International Journal of Bifurcation and Chaos, Vol. 12, No. 3 (2002) 525–534

SPATIAL DISORDER OF CELLULAR NEURAL NETWORKS — WITH BIASED TERM

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Received March 28, 2001; Revised June 4, 2001

This study describes the spatial disorder of one-dimensional Cellular Neural Networks (CNN) with a biased term by applying the iteration map method. Under certain parameters, the map is one-dimensional and the spatial entropy of stable stationary solutions can be obtained explicitly as a staircase function.

Keywords: Spatial disorder; topological entropy; Bernoulli shift; transition matrix.

1. Introduction

Cellular neural networks (CNN), a large array of nonlinear circuits, consists of only locally connected cells. This work investigates the model of one-dimensional CNN proposed by Chua and Yang [1988a, 1988b]. The circuit equation of a cell is

$$\frac{dx_i}{dt} = -x_i + z + \alpha f(x_{i-1}) + af(x_i) + \beta f(x_{i+1}), \quad i \in \mathbf{Z}^1, \tag{1}$$

where $f(x)$ is a piecewise-linear output function defined by

$$f(x) = \begin{cases} rx + m - r & \text{if } x \geq 1, \\ mx & \text{if } |x| \leq 1, \\ \ell x + \ell - m & \text{if } x \leq -1. \end{cases} \tag{2}$$

Here r, m and ℓ are non-negative real constants and the quantity z is called threshold or biased term, and is related to independent voltage sources in electric circuits. The coefficients of output function α, a and β are real constants and called the space-invariant **A**-template denoted by

$$\mathbf{A} \equiv [\alpha,\, a,\, \beta]\,. \tag{3}$$

For simplicity, f will be denoted by f_r, with $\ell = r$ and $m = 1$, i.e.

$$f_r(x) = \begin{cases} rx + 1 - r & \text{if } x \geq 1, \\ x & \text{if } |x| \leq 1, \\ rx + r - 1 & \text{if } x \leq -1. \end{cases} \tag{4}$$

CNN is applied mainly in image processing and pattern recognition [Chua & Roska, 1993; Chua & Yang, 1988a] and [Thiran *et al.*, 1995]. A basic and important class of solutions of (1) are the stable stationary solutions of (1). In particular, the

*Work partially supported by the NSC under Grant No. 89-2115-M-009-023, the Lee and MTI Center for Networking Research and the National Center for Theoretical Sciences Mathematics Division, R.O.C.
†Work partially supported by the NSC under Grant No. 89-2115-M-008-029 and the National Center for Theoretical Sciences Mathematics Division, R.O.C.

complexity of stable stationary solutions of (1) must be investigated. When the output function is f_0, i.e. $r = 0$ in (4), it is observed that much work has subsequently been done in the electrical engineering community, see [Chua & Roska, 1993, 1988a] and references therein. In addition, [Juang & Lin, 2000; Hsu & Lin, 1999, 2000] and [Hsu *et al.*, 1999] recently considered mathematical results involving the complexity of stable stationary solutions and the multiplicity of traveling wave solutions. [Juang & Lin, 2000] partitioned the parameters space (a, z) into a finite number of regions in $\mathbf{R}^2$ such that in each region (1) with $f = f_0$ has the same spatial entropy.

However, for $z = 0$ and $r \in (0, \infty)$, [Hsu & Lin, 1999] proved that (1) and (4) can release infinite different spatial entropies and the entropy function is a devil-staircase like function in r. The method used in [Hsu & Lin, 1999] considers the stationary solutions of (1) as an iteration map. In fact, if output $v = f(x)$ is taken as the unknown variable, i.e. let

$$v_i = f(x_i) \quad \text{and} \quad u_{i+1} = v_i \,. \tag{5}$$

and if f is invertible with inverse function F, then the stationary solutions of (1) can be written as one- or two-dimensional iteration maps as follows,

$$T(v) = \frac{1}{\beta}\left(F(v) - z - av\right), \tag{6}$$

when $\alpha = 0$ and $\beta \neq 0$ and

$$T_2(u, v) = \left(v, \frac{1}{\beta}\left(F(v) - z - \alpha u - av\right)\right), \tag{7}$$

when $\alpha \neq 0$ and $\beta \neq 0$.

For these maps, each bounded trajectory corresponds to the outputs of bounded stationary solutions. In practice, if the maps are chaotic, then the stationary solutions of (1) are spatially chaos. However, only stable stationary solutions of (1) should be considered and the stability results can be found in [Hsu, 2000] or [Juang & Lin, 2000]. Therefore, the set of all stable bounded orbits of T must be considered, denoted by $\mathcal{S}$, and the entropy h of $T|_{\mathcal{S}}$ must be computed. If the entropy is positive, then the stable stationary solutions of (1) are spatial chaos. For convenience, $T|_{\mathcal{S}}$ is denoted herein as T.

[Hsu & Lin, 1999] considered (6) with $z = 0$, the odd symmetry of the map T makes it much easier to investigate the complexity of T than the case of $z \neq 0$. Therefore, this work focuses on the complexity of the one-dimensional map T with $z \in \mathbf{R}^1$ by some complicated computation. According to our results, the entropy function is a staircase function. As for the two-dimensional map T_2, when r is positive and sufficiently small, the Smale Horseshoe structures of stable stationary solutions of (1) and (4) are constructed, for details, see [Hsu, 2000].

