Rare-Event Component Importance for the Consecutive-k System

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Received October 2000; revised May 2001; accepted 3 August 2001

Abstract: Various indices of component importance with respect to system reliability have been proposed. The most popular one is the Birnbaum importance. In particular, a special case called uniform Birnbaum importance in which all components have the same reliability p has been widely studied for the consecutive-k system. Since it is not easy to compare uniform Birnbaum importance, the literature has looked into the case $p = \frac{1}{2}, p \to 1$, or $p \geq \frac{1}{2}$. In this paper, we look into the case $p \to 0$ to complete the spectrum of examining Birnbaum importance over the whole range of p. © 2002 Wiley Periodicals, Inc. Naval Research Logistics 49: 159–166, 2002; DOI 10.1002/nav.10001

Keywords: consecutive-k system; system reliability; Birnbaum importance

1. INTRODUCTION

We consider a system T consisting of functionally identical components. The system state and the component state are both binary, either working or failed. We define an *aggregate state* as the union of the component-i state over all i. The functioning of the system is defined by a structural function f which maps $\{$aggregate state$\}$ to $\{$system state$\}$, i.e., $f(S) = 1$ (or 0) means the aggregate state S induces a working (failed) system. Let $R(T)$ denote the system reliability and $P(S)$ the probability of the aggregate state S. Then

$$R(T) = \sum_S f(S)P(S).$$

Note that $P(S)$ is a function of the component reliabilities $(p_1, p_2, \ldots, p_n)$; therefore, so is $R(T)$.

A component importance index measures the contribution of a (working) component towards the system reliability. The most important one is the Birnbaum importance [1] which is defined as $\partial_i R(T)$, where ∂_i is the derivative with respect to the reliability of component i. An importance index is *structural* if it depends only on $f(\cdot)$ but not on $P(\cdot)$. Note that the Birnbaum importance of component i depends both on the location of component i and the component reliabilities

$\{p_1, \ldots, p_n\}$ which appear in $R(T)$. Given a component importance index I_i for a system T, we get a structural importance index for T by applying I_i under the assumption $p_1 = p_2 = \cdots = p_n = p$ for some $p \in (0, 1)$. In the following we list several importance indices which have been studied in the literature (we write $i \to j$ to mean component i is more important than component j).

1. *Uniform Birnbaum importance [1].* $i \to j$ if

$$\partial_i R(T) \geq \partial_j R(T), \qquad \text{for all } 0 < p < 1.$$

2. *Combinatorial importance [9].* The special case of uniform Birnbaum importance with $p = \frac{1}{2}$.
3. *Half-line importance [6].* A weakened version of uniform Birnbaum importance by requiring $i \to j$ for all $p \geq \frac{1}{2}$.

Let $CS(T)$ (or $PS(T)$) denote the set of cutsets (or pathsets) of T. Let $CS_d(T)$ denote $CS(T)$ given d failed components and $PS_w(T)$ denote $PS(T)$ given w working components. Let $CS_{i,d}(T)$ denote $CS_d(T)$ given component i is failed and $PS_{i,w}(T)$ denote $PS_w(T)$ given component i is working. Furthermore, we use lower-case to denote cardinality, i.e., $cs(T)$ is the number of cutsets of T.

4. *Critical importance [2].* $i \to j$ if for any subset S not containing i and j

$$j \cup S \in CS_j(T) \Rightarrow i \cup S \in CS_i(T).$$

5. *H-importance [8].* $i \to j$ if

$$cs_{i,d}(T) \geq cs_{j,d}(T) \qquad \text{for all } d.$$

6. *Cut importance [3].* $i \to j$ if

$$(cs_{i,1}(T), cs_{i,2}(T), \cdots) \geq (cs_{j,1}(T), cs_{j,2}(T), \cdots) \qquad \text{lexicographically.}$$

7. *First-term importance [12].* Compare only the first term of the vector above.

