A Constrained Multibody System Dynamics Avoiding Kinematic Singularities*

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In the analysis of constrained multibody systems, the constraint reaction forces are normally expressed in terms of the constraint equations and a vector of Lagrange multipliers. Because it fails to incorporate conservation of momentum, the Lagrange multiplier method is deficient when the constraint Jacobian matrix is singular. This paper presents an improved dynamic formulation for the constrained multibody system. In our formulation, the kinematic constraints are still formulated in terms of the joint constraint reaction forces and moments; however, the formulations are based on a second-order Taylor expansion so as to incorporate the rigid body velocities. Conservation of momentum is included explicitly in this method; hence the problems caused by kinematic singularities can be avoided. In addition, the dynamic formulation is general and applicable to most dynamic analyses. Finally the 3-leg Stewart platform is used for the example of analysis.

Key Words: Constrained Multibody System, Kinematic Constraints, Kinematic Singularity, Constraint Reaction Force

1. Introduction

Many dynamic analysis techniques have been developed for constrained multibody systems; for example, Hiller(1), Wittenburg(2), Kane and Levinson(3), and Huston(4) have all proposed such techniques. Some of these techniques employ relative coordinates to generate the equations of motion, some are based on the Lagrange equation, such as those of Hollerbach(5) and William(6), some are based on the Newton-Euler method, as in the cases of Stepanenko(7), Orin(8), Luh(9), and Hollerach(10), and others are based on Kane's equations, such as those of Wang(11) and Wampler(12). Several computational schemes have also been introduced. Nikravesh(13) formulated the equations of motion based on the Cartesian coordinate form. Haug(14) presented a variational-vector calculus to systematically transform the Cartesian equations of motion into relative coordinate form. Using the variational-vector calculus, Bae(15) developed a recursive formulation for a constrained multibody system. In addition, he also introduced the cut joint method(16), associated with Lagrange multipliers, for finding the spanning tree of the multibody system. Haug(17) further simplified the derivation by using a state vector notation. The kinematic relations can be shown in terms of the joint-coordinates and the state-vector notation. The equations of motion can be recursively derived in terms of relative coordinates. Sheth(18) applied graph theory to analyze the topology of multibody systems using relative coordinates. The topological analysis method generates the information necessary for a recursive dynamic formulation.

Many of the above developments have addressed the issue of computational efficiency. Hollerbach(5) concluded that the recursive Newton-Euler method is more efficient than the recursive Lagrangian formulation. Kane's equation is computationally more efficient than either Lagrange's equation or Newton-Euler methods, because it eliminates the inactive constraint reaction forces. Fewer authors have

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addressed the problem of avoiding kinematic singularities. Liang[19] introduced the singular value decomposition (SVD) technique, which can usually convert singular problems into nonsingular ones. Although SVD finds the optimum solution based on the least-squares method, it is a complex technique that requires a large amount of computing time.

In this paper, an improved dynamic analysis scheme based on the solution of constraint reaction forces and moments is developed for the constrained multibody dynamic system. This method is computationally efficient and avoids the problem of the kinematic singularity.

Nomenclature

- $\boldsymbol{A}_i$: the rotational transformation matrix between the local coordinate system $x_i-y_i-z_i$ to the global coordinate system $X-Y-Z$ of the i-th link
- $\boldsymbol{B}_i$: body force of the i-th link; the vector is given w.r.t. the global coordinate system
- $\boldsymbol{d}_i^C$: position vector of point c w.r.t. the global coordinate system $X-Y-Z$
- $\boldsymbol{d}_i'^C$: vector $\boldsymbol{d}_i^C$ w.r.t. the local (link) coordinate system $x_i-y_i-z_i$
- $\boldsymbol{d}_i'^q$: vector $\boldsymbol{d}_i^q$ of point q w.r.t. the local (link) coordinate system $x_i-y_i-z_i$
- $\boldsymbol{F}_i^q$: the q-th external force on link i; the vector is given w.r.t. the global coordinate system
- $\boldsymbol{H}_i$: function of the constraint reaction forces and moments on link i; the vector is given w.r.t. the global coordinate system
- m_i : the mass of link i
- $\boldsymbol{M}_i^q$: the q-th constraint reaction moment on link i; the vector is given w.r.t. the global coordinate system
- N_i' : total moments exerted on link i in terms of the local coordinate system
- $\boldsymbol{P}_i$: function of inertia and external forces/moments of link i; the vector is given w.r.t. the global coordinate system
- $\boldsymbol{q}_i$: any arbitrary vector attached on link i; the vector is given w.r.t. the global coordinate system
- $\boldsymbol{q}_i'$: any arbitrary vector attached on link i; the vector is given w.r.t. the local coordinate system
- $\boldsymbol{Q}_i$: constraint reaction forces and moments of link i; the vector is given w.r.t. the global coordinate system
- $\boldsymbol{r}_i^C$: position vector of point c of link i w.r.t. the global coordinate
- $\Delta\boldsymbol{r}_i^C$: displacement of point c on link i w.r.t. the global coordinate
- $\boldsymbol{R}_i^q$: the q-th constraint reaction force on link i; the vector is given w.r.t. the global coordinate system
- $\boldsymbol{T}_i^q$: the q-th external moment on link i; the vector is given w.r.t. the global coordinate system
- $\boldsymbol{U}_i$: vector function of the inertia and external forces/moments of link i; the vector is given w.r.t. the global coordinate system
- $\boldsymbol{W}_i$: vector function of the constraint reaction forces and moments on link i; the vector is given w.r.t. the global coordinate system
- $\boldsymbol{Z}$: the $m\times n$ coordinate transform matrix to let $\Phi_q^T\lambda=\boldsymbol{Z}^T\boldsymbol{Q}$
- ω_i : angular velocity of link i w.r.t. the global coordinate system
- ω_i' : angular velocity of link i w.r.t. the local coordinate system $x_i-y_i-z_i$
- α_i : angular acceleration of link i w.r.t. the global coordinate system
- α_i' : angular acceleration of link i w.r.t. the local coordinate system $x_i-y_i-z_i$

2. Foundation of the Kinematic Singularity-Free Formulation

For a constrained multibody system with m kinematic constraints $\Phi=[\phi_1\ \phi_2\ \cdots\ \phi_m]^T$, n generalized coordinates $\boldsymbol{q}=[q_1\ q_2\ \cdots\ q_n]^T$, the Lagrange multiplier formulation leads to the following algebraic equations of motion[22].

$$\begin{bmatrix} \boldsymbol{M} & \boldsymbol{\Phi}_q^T \\ \boldsymbol{\Phi}_q & 0 \end{bmatrix}\begin{bmatrix} \ddot{\boldsymbol{q}} \\ -\boldsymbol{\lambda} \end{bmatrix}=\begin{bmatrix} \boldsymbol{g} \\ \gamma \end{bmatrix} \quad (1)$$

where $\boldsymbol{M}$ denotes the $m\times n$ mass matrix, $\ddot{\boldsymbol{q}}$ denotes the acceleration vector, $\boldsymbol{\lambda}$ is the $m\times 1$ vector of Lagrange multiplier, $\boldsymbol{\Phi}_q$ denotes the $m\times n$ constraint Jacobian matrix, $\boldsymbol{g}$ denotes the $n\times 1$ vector of generalized forces and $\gamma=-(\boldsymbol{\Phi}_q\dot{\boldsymbol{q}})_q\dot{\boldsymbol{q}}$. Unfortunately, the method of Lagrange multipliers, embodied in the classical Lagrange's method, Newton-Euler's method and various forms of Hamilton's principle, is deficient when the constraint Jacobian matrix $\boldsymbol{\Phi}_q$ is singular.

Our proposed formulation allows the insertion of the constraints directly into the equation of motion without introducing Lagrange undetermined multipliers. Taking the Taylor expansion on the generalized coordinate $\boldsymbol{q}$ for a coordinate increment $\Delta\boldsymbol{q}$ and, the constraint equations can be written as follows:

$$\boldsymbol{\Phi}(\boldsymbol{q})=\boldsymbol{\Phi}(\ddot{\boldsymbol{q}},\ \dot{\boldsymbol{q}},\ \boldsymbol{q},\ \Delta t)=0 \quad (2.a)$$

The acceleration vector $\ddot{\boldsymbol{q}}$ can be expressed in terms of the generalized active force vector $\boldsymbol{g}$ and the unknown constraint reaction forces/moments vector $\boldsymbol{Q}$ as follows:

$$\ddot{\boldsymbol{q}}=\boldsymbol{M}^{-1}(\boldsymbol{g}+\boldsymbol{Z}^T\boldsymbol{Q}) \quad (2.b)$$

where $\boldsymbol{Z}$ is the $m\times n$ matrix and $\boldsymbol{Z}^T\boldsymbol{Q}$ is the $m\times 1$

vector of generalized constraint reaction forces/moments. Thus, Eq.(2.a) can be rewritten as follows:

$$\boldsymbol{\Phi}(\boldsymbol{Q}, \dot{\boldsymbol{q}}, \boldsymbol{q}, \Delta t)=0 \qquad (3)$$