Carefully examining the orbits of T reveals that the entropy function h is a staircase function of r for fixed a, z and β. The main results are

Main Theorem. *Assume $\beta = 1$, $0 < z < \Gamma(a)$ (see Lemma 3.1). Denote*

$$r_\infty(z) = \frac{a + z - 2}{a^2 - 2 + az} \tag{8}$$

and $h(r)$ is the entropy function of T in (6). Then there exists $p(z) \in \mathbf{Z}^+$ and a strictly decreasing sequence $\{r_{p,p-1}(z)\}$, $p = 3, 4, \ldots, p(z)$ with

$$r_\infty(z) < r_{p,p-1} \quad \textit{and} \quad r_p < r_{p,p-1} < r_{p-1}$$

such that

(i) *If $3 \le p \le p(z)$ and $r \in [r_{p,p-1}(z), r_{p-1,p-2}(z))$ then*
$$h(r; z) = \ln \lambda_{p-1,p-2}\,.$$
Where $\lambda_{p,p-1}$ is the largest root of $\lambda^2[\lambda^{2p-3} - \sum_{i=0}^{p-3} \lambda^i \sum_{j=0}^{p-2} \lambda^j] = 0$.

(ii) *If $r \in (r_\infty(z), r_{p(z),p(z)-1})$ then $h(r; z) = \ln \lambda_{p(z),p(z)-1}$.*

(iii) *If $r \in (\overline{r}_\infty(z), r_\infty(z)]$ then $h(r; z) = 0$.*

(iv) *If $r \in [0, \overline{r}_\infty(z)]$ then $h(r; z) = \ln 2$.*

Moreover, $p(z)$ is a decreasing function of z and $\lim_{z\to 0^+} p(z) = \infty$.

The above results or the proof of the main theorem in Sec. 3 indicate that the nonzero bias z causes a situation in which map T does not have enough periodic orbits when $r \in (\overline{r}_\infty(z), r_\infty(z)]$ and it makes the entropy equal to zero. Therefore, the entropy function of T has a staircase structure as shown in Fig. 1. This differs from those results of a devil-staircase like function in [Hsu & Lin, 1999] with $z = 0$ as shown in Fig. 2. Additionally, the results of [Hsu & Lin, 1999] recalled in the following corollary can be considered as the limiting case of the main theorem when z tends to 0.

Corollary. *Assume $\beta > 0$, $z = 0$ and $a > \beta + 1$.*

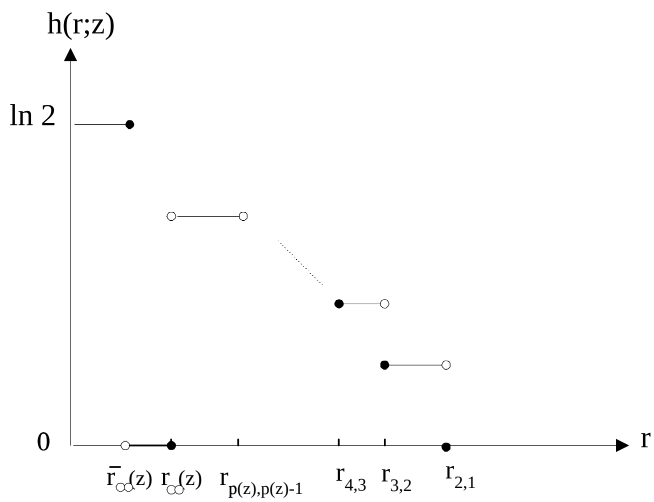

Fig. 1. Entropy of T with $z \neq 0$.

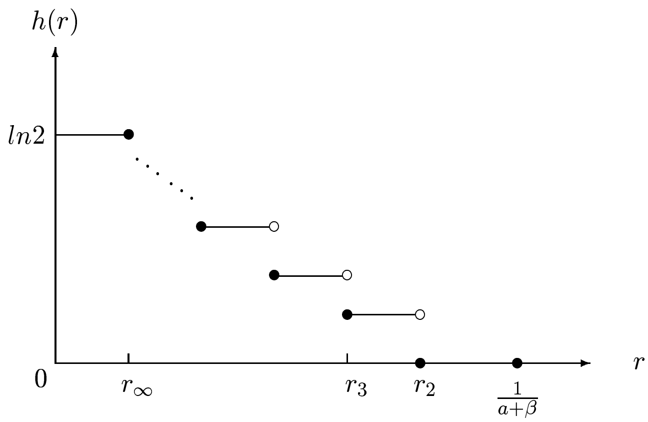

Fig. 2. Entropy of T with $z = 0$.

Denote

$$r_\infty = r_\infty(a,\, \beta) = \frac{a - \beta - 1}{a(a-1) + \beta(a-2)}\,,$$

$$r_2 = r_2(a,\, \beta) = \frac{a - \beta - 1}{a(a-1) + \beta(\beta-1)}\,,$$

and $h(r)$ *is the entropy function of* T *in (6) with* $F = F_r = f_r^{-1}$, $r > 0$. *Then there exists a strictly decreasing sequence* $\{r_p\}$, $p = 2, 3, \ldots$, *with*

$$\lim_{p\to\infty} r_p = r_\infty\,,$$

such that

(i) *If* $r_2 \le r < (1/a + \beta)$, *then* $h(r) = 0$.
(ii) *If* $r \in [r_p,\, r_{p-1})$, $p = 3, 4, \ldots$, *then* $h(r)$ *is* $\ln \lambda_p$ *where* λ_p *is the largest root of* $\lambda^{2p-2} - (\sum_{i=0}^{p-2} \lambda^i)^2 = 0$. *Moreover,* λ_p *is strictly increasing in* p *with*

$$\frac{1+\sqrt{5}}{2} = \lambda_3 < \lambda_p < 2\,, \quad \textit{for } p = 4,\, 5, \ldots$$

(iii) *If* $r \in [0,\, r_\infty]$, *then* $h(r) = \ln 2$.