Clearly, the critical importance implies the H-importance. Meng [10, 11] proved some relations with respect to the critical importance and the cut importance. It was also proved [8] that the H-importance implies uniform Birnbaum importance, which clearly implies the half-line importance, and thus the combinatorial importance. The justification of using the half-line importance is $p \geq \frac{1}{2}$ holds in most practical cases. In the more extreme case $p \to 1$, then a d-cutset is much more likely to occur than a d'-cutset for $d < d'$ and thus justifies the use of the first-term importance. The cut importance is an extension of the first-term importance by looking into more terms, but in the order of their likelihood. Thus we see that, while the critical importance, the H-importance, and the uniform Birnbaum importance are overall measures and usually too strong for comparisons, the other four indices all consider large p, or at least not small. In this paper we examine a special case from the other end of p when the number of working components is small. We call it the *rare-event* importance. There are two reasons to study the rare-event importance:

i. Showing $i \to j$ not true for the rare-event importance is an easy way to disprove a claim of $i \to j$ for the uniform Birnbaum importance.
ii. A comparison in the uniform Birnbaum importance is hard to obtain. Usually, we only establish it under the half-line importance, or under both the first-term and the combinatorial importance. If in addition, we establish the comparison under the rare-event importance, we obtain a strong indication that it might hold for all p.

In the following sections we study the rare-event importance of the consecutive-k-out-of-n system. Note that the first-term importance index of component i is simply the number of k-subsequences (with consecutive components) containing i. Thus the index is i for $i \le k-1$ or $i \ge n-k+2$, and is k otherwise. However, the rare-event importance index, surprisingly, employs interesting mathematics. We should make it clear that the rare-event importance index cannot be obtained from the first-term importance index by simply switching the roles of working and failed components, because the first-term importance index in the new system still counts the number of k-subsequences (of good components now) while the rare event importance index in the original system counts the number of minimum sets to prevent the occurrence of a k-subsequence (of failed components).

2. RARE-EVENT COMPONENT IMPORTANCE FOR THE CONSECUTIVE-*k* SYSTEM

A consecutive-k-out-of-n system is a line of n components which fails if and only if some k consecutive components all fail [7]. Component importance of this system has received a great deal of attention (see [5] for references) due to many applications.

Represent n as $n = qk + r$, where $0 \le r < k$. Then q is the minimum number of working components for a pathset to exist. Note that when $p \to 0$, then a system with exactly q working component dominates probability-wise a system with more working components. Yet a working system must have at least q working components. Thus the rare-event importance I_i^R of component i is defined to be $ps_{i,q}(k,n)$, the leading term in the system reliability. For example, for a consecutive-3-out-of-7 system (a,b,c,d,e,f,g), the minimum pathsets are $\{b,e\}$, $\{c,e\}$, and $\{c,f\}$. We now give some general formulas for I_i^R.

THEOREM 1: $ps_q(k,n) = \binom{q+k-r-1}{k-r-1}$.

PROOF: Theorem 1 is easily verified for $n \le k$. We prove the general case by induction on n. Consider $n = qk + r > k$. Suppose $r > 0$. Then

$$n - 1 = qk + (r-1).$$

Clearly, every pathset of the n-line is a pathset of the $(n-1)$-line. The additional pathsets of the $(n-1)$-line are those which has its last working component at position $n-k$ (so that the n-line is not working). The number of such pathsets is, of course, $ps_{q-1}(k, n-k-1)$. Therefore,

$$\begin{aligned} ps_q(k,n) &= ps_q(k,n-1) - ps_{q-1}(k,n-k-1) \\ &= \binom{q+k-(r-1)-1}{k-(r-1)-1} - \binom{(q-1)+k-(r-1)-1}{k-(r-1)-1} \end{aligned}$$

$$= \binom{q+k-r}{k-r} - \binom{q+k-r-1}{k-r}$$
$$= \binom{q+k-r-1}{k-r-1}.$$

Next suppose $r = 0$. The last working component must occupy one of the last k positions, say m, and the first $m-1$ components must form a working line with $q-1$ working components. Therefore,

$$\begin{aligned} ps_q(k,n) &= \sum_{i=1}^{k} ps_{q-1}(k, n-i) \\ &= \sum_{i=1}^{k} \binom{q-1+k-(k-i)-1}{k-(k-i)-1} \\ &= \sum_{i=1}^{k} \binom{q+i-2}{i-1} \\ &= \binom{q+k-1}{k-1}. \quad \square \end{aligned}$$

Represent $i = uk + v$, where $0 < v \le k$. Define $\binom{x}{y} = 0$ for $y < 0$ as usual.