Moreover, the dynamic equation, based on the solution of constraint reaction forces and moments of Eq.(3), can be derived for the constrained multibody dynamic system as follows:

$$\boldsymbol{\Phi}=\boldsymbol{\Phi}(\boldsymbol{Q}, \dot{\boldsymbol{q}}, \boldsymbol{q}, \Delta t)|_{Q=0}+\boldsymbol{\Phi}_Q|_{Q=0}\boldsymbol{Q} + \textit{higher order terms}=0$$

When the higher order terms are negligibly small, and the generalized active force $\boldsymbol{g}$ is constant during time interval Δt, we reach a linear expression for the solution of $\boldsymbol{Q}$ for small Δt as follows:

$$\boldsymbol{Q}=-\boldsymbol{\Phi}_Q^{-1}|_{Q=0}\,\Psi \qquad (4)$$

where

$$\Psi=\boldsymbol{\Phi}(\boldsymbol{Q}, \dot{\boldsymbol{q}}, \boldsymbol{q}, \Delta t)|_{Q=0} = -\boldsymbol{\Phi}_q\left[\dot{\boldsymbol{q}}\Delta t+\boldsymbol{M}^{-1}\boldsymbol{g}\frac{1}{2}\Delta t^2\right] -\frac{1}{2}\boldsymbol{\Phi}_{qq}\left[\dot{\boldsymbol{q}}\Delta t+\boldsymbol{M}^{-1}\boldsymbol{g}\frac{1}{2}\Delta t^2\right]^2$$

and

$$\boldsymbol{\Phi}_Q|_{Q=0}=\frac{1}{2}\Delta t^2\boldsymbol{M}^{-1}\boldsymbol{Z}^T\left[-\boldsymbol{\Phi}_q-\boldsymbol{\Phi}_{qq}\left[\dot{\boldsymbol{q}}\Delta t +\boldsymbol{M}^{-1}(\boldsymbol{g}+\boldsymbol{Z}^T\boldsymbol{Q})\frac{1}{2}\Delta t^2\right]\right]$$

(refer to Appendix A)

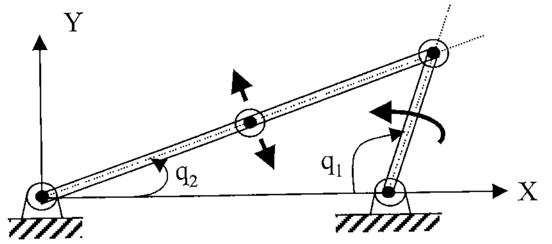
The importance of the above derivation lies in the fact that Eq.(4) is a form of the conventional momentum conservation equation. The momentum conservation employs the force as an unknown variable subjected to the kinematic constraint. In the following, we demonstrate that partial derivative matrix $\boldsymbol{\Phi}_Q$ cannot be singular even if the corresponding $\boldsymbol{\Phi}_q$ is singular. An example of planar four-bar linkage is shown in Fig. 1(a). Since the constraint Jacobian matrix $\boldsymbol{\Phi}_q$ is deficient at this singular point, the output velocity $\dot{q}_2$ can undergo infinitesimal motion even if the input is locked, i.e. $\dot{q}_1=0$. For simplicity, we assume that $\Psi=0$ of Eq.(4) in the above mechanism at the singular point. Since $\boldsymbol{\Phi}_Q$ is not deficient, a unique solution, i.e. $\boldsymbol{Q}=0$, is obtained. Accordingly, we can obtain the solution for output velocity as.

$$\dot{q}_2|_{t=0+}=\dot{q}_2|_{t=0-}.$$

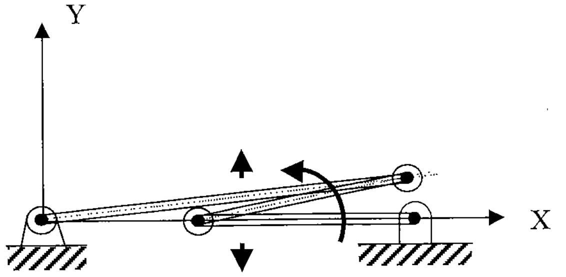
Equation (4) can be used to determine the joint reactions $\boldsymbol{Q}$ that acted on joints to produce constrained dynamic motion. This process is similar to the kinetostatic (kinematic-to-static) analysis[13] in which the determination of joint reactions can be transformed into a static analysis problem where a specific motion is sought. For the statics, the solution for joint reactions is uniquely determined unless the problem is statically indeterminate, as shown in Fig. 1(b). More detailed examples using our formulation will be provided in the later sections.

3. Spatial System

The forces and moments on the link, say link i, may be classified into three kinds: the constraint reaction forces and moments $\boldsymbol{Q}_i$, the external forces (or moments) $\boldsymbol{F}_i$ (or $\boldsymbol{T}_i$), and the body forces, such as gravity, denoted by $\boldsymbol{B}_i$. $\boldsymbol{Q}_i$ is a column vector consisting of the constraint reaction force $\boldsymbol{R}_i^k$ and moment $\boldsymbol{M}_i^k$ for all the joints k on link i, as shown in Fig. 2. The forces and moments are all given in terms of the



(a) Kinematic singularity which $\det(\boldsymbol{\Phi}_q)=0$, $\det(\boldsymbol{\Phi}_q+\boldsymbol{\Phi}_{qq}\varkappa)\neq 0$ and $\det(\boldsymbol{\Phi}_Q)\neq 0$



(b) Static indetermination which $\det(\boldsymbol{\Phi}_q)=0$, $\det(\boldsymbol{\Phi}_q+\boldsymbol{\Phi}_{qq}\varkappa)=0$ and $\det(\boldsymbol{\Phi}_Q)=0$, $\varkappa=\left[\dot{q}\Delta t+\boldsymbol{M}^{-1}(\boldsymbol{g}+\boldsymbol{Z}^T\boldsymbol{Q})\frac{1}{2}\Delta t^2\right]$ (see Appendix A)

Fig. 1 Two examples of planar 4R mechanisms

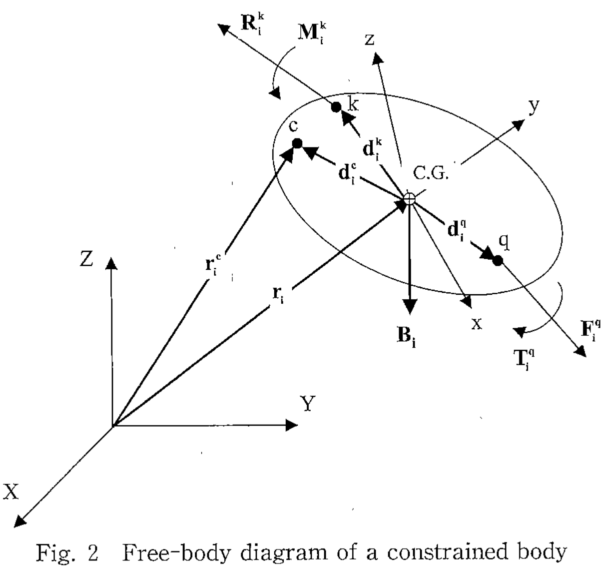


Fig. 2 Free-body diagram of a constrained body

global coordinate system.

The rotational transformation matrix $\boldsymbol{A}$ maps from the local $x-y-z$ frame to the global $X-Y-Z$ frame. Define a superscript operator' which converts a vector $\boldsymbol{\omega}$ from the global coordinates into the local coordinates $\boldsymbol{\omega}'$, i.e. $\boldsymbol{\omega}=A\boldsymbol{\omega}'$. An overhead operator $\hat{}$ which transforms the vector $\boldsymbol{\omega}$ into a skew-symmetric matrix is as follows:

$$\hat{\boldsymbol{\omega}}=\begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix}$$

The Poisson kinematic equation can be stated as follows:

$$\dot{\boldsymbol{A}}_i=\hat{\boldsymbol{\omega}}_i\boldsymbol{A}_i=\boldsymbol{A}_i\hat{\boldsymbol{\omega}}'_i \tag{5.a}$$

where $\boldsymbol{A}_i$ is the rotational transformation matrix of the $x_i-y_i-z_i$ frame relative to the $X-Y-Z$ frame. The second derivative is

$$\ddot{\boldsymbol{A}}_i=\boldsymbol{A}_i\hat{\boldsymbol{\alpha}}'_i+\boldsymbol{A}_i\hat{\boldsymbol{\omega}}'_i\hat{\boldsymbol{\omega}}'_i \tag{5.b}$$

where $\boldsymbol{\alpha}_i$ and $\boldsymbol{\alpha}'_i$ denote the angular acceleration in terms of the global coordinate system and the local coordinate system, respectively. For detailed proofs of Eqs.(5.a) and (5.b), the reader may refer to Refs. (13) and (20).