The rest of this paper is organized as follows. Section 2 introduces the basic properties of the one-dimensional map T in some range of parameters. Section 3 proves the main theorem by symbolic dynamics, indicating that the entropy function $h(r)$ is a step function under certain parameters range.

2. Iteration Map

This section considers the one-dimensional map T in (6) with $z \neq 0$. If $a > 1$, $\beta > 0$, and $m = 1$, then the inverse function F of f_r is

$$F(v;\, r) = \begin{cases} \dfrac{1}{r}v - \dfrac{1}{r} + 1 & \text{if } v \ge 1, \\ v & \text{if } |v| \le 1, \\ \dfrac{1}{r}v - 1 + \dfrac{1}{r} & \text{if } v \le -1, \end{cases} \tag{9}$$

and the map T can be rewritten as

$$T(v;\, a,\, \beta,\, r) = \begin{cases} \dfrac{1}{\beta}\left(\dfrac{1}{r}v - \dfrac{1}{r} + 1 - av - z\right) & \text{if } v \ge 1, \\ \dfrac{1}{\beta}(v - av - z) & \text{if } |v| \le 1, \\ \dfrac{1}{\beta}\left(\dfrac{1}{r}v + \dfrac{1}{r} - 1 - av - z\right) & \text{if } v \le -1. \end{cases} \tag{10}$$

Instead of $F(v;\, r)$ and $T(v;\, a,\, \beta,\, r)$, $F(v)$ and $T(v)$ will be used if it does not cause any confusion. For simplicity, assume that $\beta = 1$ and $z \ge 0$ hereinafter. The graph of T can be found in the following figure.

An elementary computation produces that

$$A = (A_1,\, A_2) = \left(\frac{rz - r + 1}{1 - ra - r},\, \frac{rz - r + 1}{1 - ra - r}\right),$$

$$B = (B_1,\, B_2) = (1,\, 1 - a - z)\,,$$

$$C = (C_1,\, C_2) = (-1,\, a - 1 - z)\,,$$

$$D = (D_1,\, D_2) = \left(\frac{rz + r - 1}{1 - ra - r},\, \frac{rz + r - 1}{1 - ra - r}\right).$$

According to [Hsu, 2000] and [Juang & Lin, 2000], any orbit $\{T^k(v)\}$ of T with $|T^k(v)| \leq 1$ for some $k \geq 0$ is unstable. Hence, only trajectories of T lying outside the unit rectangle in (u, v) plane should be considered. Therefore, assume that $B_2 < -1$ and $C_2 > 1$ while these conditions are equivalent to $2-a < z < a-2$. For further computation, we give the following notations.

Definition 2.1. Assume $a > 2$.

(i) Define functions $r_\infty(z)$ and $\overline{r}_\infty(z)$ by

$$r_\infty(z)=\frac{a+z-2}{a^2-2+az} \quad \text{and} \quad \overline{r}_\infty(z)=\frac{a-z-2}{a^2-2-az}\,. \tag{11}$$

(ii) Let $m,\ n \in \mathbf{Z}^+$, if the slope of f, $r = r_{m,n}$ satisfies

$$T^{m-1}(B_2) = -1 \quad \text{and} \quad T^{n-1}(C_2) = 1\,, \tag{12}$$

then we call map T is of (m, n)-type and denote $r_{m,m}$, $k_{m,n}$ and $\xi_{m,n}$ by

$$r_{m,m} = r_m\,,$$

$$k_{m,n} = \frac{1}{r_{m,n}} - a \quad \text{and} \quad \xi_{m,n} = k_{m,n}^{-1}\,.$$

(iii) Define polynomials $E(x;\, m)$ and $U(x;\, m)$ by

$$E(x;\, m) = a\sum_{i=1}^{m} x^i - a + 2\,, \tag{13}$$

$$U(x;\, m,\, n) = (a+z)\sum_{i=n+1}^{m} x^i + 2a\sum_{i=1}^{n} x^i - 2a + 4\,. \tag{14}$$

From Fig. 3, the relative positions of A, B, C and D are easily obtained in the following.

Lemma 2.1. *Assume $a > 2$, then $r_\infty(z)$ and $\overline{r}_\infty(z)$ are increasing and decreasing functions of z, respectively. Moreover, we have*

(1) *If $r \in (r_\infty(z),\, \infty)$, then $A_2 > C_2$ and $B_2 > D_2$.*
(2) *If $r = r_\infty(z)$ then $A_2 > C_2$ and $B_2 = D_2$.*
(3) *If $r \in (\overline{r}_\infty(z),\, r_\infty(z))$, then $A_2 > C_2$ and $D_2 > B_2$.*
(4) *If $r = \overline{r}_\infty(z)$, then $A_2 = C_2$ and $D_2 > B_2$.*
(5) *If $r \in (0,\, \overline{r}_\infty(z))$, then $A_2 < C_2$ and $D_2 > B_2$.*

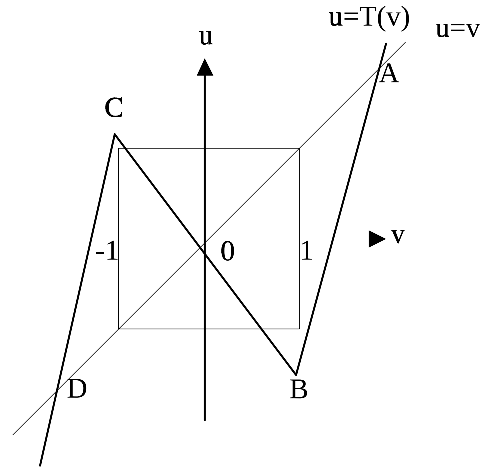

Fig. 3. Graph of T.