THEOREM 2: $ps_{i,q}(k,n) = \binom{u+k-v}{k-v}\binom{q-u+v-r-2}{v-r-1}$.

PROOF: $PS_{i,q}(k,n)$ consists of those pathsets with u working components in the first $i-1$ positions and $q-u-1$ working components in the last $n-i$ positions. In other words, it consists of the product of $PS_u(k, i-1)$ and $PS_{q-u-1}(k, n-i)$. Therefore,

$$ps_{i,q}(k,n) = ps_u(k, i-1)ps_{q-u-1}(k, n-i).$$

If $v < r+1$, then

$$n - i = (q-u)k + (r-v).$$

Hence, $ps_{q-u-1}(k, n-i) = 0$ and Theorem 2 is trivially true.

Next assume $v \ge r+1$. Then

$$n - i = (q-u-1)k + (k+r-v).$$

Therefore,

$$\begin{aligned} ps_{i,q}(k,n) &= \binom{u+k-(v-1)-1}{k-(v-1)-1}\binom{(q-u-1)+k-(k+r-v)-1}{k-(k+r-v)-1} \\ &= \binom{u+k-v}{k-v}\binom{q-u+v-r-2}{v-r-1}, \end{aligned}$$

by using Theorem 1. $\square$

There exists a relation between k consecutive components starting at $i = tk+1$ for some t.

THEOREM 3: $\sum_{i=tk+1}^{tk+k} ps_{i,q}(k,n) = ps_q(k,n)$ for each $t = 0, 1, \ldots, q-1$.

PROOF: We can prove Theorem 3 using Theorem 2; but that would be messy. Instead, we use a direct argument. Let the tth interval consist of the k positions $tk+1, tk+2, \ldots, tk+k$. A pathset of $PS_q(k,n)$ must intersect each interval or it wouldn't be a pathset. On the other hand, the pathset has q working components and there are q intervals, so it must intersect each interval exactly once. Therefore, this pathset is counted in exactly one $PS_{i,q}(k,n)$ for $tk+1 \leq i \leq tk+k$. Theorem 3 follows. □

3. COMPARISONS OF RARE-EVENT COMPONENT IMPORTANCE IN THE CONSECUTIVE-*k* SYSTEM

Recall [6] that the comparisons of the half-line importance I^h between components in a consecutive-k-out-of-n system $\{1, 2, \ldots, n\}$ are

$$I^h(1) < I^h(2) < \cdots < I^h(k-1) < I^h(k+1) < I^h(i) < I^h(2k) < I^h(k) \quad \text{for all } i > k+1 \text{ and } i \neq 2k.$$

Also,

$$I^h(2k+2) > I^h(2k+1), I^h(3k) > I^h(3k+1), \text{ and } I^h(i) < I^h(i+1) \quad \text{for } k+1 \leq i \leq 2k-1.$$

The comparisons on I_i^R are more restrictive and mostly between components k positions apart, or between the set of positions $\{uk+1\}$ and others. Some of these comparisons go beyond the comparisons of I^R, but others lag behind. In particular, the conjecture $I(2k) < I(i)$ for all $i > 2k$, suggested by its validity in I^h, is shattered by the corresponding comparison in I^R.

We first compare I_i^R with I_{i+k}^R.

THEOREM 4: I_{uk+k}^R is nonincreasing in u. Furthermore, it is decreasing except for $r = k-1$.

PROOF: By Theorem 2,

$$ps_{uk+k,q}(k,n) = \binom{q-u+k-r-2}{k-r-1}.$$

Therefore,

$$ps_{uk+k,q}(k,n) - ps_{(u+1)k+k,q}(k,n) = \begin{cases} 0 & \text{if } r = k-1, \\ \binom{q-u+k-r-3}{k-r-2} & \text{if } r < k-1, \end{cases} \geq 0.$$

Theorem 4 follows immediately. □

LEMMA 5: For $k > v \geq r+1$, $ps_{(u+1)k+v,q}(k,n) > ps_{uk+v,q}(k,n)$ if $q > (u+1)(k-r-1)/(k-v)$.