An $x_i-y_i-z_i$ frame is fixed in the mass center of a moving body. Let c be a point fixed in the $x_i-y_i-z_i$ frame, as shown in Fig. 2. The position vector of the point c with respect to the $X-Y-Z$ frame can then be obtained as follows:

$$\boldsymbol{r}_i^C=\boldsymbol{r}_i+\boldsymbol{A}_i\boldsymbol{d}_i'^C \tag{6}$$

where $\boldsymbol{r}_i$ denotes the position of the mass center of the moving body and $\boldsymbol{d}_i'^C$ denotes the position of point c in the local $x_i-y_i-z_i$ coordinates. Since the orientation of the body will change as time progresses, the $\boldsymbol{r}_i$ and $\boldsymbol{A}_i$ are functions of time. The absolute velocity can be obtained from the time derivative of the above equation as follows:

$$\dot{\boldsymbol{r}}_i^C=\dot{\boldsymbol{r}}_i+\boldsymbol{A}_i\hat{\boldsymbol{\omega}}'_i\boldsymbol{d}_i'^C. \tag{7}$$

Using Eq.(5), the second derivatives of the position vector can be written as follows:

$$\ddot{\boldsymbol{r}}_i^C=\ddot{\boldsymbol{r}}_i+\boldsymbol{A}_i\hat{\boldsymbol{d}}_i'^{CT}\boldsymbol{\alpha}'_i+\boldsymbol{A}_i\hat{\boldsymbol{\omega}}'_i\hat{\boldsymbol{\omega}}'_i\boldsymbol{d}_i'^C. \tag{8}$$

4. Equations of Motion

From Newton's second law of motion, the linear acceleration vector of the mass center, in terms of global coordinates, can be expressed as follows:

$$\ddot{\boldsymbol{r}}_i=\frac{1}{m_i}\Big(\sum_k\boldsymbol{R}_i^K+\sum_q\boldsymbol{F}_i^q+\boldsymbol{B}_i\Big) \tag{9}$$

where the vectors $\boldsymbol{R}_i^k$, $\boldsymbol{F}_i^q$ and $\boldsymbol{B}_i$ are as defined previously and m_i denotes the mass of the i-th link.

By the Euler's equations of motion, the angular acceleration is

$$\boldsymbol{\alpha}'_i=\dot{\boldsymbol{\omega}}'_i=\boldsymbol{J}_i'^{-1}(\boldsymbol{N}'_i-\hat{\boldsymbol{\omega}}'_i\boldsymbol{J}'_i\boldsymbol{\omega}'_i) \tag{10}$$

where $\boldsymbol{J}'_i$ denotes the local inertia matrix of link i. Since the local coordinate system is chosen to coincide with the principal axes of inertia at the mass center, $\boldsymbol{J}'_i$ is a diagonal matrix. $\boldsymbol{N}'_i$ denotes the total moments exerted on link i w.r.t. the local coordinate system,

$$\boldsymbol{N}'_i=\sum_k\{\hat{\boldsymbol{d}}_i'^k\boldsymbol{A}_i^T\boldsymbol{R}_i^k+\boldsymbol{A}_i^T\boldsymbol{M}_i^k\}+\sum_q\{\hat{\boldsymbol{d}}_i'^q\boldsymbol{A}_i^T\boldsymbol{F}_i^q+\boldsymbol{A}_i^T\boldsymbol{T}_i^q\} \tag{11}$$

Because the time interval Δt is very small, the translation of point c can be derived by the Taylor expansion to the second order as follows:

$$\begin{aligned}\Delta\boldsymbol{r}_i^C&\approx\dot{\boldsymbol{r}}_i^C\Delta t+\frac{1}{2}\ddot{\boldsymbol{r}}_i^C\Delta t^2\\ &\approx\dot{\boldsymbol{r}}_i\Delta t+\dot{\boldsymbol{A}}_i\boldsymbol{d}_i'^C\Delta t+\frac{1}{2}\ddot{\boldsymbol{r}}_i\Delta t^2+\frac{1}{2}\ddot{\boldsymbol{A}}_i\boldsymbol{d}_i'^C\Delta t^2\end{aligned} \tag{12}$$

However, Taylor expansion method with a finite number of terms is not an exact method, but rather an approximate method. The integration step Δt can be tuned according to the required analysis precision. In general, the accumulation error for second-order Taylor expansion of the constraint function is $O(\Delta t^2)$[21]. By first substituting Eq.(11) into Eq.(10), then carrying the result into Eq.(8), and lastly applying the result along with Eq.(9) and (7) into Eq. (12), we obtain

$$\Delta\boldsymbol{r}_i^C=\boldsymbol{U}_i^C+\boldsymbol{W}_i^C \tag{13.a}$$

where $\boldsymbol{U}_i^C$ is independent of the constraint reaction forces and moments $\boldsymbol{Q}_i$ and $\boldsymbol{W}_i^C$ is a function of $\boldsymbol{Q}_i$.

$$\begin{aligned}\boldsymbol{U}_i^C=&(\dot{\boldsymbol{r}}_i+\boldsymbol{A}_i\hat{\boldsymbol{\omega}}'_i\boldsymbol{d}_i'^C)\Delta t\\ &+\Big(\boldsymbol{A}_i\hat{\boldsymbol{d}}_i'^{CT}\boldsymbol{C}_i+\boldsymbol{A}_i\hat{\boldsymbol{\omega}}'\hat{\boldsymbol{\omega}}'_i\boldsymbol{d}_i'^C+\frac{1}{m_i}\sum_q\boldsymbol{F}_i^q\\ &+\frac{1}{m_i}\boldsymbol{B}_i\Big)\frac{\Delta t^2}{2}\end{aligned}$$

and

$$\boldsymbol{W}_i^C=\Big(\boldsymbol{A}_i\hat{\boldsymbol{d}}_i'^{CT}\boldsymbol{D}_i+\frac{1}{m_i}\sum_k\boldsymbol{R}_i^k\Big)\frac{\Delta t^2}{2} \tag{13.b}$$

where

$$\boldsymbol{C}_i=\boldsymbol{J}_i'^{-1}\Big(\sum_q\{\hat{\boldsymbol{d}}_i'^q\boldsymbol{A}_i^T\boldsymbol{F}_i^q+\boldsymbol{A}_i^T\boldsymbol{T}_i^q\}-\hat{\boldsymbol{\omega}}'_i\boldsymbol{J}'_i\boldsymbol{\omega}'_i\Big) \tag{13.c}$$

and

$$\boldsymbol{D}_i=\boldsymbol{J}_i'^{-1}\sum_k\{\hat{\boldsymbol{d}}_i'^k\boldsymbol{A}_i^T\boldsymbol{R}_i^K+\boldsymbol{A}_i^T\boldsymbol{M}_i^k\} \tag{13.d}$$

Note that $\boldsymbol{r}_i^C$, the position vector of point c at time t, is independent of the joint constraint reaction forces or moments within the time interval $[t, t+\Delta t]$. By Eqs.(13.b) and (13.d), we obtain

$$\frac{\partial\Delta\boldsymbol{r}_i^C}{\partial\boldsymbol{R}_i^k}=\frac{\partial\boldsymbol{W}_i^C}{\partial\boldsymbol{R}_i^k}=\frac{\Delta t^2}{2}\Big[\frac{1}{m_i}\boldsymbol{I}_{3*3}+\boldsymbol{A}_i\hat{\boldsymbol{d}}_i'^{CT}\boldsymbol{J}_i'^{-1}\hat{\boldsymbol{d}}_i'^k\boldsymbol{A}_i^T\Big] \tag{14.a}$$

where $\boldsymbol{I}_{3*3}$ is a 3 by 3 identity matrix, and

$$\frac{\partial\Delta\boldsymbol{r}_i^C}{\partial\boldsymbol{M}_i^k}=\frac{\partial\boldsymbol{W}_i^C}{\partial\boldsymbol{M}_i^k}=\frac{\Delta t^2}{2}[\boldsymbol{A}_i\hat{\boldsymbol{d}}_i'^{CT}\boldsymbol{J}_i'^{-1}\boldsymbol{A}_i^T] \tag{14.b}$$

The components of a vector $\boldsymbol{q}_i$ on link i are transformed from the $x_i-y_i-z_i$ coordinate system to the $X-Y-Z$ coordinate system as follows:

$$\boldsymbol{q}_i=\boldsymbol{A}_i\boldsymbol{q}'_i \tag{15}$$

Vector $\boldsymbol{q}'_i$ is the local components of vector $\boldsymbol{q}_i$ on link i defined in the $x_i-y_i-z_i$ coordinate system, and

$$\dot{\boldsymbol{q}}_i=\dot{\boldsymbol{A}}_i\boldsymbol{q}'_i=\boldsymbol{A}_i\widehat{\boldsymbol{\omega}}'_i\boldsymbol{q}'_i \quad (16)$$

Differentiating Eq. (16) with respect to time yields

$$\ddot{\boldsymbol{q}}_i=\ddot{\boldsymbol{A}}_i\boldsymbol{q}'_i=\boldsymbol{A}_i(\widehat{\boldsymbol{\alpha}}'_i+\widehat{\boldsymbol{\omega}}'_i\widehat{\boldsymbol{\omega}}'_i)\boldsymbol{q}'_i \quad (17)$$

