	1.1734	1.1087	0.9892
20	0.4924	0.3823	0.3427	0.3101	0.2599	20	0.9087	0.8278	0.7856	0.7444	0.6674
30	0.3955	0.3081	0.2765	0.2505	0.2103	30	0.7201	0.6579	0.6253	0.5933	0.5332
40	0.3391	0.2648	0.2378	0.2155	0.1811	40	0.6123	0.5604	0.5331	0.5062	0.4557
50	0.3013	0.2355	0.2116	0.1919	0.1614	50	0.5406	0.4955	0.4717	0.4482	0.4039
60	0.2736	0.2141	0.1925	0.1746	0.1469	60	0.4888	0.4485	0.4271	0.4060	0.3662
70	0.2523	0.1976	0.1777	0.1612	0.1357	70	0.4492	0.4124	0.3929	0.3736	0.3372
80	0.2352	0.1844	0.1659	0.1505	0.1267	80	0.4176	0.3837	0.3656	0.3478	0.3140
90	0.2212	0.1735	0.1561	0.1417	0.1193	90	0.3917	0.3601	0.3432	0.3266	0.2950
100	0.2094	0.1643	0.1478	0.1342	0.1131	100	0.3700	0.3403	0.3244	0.3087	0.2790
110	0.1992	0.1564	0.1407	0.1278	0.1077	110	0.3514	0.3233	0.3084	0.2935	0.2653
120	0.1904	0.1495	0.1346	0.1222	0.1030	120	0.3354	0.3087	0.2944	0.2803	0.2534
130	0.1827	0.1435	0.1292	0.1173	0.0989	130	0.3213	0.2958	0.2822	0.2686	0.2430
140	0.1758	0.1381	0.1244	0.1129	0.0952	140	0.3088	0.2844	0.2713	0.2583	0.2337
150	0.1696	0.1333	0.1200	0.1090	0.0919	150	0.2976	0.2741	0.2616	0.2491	0.2254
160	0.1640	0.1290	0.1161	0.1055	0.0889	160	0.2875	0.2649	0.2528	0.2408	0.2179
170	0.1590	0.1250	0.1126	0.1023	0.0862	170	0.2784	0.2566	0.2449	0.2332	0.2111
180	0.1544	0.1214	0.1093	0.0993	0.0838	180	0.2701	0.2489	0.2376	0.2263	0.2049
190	0.1501	0.1181	0.1064	0.0966	0.0815	190	0.2625	0.2420	0.2310	0.2200	0.1992
200	0.1462	0.1150	0.1036	0.0941	0.0794	200	0.2554	0.2355	0.2248	0.2142	0.1940
(b)						(d)					
10	0.9892	0.7968	0.7217	0.6578	0.5564	10	1.5188	1.5188	1.5188	1.5188	1.5188
20	0.6674	0.5420	0.4924	0.4501	0.3823	20	0.9999	0.9999	0.9999	0.9999	0.9999
30	0.5332	0.4346	0.3955	0.3620	0.3081	30	0.7891	0.7892	0.7891	0.7892	0.7892
40	0.4557	0.3723	0.3391	0.3106	0.2648	40	0.6692	0.6692	0.6692	0.6692	0.6692
50	0.4039	0.3305	0.3013	0.2761	0.2355	50	0.5898	0.5898	0.5898	0.5898	0.5898
60	0.3662	0.3001	0.2736	0.2508	0.2141	60	0.5325	0.5325	0.5325	0.5325	0.5325
70	0.3372	0.2766	0.2523	0.2314	0.1976	70	0.4887	0.4887	0.4887	0.4887	0.4887
80	0.3140	0.2578	0.2352	0.2158	0.1844	80	0.4540	0.4540	0.4540	0.4540	0.4540
90	0.2950	0.2423	0.2212	0.2030	0.1735	90	0.4256	0.4256	0.4256	0.4256	0.4256
100	0.2790	0.2293	0.2094	0.1921	0.1643	100	0.4017	0.4017	0.4017	0.4017	0.4017
110	0.2653	0.2182	0.1992	0.1829	0.1564	110	0.3814	0.3814	0.3814	0.3814	0.3814
120	0.2534	0.2085	0.1904	0.1748	0.1495	120	0.3637	0.3637	0.3637	0.3637	0.3637
130	0.2430	0.2000	0.1827	0.1677	0.1435	130	0.3483	0.3483	0.3483	0.3483	0.3483
140	0.2337	0.1924	0.1758	0.1614	0.1381	140	0.3346	0.3346	0.3346	0.3346	0.3346
150	0.2254	0.1856	0.1696	0.1558	0.1333	150	0.3224	0.3224	0.3224	0.3224	0.3224
160	0.2179	0.1795	0.1640	0.1507	0.1290	160	0.3114	0.3114	0.3114	0.3114	0.3114
170	0.2111	0.1740	0.1590	0.1460	0.1250	170	0.3014	0.3014	0.3014	0.3014	0.3014
180	0.2049	0.1689	0.1544	0.1418	0.1214	180	0.2923	0.2923	0.2923	0.2923	0.2923
190	0.1992	0.1642	0.1501	0.1379	0.1181	190	0.2840	0.2840	0.2840	0.2840	0.2840
200	0.1940	0.1599	0.1462	0.1343	0.1150	200	0.2763	0.2763	0.2763	0.2763	0.2763