PROOF: By Theorem 2,

$$\frac{ps_{(u+1)k+v,q}(k,n)}{ps_{uk+v,q}(k,n)} = \frac{(u+1+k-v)(q-u-1)}{(u+1)(q-u+v-r-2)}.$$

Hence, $ps_{(u+1)k+v,q}(k,n) > ps_{uk+v,q}(k,n)$ if $q > (u+1)(k-r-1)/(k-v)$. □

COROLLARY 6: If $v = r + 1 \neq k$, then I^R_{uk+v} is nondecreasing in u.

PROOF: If $v = r + 1 \neq k$, then $q > (u+1)(k-r-1)/(k-v) = u + 1$. It implies, by Lemma 5, $ps_{(u+1)k+v,q}(k,n) > ps_{uk+v,q}(k,n)$. Hence, I^R_{uk+v} is nondecreasing in u. □

THEOREM 7: $I^R_{uk+v} \leq I^R_{(u+1)k+v}$ for $1 \leq v \leq (k+1)/2$ and $(u+1)k + v \leq (n+1)/2$.

PROOF: If $v < r + 1$, then by Theorem 2, $I^R_{uk+v} = 0$ for all u and Theorem 7 is trivially true. Hence we assume $v \geq r + 1$. Note that in the range of $(u+1)k + v$ specified in Theorem 7,

$$qk + r \equiv n \geq 2[(u+1)k + v] - 1 = 2(u+1)k + 2v - 1.$$

Since

$$2v - 1 > v - 1 \geq r.$$

necessarily, $q \geq 2u + 3$. Finally, $v \leq (k+1)/2$ implies $(k-1)/(k-v) \leq 2$. Hence

$$\frac{(u+1)(k-r-1)}{k-v} \leq \frac{(u+1)(k-1)}{k-v} \leq 2(u+1) < q.$$

Theorem 7 now follows from Lemma 5. □

Although Theorem 7 deals with only the first half components, we can compare the second half by noting $I^R_i = I^R_{n+1-i}$.

The range of v in Theorem 7 cannot be pushed further in general. For example,

EXAMPLE 1: $k = 4, n = 32, ps_{15,8}(4,32) = 60 < ps_{11,8}(4,32) = 63$.

Next, we compare the rare-event importance of position $uk + 1$ with the others.

THEOREM 8: $I^R_{uk+1} < I^R_{uk}$ for all u satisfying $uk + 1 \leq (n+1)/2$.

PROOF: By Theorem 2,

$$\begin{aligned} ps_{uk+1,q}(k,n) &= \binom{u+k-1}{k-1}\binom{q-u+1-r-2}{1-r-1} \\ &= \begin{cases} \binom{u+k-1}{k-1} & \text{if } r = 0, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

For $r = 0, uk + 1 \leq (n+1)/2$ implies $q \geq 2u + 1$.

$$\begin{aligned} ps_{uk,q}(k,n) &= \binom{q-u+k-r-1}{k-r-1} \\ &> ps_{uk+1,q}(k,n). \end{aligned}$$

Hence $I^R_{uk+1} < I^R_{uk}$ for all u. □

REMARK: Theorem 8 is not true for uniform Birnbaum importance [6].

THEOREM 9: $I^R_{uk+1} \le I^R_j$ for all $uk+1 < j \le (n+1)/2$.