Incorporating a small time increment Δt, we have

$$\Delta\boldsymbol{q}_i\approx\dot{\boldsymbol{q}}_i\Delta t+\frac{1}{2}\ddot{\boldsymbol{q}}_i\Delta t^2=\boldsymbol{P}_i+\boldsymbol{H}_i \quad (18)$$

Where by Eq. (8), we obtain

$$\boldsymbol{P}_i=\widehat{\boldsymbol{\omega}}_i\boldsymbol{A}_i\boldsymbol{q}'_i\Delta t+(\boldsymbol{A}_i\widehat{\boldsymbol{q}}'^T_i\boldsymbol{C}_i+\boldsymbol{A}_i\widehat{\boldsymbol{\omega}}'_i\widehat{\boldsymbol{\omega}}'_i\boldsymbol{q}'_i)\frac{\Delta t^2}{2} \quad (19.a)$$

and

$$\boldsymbol{H}_i=\boldsymbol{A}_i\widehat{\boldsymbol{q}}'^T_i\boldsymbol{D}_i\frac{\Delta t^2}{2} \quad (19.b)$$

where $\boldsymbol{C}_i$ and $\boldsymbol{D}_i$ are defined as in Eqs. (13.c) and (13.d). Since $\boldsymbol{q}_i(t)$ stands for the vector at time t strictly, which is independent of the joint constraint reaction forces or moments in the interval $[t, t+\Delta t]$, we obtain

$$\frac{\partial\Delta\boldsymbol{q}_i}{\partial\boldsymbol{R}^k_i}=\frac{\partial\boldsymbol{H}_i}{\partial\boldsymbol{R}^k_i}=\frac{\Delta T^2}{2}\boldsymbol{A}_i\widehat{q}'^T_i\boldsymbol{J}'^{-1}_i\widehat{\boldsymbol{d}}'^k_i\boldsymbol{A}^T_i \quad (20.a)$$

and

$$\frac{\partial\Delta\boldsymbol{q}_i}{\partial\boldsymbol{M}^k_i}=\frac{\partial\boldsymbol{H}_i}{\partial\boldsymbol{M}^k_i}=\frac{\Delta t^2}{2}\boldsymbol{A}_i\widehat{\boldsymbol{q}}'^T_i\boldsymbol{J}'^{-1}_i\boldsymbol{A}^T_i \quad (20.b)$$

5. Spherical Joint

A spherical joint that coincides two links i and j, as shown in Fig. 3, must be subject to the following three kinematic constraints due to the joint position c:

$$\Delta\boldsymbol{r}^c_i+\boldsymbol{r}^c_i=\Delta\boldsymbol{r}^c_j+\boldsymbol{r}^c_j \quad (21)$$

When Eq. (13) is substituted into the above equation, it yields

$$\boldsymbol{W}^c_i-\boldsymbol{W}^c_j=(\boldsymbol{r}^c_j-\boldsymbol{r}^c_i)+(\boldsymbol{U}^c_j-\boldsymbol{U}^c_i) \quad (22)$$

As indicated previously, $\boldsymbol{Q}_i$ and $\boldsymbol{Q}_j$ are used to denote the constraint reaction force and moment vectors on link i and link j, respectively, during the time interval $[t, t+\Delta t]$. Vector $\boldsymbol{Q}_i$ consists of the constraint reaction forces $\boldsymbol{R}^k_i$ and moments $\boldsymbol{M}^k_i$ for all the joints k on link i, and vector $\boldsymbol{Q}_j$ consists of the forces and moments for the joints on link j. According to Eqs. (13.b) and (13.d), the function $\boldsymbol{W}^C_i$ is a linear function of individual constraint reaction force $\boldsymbol{R}^k_i$ and moment $\boldsymbol{M}^k_i$. By Eq. (14), Eq. (22) can be expressed in terms of the constraint reaction forces and moments on link i and link j as the following linear relation:

$$\frac{\partial\boldsymbol{W}^C_i}{\partial\boldsymbol{Q}_i}\boldsymbol{Q}_i-\frac{\partial\boldsymbol{W}^C_j}{\partial\boldsymbol{Q}_j}\boldsymbol{Q}_j=(\boldsymbol{r}^C_j-\boldsymbol{r}^C_i)+(\boldsymbol{U}^C_j-\boldsymbol{U}^C_i) \quad (23)$$

The partial derivative terms for the above equation are given in Eqs. (14.a) and (14.b). On the right-hand side of Eq. (23), the vectors $\boldsymbol{U}^C_i$ and $\boldsymbol{U}^C_i$ contain the linear velocity, the angular velocity, the body force, and the external forces/moments. Both are known at time t and they embody the momentum conservation law for each individual link i and j.

6. Spatial Revolute Joint

Five kinematic constraint equations are needed to define a revolute joint. A revolute joint between link i and j is shown in Fig. 4. Similar to the spherical joint, the revolute joint must also be subject to three translational constraint equations in Eq. (23). Furthermore, the unit vector $\boldsymbol{q}_i$ of link i and unit vector $\boldsymbol{q}_j$ of link j on the revolute joint must remain parallel, hence we have two independent equations as follows:

$$(\boldsymbol{q}_i+\Delta\boldsymbol{q}_i)\times(\boldsymbol{q}_j+\Delta\boldsymbol{q}_j)=0 \quad (25)$$

When we substitute Eq. (18) into the above equation, the resultant equation in matrix form becomes:

$$(\widehat{\boldsymbol{q}}_i+\widehat{\boldsymbol{H}}_i+\widehat{\boldsymbol{P}}_i)(\boldsymbol{q}_j+\boldsymbol{H}_j+\boldsymbol{P}_j)=0 \quad (26)$$

Since

$$\widehat{\boldsymbol{H}}_j\boldsymbol{P}_i=-\widehat{\boldsymbol{P}}_i\boldsymbol{H}_j$$

By neglecting the second order term $\widehat{\boldsymbol{H}}_i\boldsymbol{H}_j$, we can expand Eq. (26) into

$$(\widehat{\boldsymbol{q}}_j+\widehat{P}_j)\frac{\partial\boldsymbol{H}_i}{\partial\boldsymbol{Q}_i}\boldsymbol{Q}_i-(\widehat{\boldsymbol{q}}_i+\widehat{\boldsymbol{P}}_i)\frac{\partial\boldsymbol{H}_j}{\partial\boldsymbol{Q}_j}\boldsymbol{Q}_j=(\widehat{\boldsymbol{q}}_i+\widehat{\boldsymbol{P}}_i)(\boldsymbol{q}_j+\boldsymbol{P}_j) \quad (27)$$

The partial derivatives in Eq. (27) can be expressed in terms of the derivatives, as shown in Eq. (20).