Proof. By elementary computation, we have

$$r'_\infty(z) = \frac{2a-2}{(a^2-2+az)^2}$$

and

$$\overline{r}'_\infty(z) = \frac{2-2a}{(a^2-2-az)^2}$$

and $r_\infty(z)$ and $\overline{r}_\infty(z)$ are increasing and decreasing functions of z respectively. The proofs from (1) to (5) are also simple and omitted. ■

The proof of the main theorem in Sec. 3 indicates that the case of (1) in Lemma 2.2 is more interesting and complicated.

3. Proof of Main Theorem

In this section, we prove the main theorem by introducing some lemmas. If $z > 0$, the following lemmas will show that unique $r_{m,m-1}$ lies between $r_{m,m}$ and $r_{m-1,m-1}$ such that (12) holds.

Lemma 3.1. *Assume $m \geq 3$ and define $\Gamma(a)$ by*

$$\Gamma(a) \equiv \min\left\{a-2,\ \frac{-a^3+6a^2-4a}{3a^2-6a+4}\right\}.$$

If $0 < z < \Gamma(a)$, $p > q$ and $r_{p,q}$ satisfies (12) with $r_{m,m} < r_{p,q} < r_{m-1,m-1}$, then $p = m$ and $q = m-1$.

Proof. First, we claim that $U(\xi_{p,q};\, p,\, q) = 0$ and $E(\xi_{m,m};\, m) = 0$. By simple computation, it is obvious that

$$T^{-1}(1)=\frac{rz+1}{1-ra} \quad \text{and} \quad T^{-1}(-1)=\frac{rz-1}{1-ra}\,. \tag{15}$$

Define R and L by

$$R = T^{-1}(1) - 1 \quad \text{and} \quad L = 1 - T^{-1}(-1)\,. \qquad (16)$$

If $p > q$ and $r = r_{p,q}$ satisfies (12), then it is not difficult to compute that $\xi_{p,q}$ satisfies

$$\frac{L(1-\xi_{p,q}^{q})}{1-\xi_{p,q}} + \frac{R(1-\xi_{p,q}^{p})}{1-\xi_{p,q}} = 2a - 4\,. \qquad (17)$$

By (15) and (16), we know that

$$R + L = \frac{2\xi_{p,q}}{r_{p,q}} - 2\,, \quad R = \frac{r_{p,q}(z+a)}{1 - r_{p,q}a}\,,$$

and (17) can be rewritten as

$$\left(\frac{2\xi_{p,q}}{r_{p,q}} - 2\right)\sum_{j=0}^{q-1}\xi_{p,q}^{j} + R\sum_{j=0}^{p-1}\xi_{p,q}^{j} = 2a - 4\,, \qquad (18)$$

$$\xi_{p,q}(a+z)\sum_{j=q}^{p-1}\xi_{p,q}^{j} + \left(\frac{2\xi_{p,q}}{r_{p,q}} - 2\right)\sum_{j=0}^{q-1}\xi_{p,q}^{j} = 2a - 4\,. \qquad (19)$$

According to the definition of $\xi_{p,q}$, we have $U(\xi_{p,q};\, p,\, q) = 0$. Similarly, we have $E(\xi_{m,m};\, m) = 0$. Next, we show that $r_{m,m-1}$ satisfies (12) and $r_{m,m} < r_{m,m-1} < r_{m-1,m-1}$. Since $z > 0$ and $\xi_{m,m-1} > 0$, by (13), (14) and (19), we have

$$a\sum_{i=1}^{m-1}\xi_{m,m-1}^{i} < a - 2\,, \quad a\sum_{i=1}^{m-1}\xi_{m-1,m-1}^{i} = a - 2\,, \qquad (20)$$

and

$$a\sum_{i=1}^{m}\xi_{m,m-1}^{i} > a - 2\,, \quad a\sum_{i=1}^{m}\xi_{m,m}^{i} = a - 2\,. \qquad (21)$$

From (20) and (21), $r_{m,m-1}$ satisfies (12) and $r_{m,m} < r_{m,m-1} < r_{m-1,m-1}$, for $m > 2$.

Now, we claim that no $r_{p,q}$ satisfies (12) and $r_{m,m} < r_{p,q} < r_{m-1,m-1}$ except for $p = m$ and $q = m-1$. For convenience, let $h = r_{m-1,m-1}$, $k = r_{m,m}$ and $\xi = r_{p,q}$, where $p = m + n$, $q = m - n - 1$ and $1 \le n < m - 2$. By (14) and elementary computation, we have

$U(h;\, p,\, q) < 0$ if and only if

$$2a - (a+z)h^{n} + (z-a)h^{-(n+1)} < 0 \qquad (22)$$

and

$U(k;\, p,\, q) < 0$ if and only if

$$2a - (a+z)k^{n+1} + (z-a)k^{-n} < 0\,. \qquad (23)$$

Obviously $U'(x;\, p,\, q) > 0$ and if $U(h;\, p,\, q)U(k;\, p,\, q) > 0$; by intermediate value theorem, no ξ lies between h and k and satisfies (12). Therefore, we claim that $U(h;\, p,\, q) < 0$ and $U(k;\, p,\, q) < 0$, if a, z satisfy $0 < z < \Gamma(a)$. Denote $P(x)$ and $Q(x)$ by

$$P(x) = 2a - (a+z)x^{n+1} + (z-a)x^{-n}$$

and

$$Q(x) = 2a - (a+z)x^{n} + (z-a)x^{-(n+1)}\,,$$

then $P(x)$ and $Q(x)$ are concave functions in $(0, 1]$ and $P(1) = Q(1) = 0$. By elementary computation or [Hsu & Lin, 1999], we know that $r_{2,2} = r_2 = (a-2)/(a^2 - a)$ and

$$0 < k < h < \frac{1}{\dfrac{1}{r_{2,2}} - a} = \frac{a-2}{a}\,. \qquad (24)$$

If a, z satisfy $0 < z < \Gamma(a)$, we have $P((a-2)/a) < 0$. Since $P(x)$ is concave, by (23) we obtain that $U(k;\, p,\, q) < 0$. Furthermore, the zero of $Q(x)$ is obviously larger than the zero of $P(x)$ in $(0, 1)$. By the concavity of $Q(x)$, we also obtain $P((a-2)/a) < 0$ and this implies $U(h;\, p,\, q) < 0$. Hence, the proof is complete. ■

Corollary 3.1. *Under the same assumptions of Lemma 3.1, we have* $r_{m+1,m} < r_{m,m-1}$ *for all integer* $m > 1$.