PROOF: If $r \ne 0$, then by Theorem 2, $I^R_{uk+1} = 0$ for all u, and Theorem 9 is trivially true. Consider $r = 0$. First, we compare I^R_{uk+1} with I^R_{uk+v} for $2 \le v \le k$:

$$\frac{ps_{uk+v,q}(k,n)}{ps_{uk+1,q}(k,n)} = \frac{(k-v+1)(k-v+2)\cdots(k-1)(q-u)(q-u+1)\cdots(q-u+v-2)}{(u+k-v+1)(u+k-v+2)\cdots(u+k-1)\cdot 1\cdot 2\cdots(v-1)}.$$

Since the ratio is increasing in q, and $q \ge 2u+1$, we only need to consider $q = 2u+1$:

$$\frac{ps_{uk+v,q}(k,n)}{ps_{uk+1,q}(k,n)} = \frac{(k-v+1)(k-v+2)\cdots(k-1)(u+1)(u+2)\cdots(u+v-1)}{(u+k-v+1)(u+k-v+2)\cdots(u+k-1)\cdot 1\cdot 2\cdots(v-1)}.$$

Since

$$(k-v+l)(u+l) \ge l(u+k-v+l) \qquad \text{for } 1 \le l \le v-1,$$

$I^R_{uk+1} < I^R_{uk+v}$. By Theorem 7, I^R_{uk+1} is nondecreasing. Hence $I^R_{uk+1} \le I^R_j$ for all $j > uk+1$. □

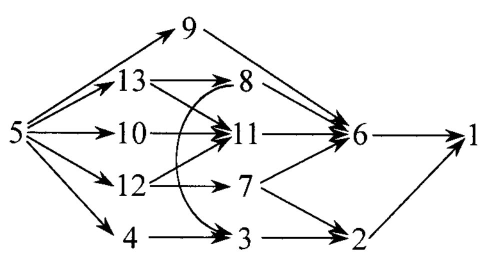
REMARK: Theorem 9 holds for half-line importance only for $u = 1$ [6].

We illustrate the above results by the consecutive 5-out-of-25 system $\{1, 2, \ldots, 25\}$. We only compare the first thirteen components and use $i \to j$ to indicate $I^R_i > I^R_j$.

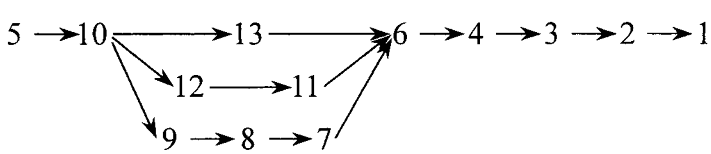
By Theorem 4, we have $5 \to 10$. By Corollary 6, we have $11 \to 6 \to 1$. By Theorem 7, we have $12 \to 7 \to 2$ and $13 \to 8 \to 3$. By Theorem 9, we have 7, 8, 9, 10, 11, 12, $13 \to 6$ and 12, $13 \to 11$. We also know the following from comparisons of uniform Birnbaum importance [6]:

$$5 \to 4 \to 3 \to 2 \to 1, \quad 5 \to i \text{ for all } i, \quad 10 \to 11, \quad 7 \to 6, \quad 10 \to 9.$$

Combining, we have



For comparison, the graph for I^h is



$I_{2k} > I_i$ for all $i > 2k$ was proved [6] for the half-line importance, and its holding for the uniform case was conjectured there. By showing $I^R_{2k} < I^R_i$ for some k and $i > 2k$, we disprove the conjecture:

EXAMPLE 2: For $k = 5, n = 25, ps_{10,5}(5, 25) = 35 < ps_{13,5}(5, 25) = 36$.

4. SOME CONCLUDING REMARKS

While the concept of importance index is universal for all systems, it can be extremely difficult to compute efficiently for most systems. A notable exception is the consecutive-k system for which some break-throughs in comparing component importance have been recently obtained [6, 9]. In this paper we provide one more such comparison, but from the other end of the p-spectrum for which no general method is known. This new piece of information has allowed us [4, 6] to determine the uniform-Birnbaum-importance comparability in several cases.

The background system does not have to be an engineering system. It could be a human organization where the components are various agents. For example, the agents can be the president, the senate, and the congress in a study of their relative power of passing a bill. Of course, in this particular example, the geometric structure existing in a consecutive-k system is not there. We hope that our study of component importance with the simple geometric structure here can be extended to more complicated geometric structure, and, perhaps, can shed light to problems where the structures are not geometric.

ACKNOWLEDGMENTS

We thank a reviewer for a careful reading and many helpful suggestions. H.-W. Chang acknowledges the support in part by the National Science Council under Grant NSC 89-2118-E-036-022.

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