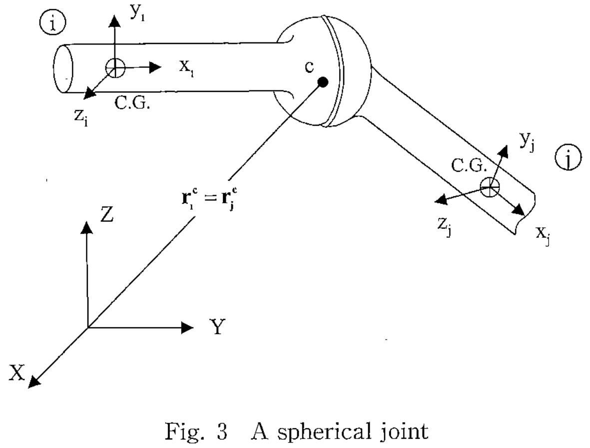


Fig. 3 A spherical joint

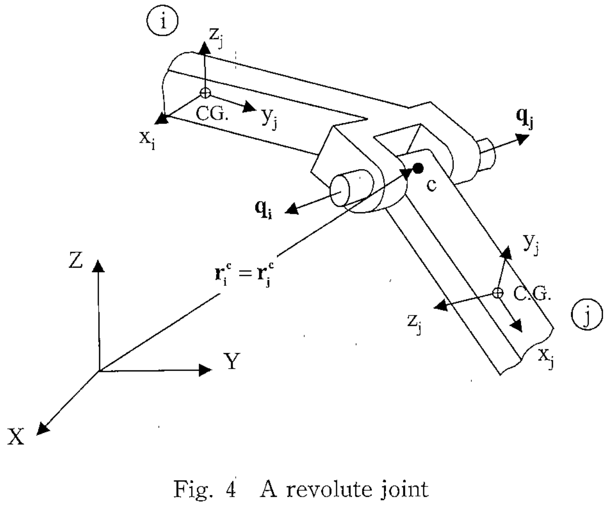


Fig. 4 A revolute joint

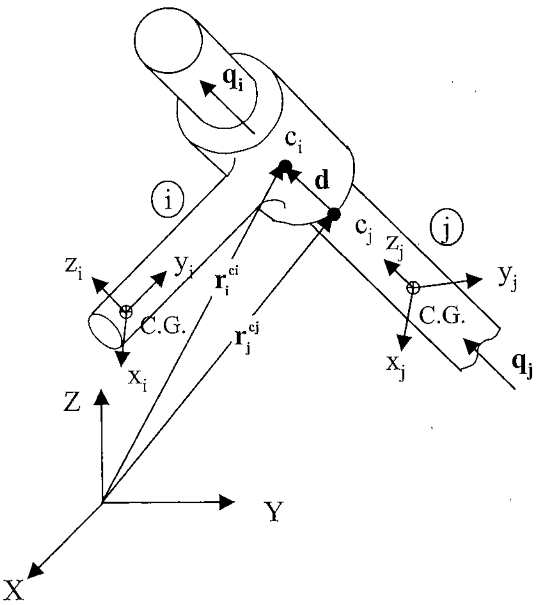


Fig. 5 A cylindrical joint

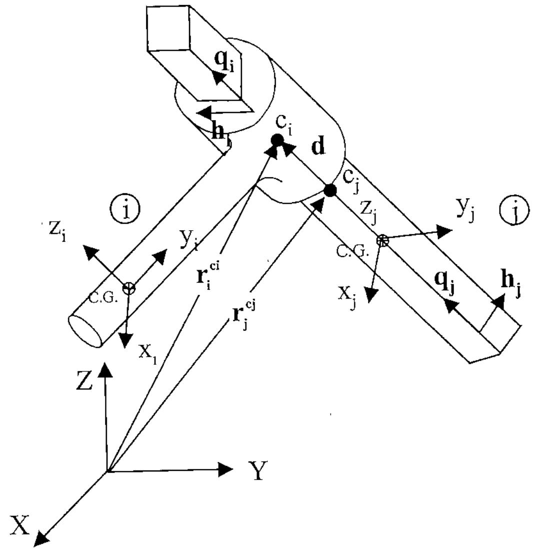


Fig. 6 A prismatic joint

7. Spatial Cylindrical Joint

In Fig. 5, the vectors $\boldsymbol{q}_i$ and $\boldsymbol{q}_j$ are located on the common axis of rotation. The variable $\boldsymbol{d}$ denotes the distance vector from point c_i of link i to point c_j of link j. Since the vector $\boldsymbol{q}_i$ of link i and the vector $\boldsymbol{q}_j$ of link j are parallel, the constraint equations of Eq. (27) must hold. In addition, the vectors $\boldsymbol{q}_i$ and $\boldsymbol{d}$ are parallel, so we have

$$(\boldsymbol{q}_i+\Delta\boldsymbol{q}_i)\times(\boldsymbol{d}+\Delta\boldsymbol{d})=0 \qquad (28)$$

where by definition

$$\boldsymbol{d}+\Delta\boldsymbol{d}=(\boldsymbol{r}_i^{ci}-\boldsymbol{r}_j^{cj})+(\Delta\boldsymbol{r}_i^{ci}-\Delta\boldsymbol{r}_j^{cj}) \qquad (29)$$

Substituting Eqs. (13) and (19) into Eq. (28), we obtain

$$(\boldsymbol{q}_i+\boldsymbol{P}_i+\boldsymbol{H}_i)\times[(\boldsymbol{r}_i^{ci}-\boldsymbol{r}_j^{cj})+(\boldsymbol{U}_i^{ci}-\boldsymbol{U}_j^{cj})+(\boldsymbol{W}_i^{ci}-\boldsymbol{W}_j^{cj})]=0 \qquad (30)$$

Substituting Eqs. (14) and (20) into Eq. (30), we obtain the following matrix form:

$$\left\{[(\hat{\boldsymbol{r}}_i^{ci}-\hat{\boldsymbol{r}}_j^{cj})+(\hat{\boldsymbol{U}}_i^{ci}-\hat{\boldsymbol{U}}_j^{cj})]\frac{\partial\boldsymbol{H}_i}{\partial\boldsymbol{Q}_j}-(\hat{\boldsymbol{q}}_i+\hat{\boldsymbol{P}}_i)\frac{\partial\boldsymbol{W}_i}{\partial\boldsymbol{Q}_j}\right\}\boldsymbol{Q}_i+\left\{(\hat{\boldsymbol{q}}_i+\hat{\boldsymbol{P}}_i)\frac{\partial\boldsymbol{W}_j}{\partial\boldsymbol{Q}_j}\right\}\boldsymbol{Q}_j=[(\hat{\boldsymbol{r}}_i^{ci}-\hat{\boldsymbol{r}}_j^{cj})+(\hat{\boldsymbol{U}}_i^{ci}-\hat{\boldsymbol{U}}_j^{cj})](\boldsymbol{q}_i+\boldsymbol{P}_i) \qquad (31)$$

Equations (27) and (31) yield four linear equations in terms of the constraint reaction force and moment vectors $\boldsymbol{Q}_i$ and $\boldsymbol{Q}_j$.

8. Spatial Prismatic Joint

As shown in Fig. 6, the prismatic joint is similar to the cylindrical joint except that the two adjacent links are not allowed to have relative rotation. Therefore, the prismatic joint requires one more kinematic constraint equation; for instance, the vector $\boldsymbol{h}_i$ on link i and the vector $\boldsymbol{h}_j$ on link j must remain perpendicular at all times. The constraint equation can be derived from the vector inner product condition as follows:

$$(\boldsymbol{h}_i+\Delta\boldsymbol{h}_i)\cdot(\boldsymbol{h}_j+\Delta\boldsymbol{h}_j)=0 \qquad (32)$$

Substituting Eqs. (19) and (20) into Eq (32), we have

$$\left\{(\boldsymbol{h}_j+\boldsymbol{P}_j)^T\frac{\partial\boldsymbol{H}_i}{\partial\boldsymbol{Q}_i}\right\}\boldsymbol{Q}_i+\left\{(\boldsymbol{h}_i+\boldsymbol{P}_i)^T\frac{\partial\boldsymbol{H}_j}{\partial\boldsymbol{Q}_j}\right\}\boldsymbol{Q}_j=-(\boldsymbol{h}_i+\boldsymbol{P}_i)^T(\boldsymbol{h}_j+\boldsymbol{P}_j) \qquad (33)$$

where the partial derivatives have been derived in Eqs. (14.a) and (14.b).

Vectors $\boldsymbol{h}_i$, $\boldsymbol{h}_j$, $\boldsymbol{P}_i$, and $\boldsymbol{P}_j$ are known at time t.

9. Constraint Reaction Force and Moment Balance

For all types of kinematic pairs, the constraint reaction force and moments must be balanced within the pairs. The constraint reaction force balance equation for joint k is written as follows:

$$\boldsymbol{R}_i^k=-\boldsymbol{R}_j^k \qquad (34)$$

and the constraint reaction moment balance equation is

$$\boldsymbol{M}_i^k=-\boldsymbol{M}_j^k \qquad (35)$$

10. Dynamic Equation of Motion and Analysis Recursion

The joint constraint force and moments, for multibody dynamics as shown in Eqs. (23), (27), (31) and (33) may be formulated as the following linear relation:

$$\begin{bmatrix}\left(\dfrac{\partial f(\boldsymbol{W},\boldsymbol{H})}{\partial\boldsymbol{Q}}\right)_{Joint\ 1}\\ \vdots \\ \left(\dfrac{\partial f(\boldsymbol{W},\boldsymbol{H})}{\partial\boldsymbol{Q}}\right)_{Joint\ n}\\ \vdots\end{bmatrix}\begin{bmatrix}(\boldsymbol{Q})_{Joint\ 1}\\ \vdots \\ (\boldsymbol{Q})_{Joint\ n}\\ \vdots\end{bmatrix}$$