Now, if z is fixed, since $\lim_{p\to\infty} r_p = r_\infty$ and $r_\infty(z)$ is an increasing function of z, by Lemma 3.1, we obtain that there exists a maximal positive integer $p(z)$ such that (12) holds for sequence $\{r_{p,p-1}(z)\}$ with $p = 3, 4, \ldots, p(z)$ and no $r_{m,m-1}(z)$ satisfies (12) with $m > p(z)$. As demonstrated later this observation reveals the staircase structure of entropy function h of T. For completeness, this study recalls the definitions and some results of entropy for a dynamical system. Details can be found in [Bowen, 1973] or [Afraimovich & Hsu, 1998, Sec. 6].

Definition 3.1. Let $G : \mathbf{X} \to \mathbf{X}$ be a dynamical system on the complete metric space $\mathbf{X}$ and $S \subset \mathbf{X}$ be an invaraint set.

(i) The set $\Gamma_n(x) = \{G^k(x)\}_{k=0}^{n-1}$ is called an orbit segment of temporal length n. Two segments $\Gamma_n(x)$ and $\Gamma_n(y)$ are said to be (n, ε)-separated if there exists $k \in \mathbf{Z}^1$, $0 \le k \le n-1$, such that $\mathrm{dist}(G^k(x), G^k(y)) \ge \varepsilon$.

(ii) Let $S_{n,\varepsilon}$ be a set of segments of temporal length n such that

(a) if $\Gamma_n(x), \Gamma_n(y) \in S_{n,\varepsilon}$, then they are (n, ε)-separated;

(b) if $w \in S$ and $\Gamma_n(w) \notin S_{n,\varepsilon}$, then there is $x \in S$ such that $\Gamma_n(x) \in S_{n,\varepsilon}$ and $\mathrm{dist}(G^k x, G^k w) < \varepsilon$ for each $k = 0, 1, \ldots, n-1$.

Define $\tilde{C}_{n,\varepsilon} = \sharp S_{n,\varepsilon}$, the number of elements of the set $S_{n,\varepsilon}$ and $C_{n,\varepsilon} = \inf_{S_{n,\varepsilon}} \tilde{C}_{n,\varepsilon}$. Then, the entropy function of G, denoted by $h(G)$, is defined as follows:

$$h(G) = \lim_{\varepsilon\to 0} \overline{\lim_{n\to\infty}} \frac{\ln C_{n,\varepsilon}}{n}. \tag{25}$$

Proposition 3.1. ([Afraimovich & Hsu, 1998, Sec. 2.4; Robinson, 1995]). *Let $\sigma_M : \Sigma_M \to \Sigma_M$ be a subshift of finite type with the transition matrix M on N symbols. Denoted by K_n the number of admissible words of length $n+1$, the entropy of σ_M is equal to*

$$h(\sigma_M) = \lim_{n\to\infty} \frac{\ln K_n}{n} = \ln|\lambda_1|,$$

where λ_1 is the real eigenvalue of M such that $|\lambda_1| \ge |\lambda_j|$ for all other eigenvalues λ_j of M.

By Proposition 3.4, we must find a subshift of finite type such that T is topologically conjugate to the subshift. The subshift can be constructed by finding some subintervals of $I\backslash(-1, 1)$ with the covering relation as shown in the proof of the main theorem later.

Definition 3.2. An interval I_1 T-covers an interval I_2 provided $I_2 \subseteq T(I_1)$. This study writes $I_1 \to I_2$.

Proof of Main Theorem. First, we consider the case $r > r_\infty(z)$, i.e. $A_2 > C_2$ and $B_2 > D_2$. Let $R_1^+(r)$ and $R_1^-(r)$ be the first components of the intersection points of $\overline{AB}$ with $u = +1$ and $u = -1$, respectively. A simple computation produces

$$R_1^-(r) = \frac{1-2r+rz}{1-ra} \quad \text{and} \quad R_1^+(r) = \frac{1+rz}{1-ra}. \tag{26}$$

Then, the continuity of $T(v;\, r)$ with respect to r and Lemma 3.1 make it easy to prove that for any positive integer $2 < p \le p(z)$, there exists a unique $r_{p,p-1} > 0$ such that $\{T^i(C_2;\, r_{p,p-1})\}_{i=-\infty}^{i=\infty}$ is a $2p-1$-periodic orbit, i.e. of $(p,\, p-1)$ type, where $p(z)$ is the largest integer such that $r_{p(z)}$ less than $r_\infty(z)$. Restated, after $2p-1$ iteration, $(v,\, T(v;\, r_{p,p-1}))$ maps C to B and B to C, respectively.