unknown. In practice, we can use the UMVUE, $\tilde{C}_{ia} = [(\bar{X}-T)/D]^2 - (S_{n-1})^2/(nD^2)$, recommended by Pearn and Lin [5] to estimate C_{ia} (see also Greenwich and Jahr-Schaffrath [1]) where $S_{n-1} = [\sum_{i=1}^{n}(X_i - \bar{X})^2/(n-1)]^{1/2}$ is the conventional estimator of the process standard deviation σ. We note that $\delta = n(\mu - T)^2/\sigma^2$, will also be estimated as $\hat{\delta} = n(\bar{X}-T)^2/(S_{n-1})^2$. Thus, if $\hat{C}_{pp} \le \chi^2_{n,\alpha}(\hat{\delta})(C - \tilde{C}_{ia})/n$, where C is the recommended maximum value, then we claim that the process satisfies the quality requirement for at least $100(1-\alpha)\%$ of the time.

A simple procedure, based on the recommended maximum value C, for judging whether the process satisfies the preset quality requirement is presented in the following. The SAS software package can be used for generating the values $\chi^2_{n,\alpha}(\hat{\delta})(C - \tilde{C}_{ia})/n$ using the command CINV(α, n, $\hat{\delta}$), with input of the process information, USL, LSL, T, C, α-risk, $\bar{X}$, S^2_{n-1}, and $\tilde{C}_{ia}$.

Procedure

Step 1. Decide the quality requirement C (normally set to 1.00, 0.56, 0.44, 0.36 and 0.25) and the α-risk (normally set to 0.01, 0.025 or 0.05), the chance of wrongly concluding an incapable process as capable.

Step 2. Calculate the values of $\hat{\delta} = n(\bar{X} - T)^2/s^2$, $\tilde{C}_{ia}$ and $\hat{C}_{pp}$ from the sample data.

Step 3. Conclude that the process satisfies the quality requirement if the $\hat{C}_{pp}$ value is less than $\chi^2_{n,\alpha}(\hat{\delta})(C - \tilde{C}_{ia})/n$. Otherwise, we do not have enough information to conclude that the process is capable.

For example, consider a manufacturing process with an upper specification limit $USL = 20$, a lower specification $LSL = 10$, a target value $T = 15$, capability requirement is set to $C = 1.00$. For the calculated sample mean $\bar{X} = 14.5$ and sample variance $S^2_{n-1} = 2.0$, under the risk $\alpha = 0.05$, we calculate $D = \min\{(USL - T)/3, (T - LSL)/3\}$ and $\hat{\delta} = n(\bar{X} - T)^2/s^2$, $\tilde{C}_{ia}$. Running the SAS program with those input data, we obtain the critical value $\chi^2_{n,\alpha}(\hat{\delta})(C - \tilde{C}_{ia})/n = 0.7246$ for $n = 50$. Thus, if the calculated value $\hat{C}_{pp} < 0.7246$, then we conclude that the process satisfies the quality requirement.

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