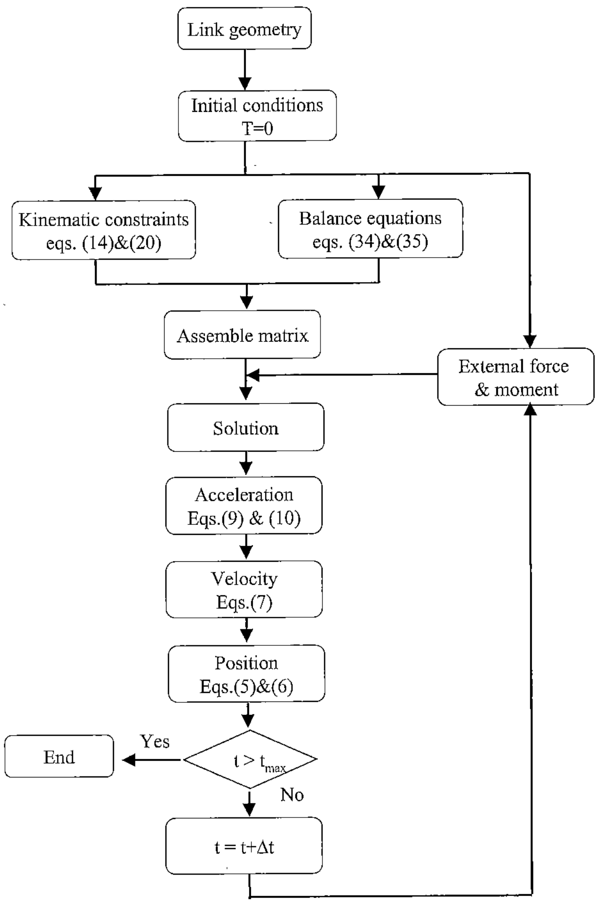


Fig. 7 Flow chart of the dynamic simulation

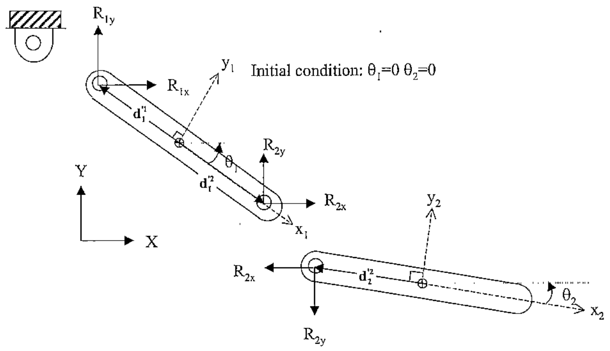
$$=\begin{bmatrix}(\Psi(\boldsymbol{r}, \boldsymbol{U}, \boldsymbol{q}, \boldsymbol{P}))_{Joint\ 1}\\ \vdots \\ (\Psi(\boldsymbol{r}, \boldsymbol{U}, \boldsymbol{q}, \boldsymbol{P}))_{Joint\ n}\\ \vdots\end{bmatrix} \tag{36}$$

where subscript *Joint n* denotes the n-th joint in the multibody system. The algorithm for the dynamic analysis is illustrated in Fig. 7. Based on the Gaussian-elimination method, a unique set of constraint forces $\boldsymbol{Q}$ is obtained when these constraint equations are considered simultaneously. Furthermore, the acceleration, velocity and position analyses can be accomplished in a straightforward manner using Eqs. (5)-(10).

11. Planar System Example

Figure 8 depicts a double pendulum mechanism. The joint reaction-force vectors of link 1 and 2 are $\boldsymbol{Q}_1=[R_{1x}\ R_{1y}\ R_{2x}\ R_{2y}]^T$ and $\boldsymbol{Q}_2=[-R_{2x}\ -R_{2y}]^T$ respectively. Note that we have ignored the z-component reaction force to ensure a planar motion. From Eq. (14.a), we obtain

$$\frac{\partial \Delta \boldsymbol{r}_1^1}{\partial \boldsymbol{Q}_1}=\frac{\partial \boldsymbol{W}_1^1}{\partial \boldsymbol{Q}_1}$$



	Link1	Link 2
Mass m (kg)	14.59	14.59
Inertia J' (kg-m²)	$[0\ \ 0\ \ 1.36]$	$[0\ \ 0\ \ 1.36]$
Position vector d_i^c (m)	$d_1^1=\begin{bmatrix}-0.61\\0\\0\end{bmatrix}$ $d_1^2=\begin{bmatrix}0.61\\0\\0\end{bmatrix}$	$d_2^2=\begin{bmatrix}-0.61\\0\\0\end{bmatrix}$
C. G. position	At the middle position of each link	
Δt=0.001 s		

Fig. 8 A planar double pendulum

$$=\frac{\Delta t^2}{2}\begin{bmatrix}1+4\sin^2\theta_1 & 2\sin 2\theta_1 & 1-4\sin^2\theta_1 & -2\sin 2\theta_1\\ 2\sin 2\theta_1 & 1+4\cos^2\theta_1 & -2\sin 2\theta_1 & 1-4\cos^2\theta_1\end{bmatrix} \tag{37}$$

where the subscript $\boldsymbol{W}_1^1$ denotes the link number and the superscript represents the joint number. Similarly, we obtain

$$\frac{\partial \boldsymbol{W}_1^2}{\partial \boldsymbol{Q}_1}=\frac{\Delta t^2}{2}\begin{bmatrix}1-4\sin^2\theta_1 & -2\sin 2\theta_1 & 1+4\sin^2\theta_1 & 2\sin 2\theta_1\\ -2\sin 2\theta_1 & 1-4\cos^2\theta_1 & 2\sin 2\theta_1 & 1+4\cos^2\theta_1\end{bmatrix} \tag{38.a}$$

$$\frac{\partial \boldsymbol{W}_2^2}{\partial \boldsymbol{Q}_2}=\frac{\Delta t^2}{2}\begin{bmatrix}1+4\sin^2\theta_2 & 2\sin 2\theta_2\\ 2\sin 2\theta_2 & 1+4\cos^2\theta_2\end{bmatrix} \tag{38.b}$$

In order to solve for all the joint reaction forces $\boldsymbol{Q}=[R_{1x}\ R_{1y}\ R_{2x}\ R_{2y}]^T$, we can obtain the following constraint equations according to Eq. (36):

$$\left[\frac{\partial \boldsymbol{f}(\boldsymbol{W})}{\partial \boldsymbol{Q}}\right]_{4\times 4}[\boldsymbol{Q}]_{4\times 1}=[\Psi(\boldsymbol{r}, \boldsymbol{U})]_{4\times 1} \tag{39}$$

where the partial derivative matrix can be assembled from Eqs. (37) to (38) that

$$\left[\frac{\partial \boldsymbol{f}(\boldsymbol{W})}{\partial \boldsymbol{Q}}\right]=\frac{\Delta t^2}{2}\begin{bmatrix}1+4\sin^2\theta_1 & 2\sin 2\theta_1 & 1-4\sin^2\theta_1 & -2\sin 2\theta_1\\ 2\sin 2\theta_1 & 1+4\cos^2\theta_1 & -2\sin 2\theta & 1-4\cos^2\theta_1\\ 1-4\sin^2\theta_1 & -2\sin 2\theta_1 & 4\sin^2\theta_1-4\sin^2\theta_2 & 2\sin 2\theta_1-\sin 2\theta_2\\ -2\sin 2\theta_1 & 1-4\cos^2\theta_1 & 2\sin 2\theta_1-2\sin 2\theta_2 & 4\cos^2\theta_1-4\cos^2\theta_2\end{bmatrix}$$

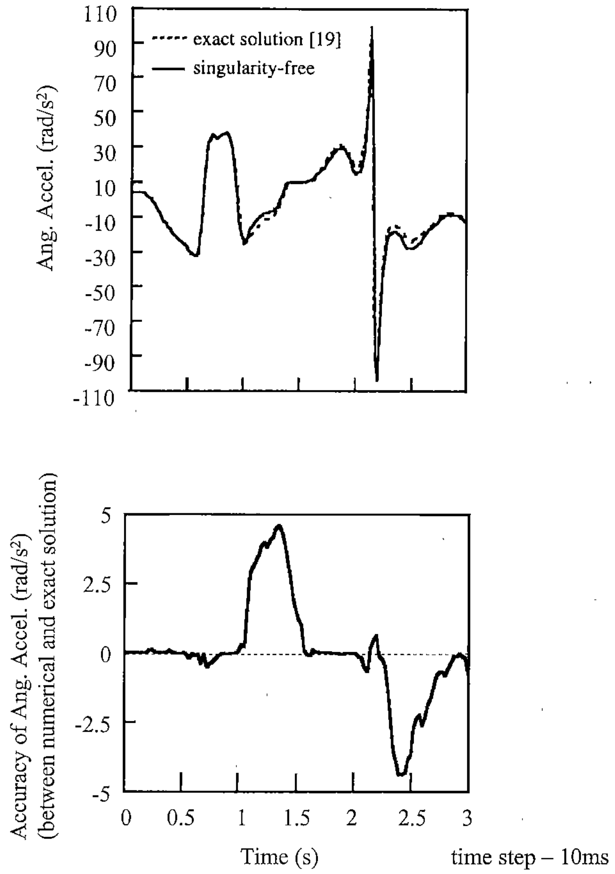


Fig. 9 Numerical analysis for link 2 of the planar double pendulum

At the known kinematic singularity $\theta_1=\theta_2$, it can be verified from the above equation that

$$\det\left(\frac{\partial f}{\partial Q}\right)=\frac{9}{2}\Delta t^2$$

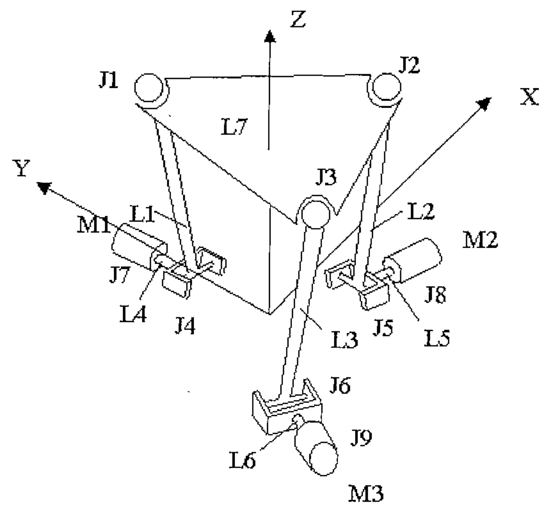
the matrix $\frac{\partial f}{\partial Q}$ is not singular for any non-zero integration step Δt. Thus, we ensure a singularity-free condition. The results listed in Fig. 9 are identical to those given in Ref.(19), which uses the SVD algorithm. However, we will only require the normal Gaussian-elimination method to solve the linear equations in Eq.(39). It presents a significant improvement in computational simplicity and efficiency.