Denote

$$R^+ = (R_1^+,\, R_2^+) = \overline{AB} \cap \{u = 1\},$$

$$R^- = (R_1^-,\, R_2^-) = \overline{AB} \cap \{u = -1\},$$

$$L^+ = (L_1^+,\, L_2^+) = \overline{CD} \cap \{u = 1\},$$

$$L^- = (L_1^-, L_2^-) = \overline{CD} \cap \{u = -1\},$$

$$\Omega_r = \left\{(v,\, u) \,|\, |v| \le \frac{ra-2r+1}{1-ra} \right.$$

$$\left. \text{and } |u| \le \frac{ra-2r+1}{1-ra} \right\},$$

here $\{u = D_2\} \cap \overline{CD} = (((2r-ra-1)/(1-ra)),\, 1-a-z)$ and $\Omega_r \subset \Omega$. Figures 4 and 5 give the five-periodic orbit and seven-periodic orbit of T at $r_{3,2}$ and $r_{4,3}$, respectively. Now, if $3 \le p \le p(z)$ and $r_{p,p-1} \le r < r_{p-1,p-2}$ or $r_\infty(z) < r < r_{p(z),p(z)-1}$, define the $2p-1$ *stable subintervals* by

$$I_{p+1} = (1,\, R_2^-),$$

$$I_{p+k} = (T^{-k+1}(R_2^+),\, T^{-k}(R_2^-)) \quad \text{for } k = 1 \text{ to } p-2.$$

and

$$I_p = (L_2^+,\, -1),$$

$$I_{p-k} = (T^{-k}(L_2^+),\, T^{-k+1}(L_2^-)) \quad \text{for } k = 1 \text{ to } p-1.$$

The $2p-1$ subintervals have the following covering relation:

$$I_i \to I_{i+1} \quad \text{for } i = 1 \text{ to } p-1,$$

$$I_p \to I_j \quad \text{for } j = p+1 \text{ to } 2p-2,$$

$$I_{p+1} \to I_k \quad \text{for } k = 2 \text{ to } p,$$

$$I_l \to I_{l-1} \quad \text{for } l = p+2 \text{ to } 2p-1.$$

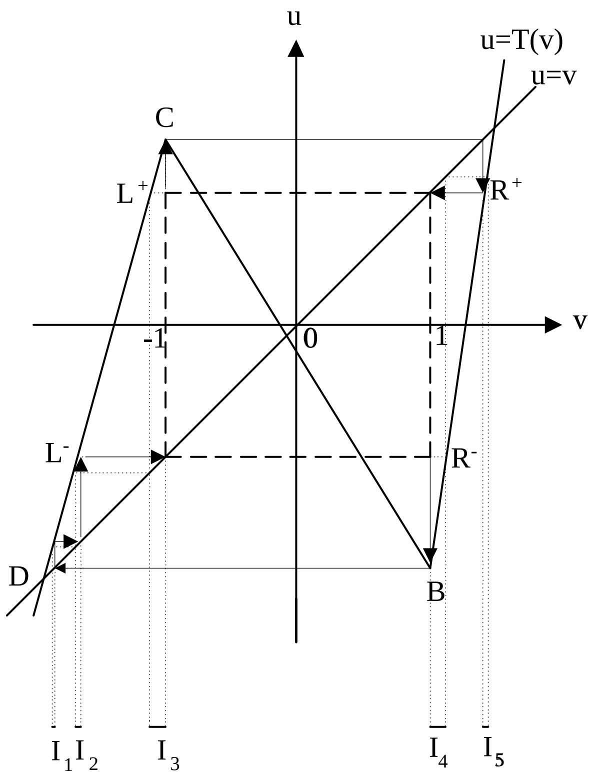

Fig. 4. Graph of T in $(3, 2)$ type and its *stable subintervals*.

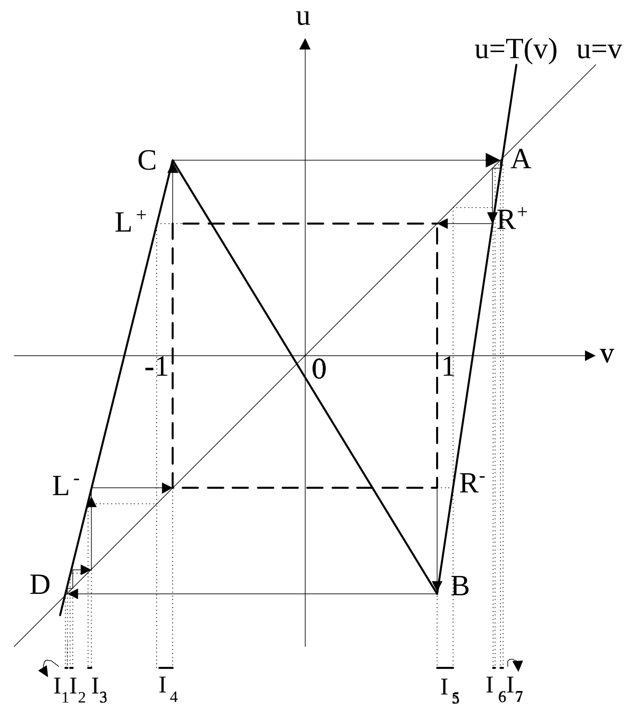

Fig. 5. Graph of T in $(4, 3)$ type and its *stable subintervals*.