12. Spatial System Example

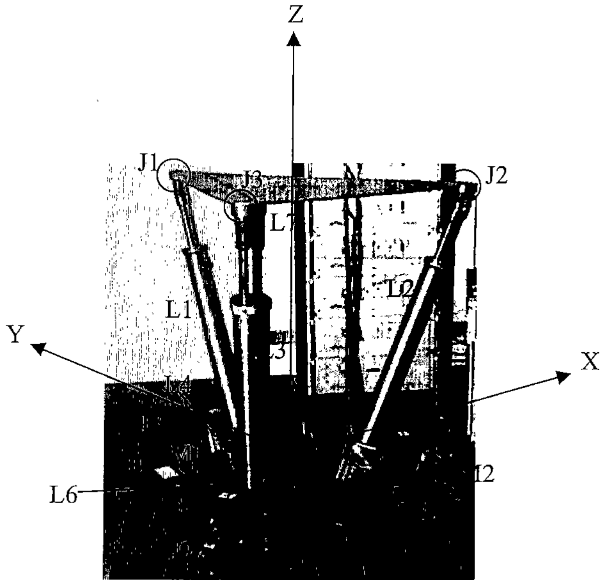
A 3 leg Stewart platform is shown in Fig. 10, where the joints J7, J8, and J9 are replaced by motors. The joint reaction force and moment vector $\boldsymbol{Q}_i$ of link i, consisting of the constraint reaction force $\boldsymbol{R}_i^k$ and moment $\boldsymbol{M}_i^k$ for all the joints k on link i, can be rewritten as follows:

$$\boldsymbol{Q}_i=[\boldsymbol{R}_{ix}^i, \boldsymbol{R}_{iy}^i, \boldsymbol{R}_{iz}^i, \boldsymbol{R}_{ix}^{i+3}, \boldsymbol{R}_{iy}^{i+3}, \boldsymbol{R}_{iz}^{i+3}, \boldsymbol{M}_{ix}^{i+3}, \boldsymbol{M}_{iy}^{i+3}, \boldsymbol{M}_{iz}^{i+3}]^T$$

$$\boldsymbol{Q}_{i+3}=[-\boldsymbol{R}_{1x}^{i+3}, -\boldsymbol{R}_{1y}^{i+3}, -\boldsymbol{R}_{iz}^{i+3}, -\boldsymbol{M}_{1x}^{i+3}, -\boldsymbol{M}_{1y}^{i+3},$$



(a) The 3 leg Stewart platform mechanism



(b) The Photograph of the 3 leg Stewart platform

Link		L1	L2	L3	L4	L5	L6	L7
Mass (kg)		5	5	5	1	1	1	10
Inertia (kg*m²)	I_x	0.07	0.07	0.07	0.01	0.01	0.01	2.5
	I_y	0.7	0 7	0.7	0 1	0.1	0.1	5
	I_z	0.7	0.7	0.7	0.1	0.1	0.1	2.5
Mass Center (m)	x	0	0.39	-0.39	0	0.292	-0.292	0
	y	0 450	-0 225	-0.225	0.337	-0.168	-0.168	0
	z	0.355	0 355	0.355	-0.013	-0.013	-0.013	0.71

(c) Link parameters

Joint		J1	J2	J3	J4	J5	J6	J7	J8	J9
Joint type		S	S	S	R	R	R	R	R	R
Position	x	0	0.53	-0.53	0	0.25	-0.25	0.33	0	-0.33
	y	0.61	-0.31	-0 31	0.29	-0.14	-0.14	-0.19	0.39	-0.19
	z	0.71	0.71	0.71	0	0	0	-0.03	-0.03	-0.03
Direction	θ_x				1	-0.5	-0.5	0	-0.84	0.84
	θ_y				0	-0 87	0.867	-0 97	0.48	0.48
	θ_z				0	0	0	0.26	0.26	0.26

(d) Joint parameters

Fig. 10 Link and joint parameters for the 3 leg Stewart platform

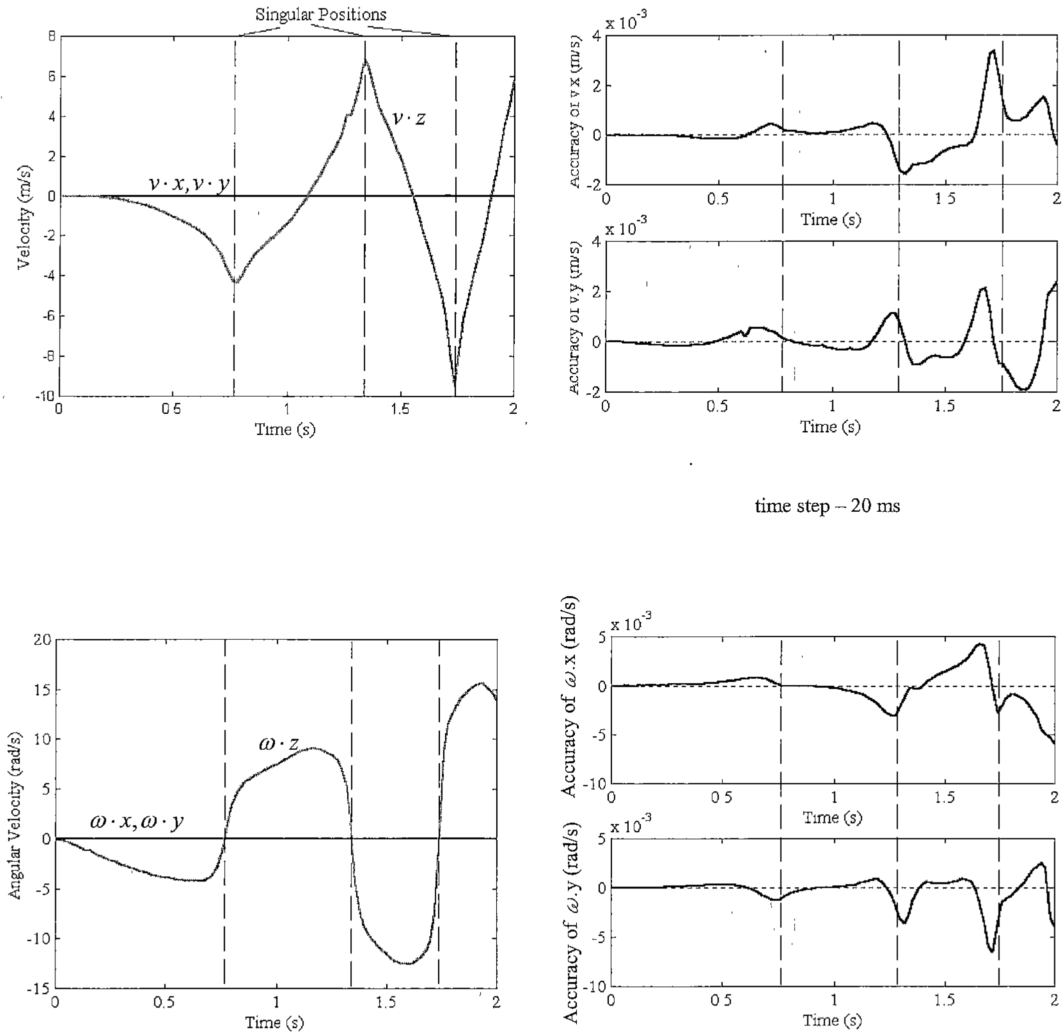


Fig. 11 Numerical analysis for link 7 of the 3 leg Stewart platform

$$-\boldsymbol{M}_{iz}^{i+3},\ \boldsymbol{R}_{(i+3)x}^{i+6},\ \boldsymbol{R}_{(i+3)y}^{i+6},\ \boldsymbol{R}_{(i+3)z}^{i+6},\ \boldsymbol{M}_{(i+3)x}^{i+6},\ \boldsymbol{M}_{(i+3)y}^{i+6},\ \boldsymbol{M}_{(i+3)z}^{i+6}]$$

$$\boldsymbol{Q}_7=[-\boldsymbol{R}_{1x}^1,\ -\boldsymbol{R}_{1y}^1,\ -\boldsymbol{R}_{1z}^1,\ -\boldsymbol{R}_{2x}^2,\ -\boldsymbol{R}_{2y}^2,\ -\boldsymbol{R}_{2z}^2,\ -\boldsymbol{R}_{3x}^3,\ -\boldsymbol{R}_{3y}^3,\ -\boldsymbol{R}_{3z}^3]^T$$

where $i=1\sim3$. Moreover, the partial derivative matrix can be assembled from Eq. (36) as follows:

$$\left[\frac{\partial f(\boldsymbol{W},\boldsymbol{H})}{\partial \boldsymbol{Q}}\right]_{45*45}[\boldsymbol{Q}]_{45*1}=[\Psi(\boldsymbol{r},\boldsymbol{U},\boldsymbol{q},\boldsymbol{P})]_{45*1}$$

Each motor generates a constant torque of 10 N-m counterclockwise. The rotor shafts are L4, L5, and L6, respectively. The analysis is performed using an integration step of 0.002 seconds. Following the analysis procedure shown in Fig. 7, the dynamic result for link 7 (the upper platform) is obtained, as shown in Fig. 11. In Fig. 11, the derivative of linear velocity v_z is discontinuous at times of 0.76, 1.32 and 1.72 seconds, where the kinematic singularity occurs. At these positions, our dynamic formulation has successfully surpassed the rank deficiency problem.