Therefore, we obtain the following transition matrix $M \equiv M[p, p-1]$ of the $2p-1$ subshifts of finite type.

$$M = \begin{bmatrix}
0 & 1 & 0 & & \bullet & \bullet & & \bullet & & \bullet & & \bullet & & 0 \\
0 & 0 & 1 & 0 & \bullet & \bullet & & \bullet & & \bullet & & \bullet & & 0 \\
\bullet & & \bullet & & \bullet & \bullet & & \bullet & & \bullet & & \bullet & & \bullet \\
\bullet & & \bullet & & \bullet & \bullet & & \bullet & & \bullet & & \bullet & & \bullet \\
0 & & \bullet & & \bullet & 0 & 1 & 0 & & \bullet & & \bullet & & 0 \\
\bullet & & \bullet & & \bullet & \bullet & 0 & 1 & & \bullet & & \bullet & 1 & 0 \\
0 & 1 & \bullet & & \bullet & \bullet & 1 & 0 & & \bullet & & \bullet & & 0 \\
0 & & \bullet & & \bullet & \bullet & 0 & 1 & 0 & \bullet & & \bullet & & 0 \\
\bullet & & \bullet & & \bullet & \bullet & & \bullet & & \bullet & & \bullet & & \bullet \\
\bullet & & \bullet & & \bullet & \bullet & & \bullet & & \bullet & & \bullet & & \bullet \\
0 & & \bullet & & \bullet & \bullet & & \bullet & & \bullet & 0 & 1 & 0 & 0 \\
0 & & \bullet & & \bullet & \bullet & & \bullet & & \bullet & & 0 & 1 & 0
\end{bmatrix}$$

pth row with $p-2$ terms of 1

$(p+1)$th row with $p-1$ terms of 1

$(2p-1) \times (2p-1)$

This study defines spaces Σ_{2p-1} and Σ_M by

$$\Sigma_{2p-1} = \{1, 2, \ldots, 2p-1, 2p-1\}^N,$$

$$\Sigma_M = \{s \in \Sigma_{2p-1} : M_{s_k s_{k+1}} = 1 \quad \text{for } k = 0, 1, 2, \ldots\},$$

with a metric on Σ_M by

$$d(s, t) = \sum_{k=0}^{\infty} \frac{\delta(s_k, t_k)}{3^k},$$

for $s = (s_0, s_1, \ldots)$ and $t = (t_0, t_1, \ldots)$ in Σ_M, where

$$\delta(i, j) = \begin{cases} 0 & \text{if } i = j, \\ 1 & \text{if } i \neq j. \end{cases}$$

Let $\sigma_M : \Sigma_M \to \Sigma_M$ be the subshift of finite type for the matrix M, i.e. $\sigma(s) = t$ where $t_k = s_{k+1}$. Therefore, if $r_{p,p-1} \leq r < r_{p-1,p-2}$ then there exists an invariant subset Λ_p in Ω such that $T|_{\Lambda_p}$ is topological conjugate to the $2p-1$ subshift (Σ_M, σ_M) with entropy h equal to $\ln \lambda_{p,p-1}$, where $\lambda_{p,p-1}$ is the positive maximal root of characteristic polynomial of M. To derive $\lambda_{p,p-1}$, we need the following lemma.

Lemma 3.2. *Given $p \in \mathbf{Z}^1$ and $p > 1$, then the characteristic polynomial $g(x;\, p,\, p-1)$ of transition matrix $M[p,\, p-1]$ is*

$$g(x;\, p,\, p-1) = x^2\left(x^{2p-3} - \sum_{i=0}^{p-3} x^i \sum_{j=0}^{p-2} x^j\right).$$

Proof. By elementary matrix computation, see Appendix A, we obtain

$$g(x;\, p,\, p-1) = xg(x;\, p-1,\, p-1) - x^2 \sum_{i=0}^{p-3} x^i\,,$$

where, $g(x;\, p-1,\, p-1)$ is the characteristic polynomial of M with $z = 0$, for details see [Hsu & Lin, 1999]. In [Hsu & Lin, 1999], we also have $g(x;\, p-1,\, p-1) = x^2[x^{2p-4} - (\sum_{i=0}^{p-3} x^i)^2]$. Therefore, the result follows by simple computation. ■

By Lemmas 3.1 and 3.6, we prove results (i) and (ii) of the main theorem. As for the assumption (iii) of the main theorem, it is equivalent to the conditions of (2) and (3) in Lemma 2.2. By the same arguments, we obtain the entropy h of T is zero, see e.g. Fig. 6. In case (iv), which is equivalent to the conditions of (4) and (5) in Lemma 2.2, we know that $D_2 > B_2$ and $C_2 \geq A_2$ in Fig. 7 such that the behavior of the map T resembles that of the logistic map as discussed in [Robinson, 1995, Theorem 5.2]. Therefore, there exists an invariant Cantor set such that T is topologically conjugate to a one-sided Bernoulli shift of two symbols. Since the entropy of the one-sided Bernoulli shift of two symbols is ln 2, the result follows by Proposition 3.4.

Finally, since $\lim_{z\to 0} r_\infty(z) = r_\infty$, by Lemma 3.1 we obtain that $p(z)$ is a decreasing function of z with $\lim_{z\to 0} p(z) = 0$. The proof is complete. ■

Remark

(i) If we consider the output function is not symmetric, i.e. $r \neq \ell$ in (2), then Lemma 3.1 is no longer valid. In fact, there exists many different m, n such that $r = r_{m,n}$ lies between r_p and r_{p-1} for any $p \geq 3$ and T is of (m, n) type. Hence, by similar arguments in the proof of the main theorem, we also obtain transition matrix $M[m, n]$ such that the corresponding

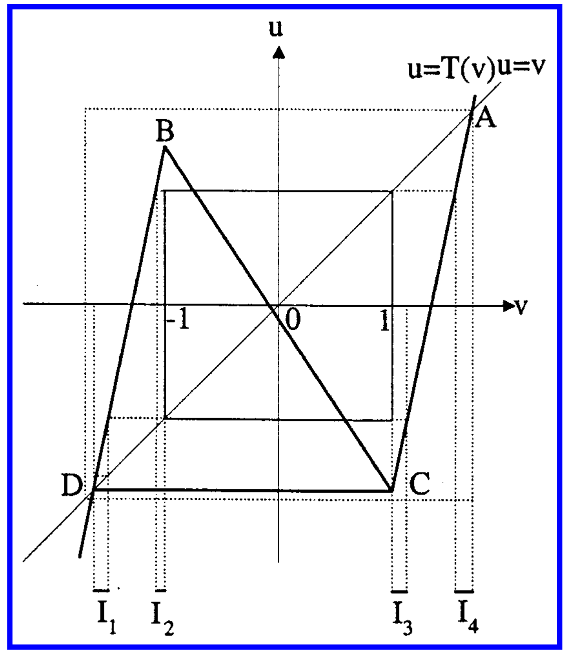

Fig. 6. Graph of T with $r = r_\infty(z)$ and its *stable subintervals.*

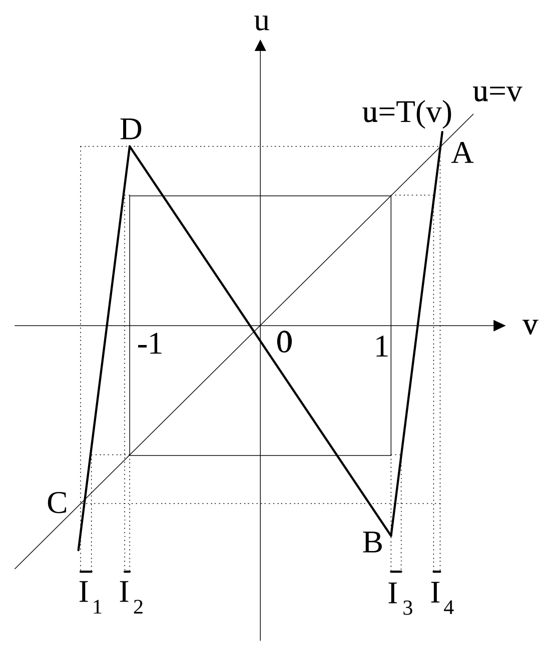

Fig. 7. Graph of T with $r = \bar{r}_\infty(z)$ and its *stable subintervals.*

characteristic ploynomial $g(x;\, m,\, n)$ is

$$g(x;\, m,\, n) = x^2\left(x^{m+n-2} - \sum_{i=0}^{m-2} x^i \sum_{j=0}^{n-2} x^j\right). \tag{27}$$

(ii) By some further computation, the ordering relation of the maximal root $\lambda_{m,n}$ of $g(x;\, m,\, n)$ can also be obtained as following lemma.

Lemma 3.3. *Given* $(m_1,\, n)$, $(m_2,\, n+1)$ *and* $m_1 > m_2$, *then* $g(\lambda_{m_1,n};\, m_2, n+1) < 0$. *Moreover, we have*

(1) *If* $n_1 > n_2$ *then* $\lambda_{m_1,n_1} > \lambda_{m_2,n_2}$.
(2) *If* $n_1 = n_2$ *and* $m_1 > m_2$ *then* $\lambda_{m_1,n_1} > \lambda_{m_2,n_2}$.

Proof. Since

$$\begin{aligned} & x^{m_1-m_2+n} g(\lambda_{m_1,n};\, m_2, n+1) \\ &\quad = \sum_{i=0}^{n-2} x^i \left[\sum_{i=n+1}^{m_1-m_2+n-1} x^i - \sum_{i=0}^{m_1-2} x^i\right] \\ &\qquad - \sum_{m_1-m_2+2n-1}^{m_1+n-3} x^i < 0\,, \end{aligned}$$

the results follows. ■

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Appendix

To compute the $g(\lambda;\, p,\, p-1)$ of M in the proof of the main theorem, this work only computes the special case when $m = 6$. For other m, $g(\lambda;\, p,\, p-1)$ can be obtained analogously.

If $m = 6$ then

$$\det[M(6,\,5)] = \det\begin{bmatrix} -\lambda & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\lambda & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\lambda & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\lambda & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\lambda & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\lambda & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & -\lambda & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -\lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -\lambda & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -\lambda & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -\lambda \end{bmatrix}$$

$$= \det\begin{bmatrix} -\lambda & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\lambda & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\lambda & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\lambda & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\lambda & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\lambda & 1 & 1 & 1 & 1 & 0 \\ \lambda & 0 & 1 & 1 & 1 & 1 & -\lambda & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -\lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -\lambda & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -\lambda & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -\lambda \end{bmatrix}$$

$$= -\lambda g(\lambda;\,5,\,5) + \lambda^2 \det\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -\lambda & 0 & 0 \\ 0 & 1 & -\lambda & 0 \\ 0 & 0 & 1 & -\lambda \end{bmatrix}$$

$$= -\lambda g(\lambda;\,5,\,5) + \lambda^2 \det\begin{bmatrix} -1-\lambda-\lambda^2 & -1 \\ 1 & -\lambda \end{bmatrix}$$

Hence $g(\lambda;\,6,\,5) = \det[M(6,\,5)] = -\lambda g(\lambda;\,5,\,5) + \lambda^2 \sum_{i=0}^{3} \lambda^i$. Induction produces

$$g(\lambda;\,p,\,p-1) = -\lambda g(\lambda;\,p-1,\,p-1) + \lambda^2 \det\begin{bmatrix} -\lambda^{p-4} - \lambda^{p-2} \cdots - 1 & -1 \\ 1 & -\lambda \end{bmatrix}$$

$$= -\lambda g(\lambda;\,p-1,\,p-1) + \lambda^2 \sum_{i=0}^{p-3} \lambda^i .$$

By [Hsu & Lin, 1999], we know that

$$g(\lambda;\,p-1,\,p-1) = \lambda^2 \left[\lambda^{2p-4} - \left(\sum_{i=0}^{p-3} \lambda^i\right)^2\right],$$

and the formula of Lemma 3.6 is obtained by simple computation.

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