13. Conclusion

We have presented an improved dynamic formulation based on a second-order Taylor expansion with a given integration step, which may be tuned according to the required analysis precision. In our formulation, the constraint reaction force and moment must simultaneously account for the conservation of momentum of the rigid bodies. Therefore, the proposed formulation avoids the kinematic singularity found in previous dynamic analysis methods. In addition, all the kinematic constraint equations and constraint reaction force and moment balance equations are formulated in terms of linear algebraic equations, thus simplifying the computations compared with the SVD methods involved in the analysis. The number of unknowns in our formulation is equal to the number of total joint constraints in the system and the computational efficiency is also attained.

Acknowledgments

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Appendix

For a system with m kinematic constraints $\boldsymbol{\Phi}$, we may obtain m constraint equations as follows:

$$\boldsymbol{\Phi} \equiv \boldsymbol{\Phi}(\boldsymbol{q}) = 0 \qquad \text{(A1)}$$

The Taylor series expansion of Eq. (A1) about $\boldsymbol{q}$ is

$$\boldsymbol{\Phi}(\boldsymbol{q}+\Delta\boldsymbol{q}) = \boldsymbol{\Phi}(\boldsymbol{q}) + \boldsymbol{\Phi}_q\Delta\boldsymbol{q} + \frac{1}{2}\boldsymbol{\Phi}_{qq}(\Delta\boldsymbol{q})^2 + \textit{higher-order terms} \qquad \text{(A2)}$$

Using $\boldsymbol{\Phi}(\boldsymbol{q}+\Delta\boldsymbol{q})=0$ and eliminating the higher-order terms for small $\Delta\boldsymbol{q}$, we find that

$$\boldsymbol{\Phi}(\boldsymbol{q}) \approx -\boldsymbol{\Phi}_q\Delta\boldsymbol{q} - \frac{1}{2}\boldsymbol{\Phi}_{qq}(\Delta\boldsymbol{q})^2 \qquad \text{(A3)}$$

Similarly, taking the Taylor series expansion of $\boldsymbol{q}$ about t and neglecting the high-order terms for a time increment Δt, we may obtain

$$\boldsymbol{q}(t+\Delta t) \approx \boldsymbol{q}(t) + \dot{\boldsymbol{q}}\Delta t + \frac{1}{2}\ddot{\boldsymbol{q}}\Delta t^2 \qquad \text{(A4)}$$

We note that

$$\Delta\boldsymbol{q} \equiv \boldsymbol{q}(t+\Delta t) - \boldsymbol{q}(t) \qquad \text{(A5)}$$

Substituting Eqs. (A4) and (A5) into Eq. (A3), the constraint equation $\boldsymbol{\Phi}(\boldsymbol{q})$ yields

$$\boldsymbol{\Phi}(\boldsymbol{q}) \approx -\boldsymbol{\Phi}_q\left(\dot{\boldsymbol{q}}\Delta t + \frac{1}{2}\ddot{\boldsymbol{q}}\Delta t^2\right) - \frac{1}{2}\boldsymbol{\Phi}_{qq}\left(\dot{\boldsymbol{q}}\Delta t + \frac{1}{2}\ddot{\boldsymbol{q}}\Delta t^2\right)^2 \qquad \text{(A6)}$$

Thus, the constraint equation $\boldsymbol{\Phi}(\boldsymbol{q})$ may be evaluated by the acceleration $\ddot{\boldsymbol{q}}$, velocity $\dot{\boldsymbol{q}}$ and time increment Δt.

According to the Lagrange equation, the acceleration vector $\ddot{\boldsymbol{q}}$ may be expressed in terms of the generalized forces $\boldsymbol{g}$ and the unknown constraint reaction forces/moment vector $\boldsymbol{Q}$ as follows: [22]

$$\ddot{\boldsymbol{q}} = \boldsymbol{M}^{-1}(\boldsymbol{g} + \boldsymbol{\Phi}_q^T\boldsymbol{\lambda}) = \boldsymbol{M}^{-1}(\boldsymbol{g} + \boldsymbol{Z}^T\boldsymbol{Q}) \qquad \text{(A7)}$$

where $\boldsymbol{Z}$ is the coordinate transform matrix and $\boldsymbol{\Phi}_q^T\boldsymbol{\lambda} = \boldsymbol{Z}^T\boldsymbol{Q}$.

Substituting Eq. (A7) into Eq. (A6), the constraint equation $\boldsymbol{\Phi}(\boldsymbol{q})$ may be expressed by $\boldsymbol{Q}$ as follows.

$$\boldsymbol{\Phi}(\boldsymbol{q}) = -\boldsymbol{\Phi}_q\left[\dot{\boldsymbol{q}}\Delta t + \boldsymbol{M}^{-1}(\boldsymbol{g}+\boldsymbol{Z}^T\boldsymbol{Q})\frac{1}{2}\Delta t^2\right] - \frac{1}{2}\boldsymbol{\Phi}_{qq}\left[\dot{\boldsymbol{q}}\Delta t + \boldsymbol{M}^{-1}(\boldsymbol{g}+\boldsymbol{Z}^T\boldsymbol{Q})\frac{1}{2}\Delta t^2\right]^2 \qquad \text{(A8)}$$

Thus,

$$\begin{aligned}\boldsymbol{\Phi}_Q|_{Q=0} &= \frac{d\boldsymbol{\Phi}}{d\boldsymbol{Q}}\Big|_{Q=0} \\ &= -\boldsymbol{\Phi}_q(\boldsymbol{M}^{-1}\boldsymbol{Z}^T)\frac{1}{2}\Delta t^2 - \boldsymbol{\Phi}_{qq}\left[\dot{\boldsymbol{q}}\Delta t + \boldsymbol{M}^{-1}(\boldsymbol{g} + \boldsymbol{Z}^T\boldsymbol{Q})\frac{1}{2}\Delta t^2\right](\boldsymbol{M}^{-1}\boldsymbol{Z}^T)\frac{1}{2}\Delta t^2 \\ &= \frac{1}{2}\Delta t^2\boldsymbol{M}^{-1}\boldsymbol{Z}^T\left[-\boldsymbol{\Phi}_q - \boldsymbol{\Phi}_{qq}\left[\dot{\boldsymbol{q}}\Delta t + \boldsymbol{M}^{-1}\boldsymbol{g}\frac{1}{2}\Delta t^2\right]\right] \\ &= -\frac{1}{2}\Delta t^2\boldsymbol{M}^{-1}\boldsymbol{Z}^T[\boldsymbol{\Phi}_q + \boldsymbol{\Phi}_{qq}\boldsymbol{\varkappa}]\end{aligned} \qquad \text{(A9)}$$

where

$$\boldsymbol{\varkappa} = \left[\dot{\boldsymbol{q}}\Delta t + \boldsymbol{M}^{-1}(\boldsymbol{g}+\boldsymbol{Z}^T\boldsymbol{Q})\frac{1}{2}\Delta t^2\right]$$

Since Eq. (A8) is 2nd order function of $\boldsymbol{Q}$, and $\boldsymbol{Q}^2$ is associated with Δt^4 which is concerned small and can be ignored.

$$\begin{aligned}\Psi &= \boldsymbol{\Phi}(\boldsymbol{Q}, \dot{q}, q, \Delta t)|_{Q=0} \\ &= -\boldsymbol{\Phi}_q\left[\dot{\boldsymbol{q}}\Delta t + \boldsymbol{M}^{-1}\boldsymbol{g}\frac{1}{2}\Delta t^2\right] - \frac{1}{2}\boldsymbol{\Phi}_{qq}\left[\dot{\boldsymbol{q}}\Delta t + \boldsymbol{M}^{-1}\boldsymbol{g}\frac{1}{2}\Delta t^2\right]^2\end{aligned} \qquad \text{(A10)}$$

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