Discrete-Time Optimal Fuzzy Controller Design: Global Concept Approach

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***Abstract*—In this paper, we propose a systematic and theoretically sound way to design a global optimal discrete-time fuzzy controller to control and stabilize a nonlinear discrete-time fuzzy system with finite or infinite horizon (time). A *linear-like* global system representation of discrete-time fuzzy system is first proposed by viewing a discrete-time fuzzy system in global concept and unifying the individual matrices into synthetical matrices. Then, based on this kind of system representation, the discrete-time optimal fuzzy control law which can achieve *global minimum* effect is developed theoretically. A nonlinear two-point-boundary-value-problem (TPBVP) is derived as the necessary and sufficient condition for the nonlinear quadratic optimal control problem. To simplify the computation, a multistage decomposition of optimization scheme is proposed and then a *segmental* recursive Riccati-like equation is derived. Moreover, in the case of time-invariant fuzzy systems, we show that the optimal controller can be obtained by just solving discrete-time algebraic Riccati-like equations. Grounding on this, several fascinating characteristics of the resultant closed-loop fuzzy system can be elicited easily. The stability of the closed-loop fuzzy system can be ensured by the designed optimal fuzzy controller. The optimal closed-loop fuzzy system can not only be guaranteed to be exponentially stable, but also be stabilized to any desired degree. Also, the total energy of system output is absolutely finite. Moreover, the resultant closed-loop fuzzy system possesses an infinite gain margin; that is, its stability is guaranteed no matter how large the feedback gain becomes. An example is given to illustrate the proposed optimal fuzzy controller design approach and to demonstrate the proved stability properties.**

***Index Terms*—Degree of stability, finite energy, gain margin, global minimum, Riccati-like equation, two-point-boundary-value-problem.**

I. INTRODUCTION

ALTHOUGH the researches in fuzzy modeling and fuzzy control have been quite matured [4], [5], [10], [14], [16], [18], it seems that the field of optimal fuzzy control is nearly open [20]. The goal of this work is to propose a systematic and theoretically sound scheme for designing a global optimal fuzzy controller to control and stabilize a discrete-time fuzzy system with finite or infinite horizon.

Stability and *optimality* are the most important requirements for any control system. Most of the existed works on the stability analysis of fuzzy control are based on Takagi–Sugeno (T–S) type fuzzy model combined with parallel distribution compensation (PDC) concept [16] and application of Lyapunov's method for stability analysis. Tanaka and coworkers reduced the stability analysis and control design problems to linear matrix inequality (LMI) problems [18], [17]. They also dealt with uncertainty issue [14]. This approach had been applied to several control problems such as control of chaos [17] and of articulated vehicle [15]. A frequency shaping method for systematic design of fuzzy controllers was also done by [13]. [10] developed a separation scheme to design fuzzy observer and fuzzy controller independently. Methods based on grid-point approach [9] and circle criteria [8], [12] were introduced to do stability analysis of fuzzy control, too. [19] adopted a supervisory controller and introduced stability and robustness measures. [5] proposed a decomposition principle to design a discrete-time fuzzy control system and an equivalent principle to do stability analysis. On the issue of optimal fuzzy control, [20] developed an *optimal* fuzzy controller to stabilize a *linear continuous* time-invariant system via Pontryagin minimum principle. Although fuzzy control of linear systems could be a good *starting point* for a better understanding of some issues in fuzzy control synthesis, it does not have much practical implications since using the fuzzy controller designed for a linear system directly as the controller may not be a good choice [20]. Moreover, the cited stability criteria may be simple, but rough to do systematic analysis and also may result in a controller with less flexibility.

Even with the aforementioned research results on the theoretic aspect of fuzzy control, the field of optimal fuzzy control for continuous system is still nearly open [20] and that for discrete-time system is fully open. Tanaka and others' works mentioned in the above always treat the stability of general linear feedback fuzzy controllers. The continuous optimal controller constructed by [20] is suitable only to be a rough or initial controller, since the system concerned is linear. All of them viewed the fuzzy system by individual rules, i.e., from the *local concept*. It is difficult for researchers to provide a *theoretical demonstration* on that the designed controller can reach *global minimum effect*, if the design scheme is based on local concept approach.

Technical contributions of this paper can be described as follows. The *entire* fuzzy system representation is proposed to maturate the *formulation* and simplification of the *quadratic optimal fuzzy control* problem. This original global-concept approach might initiate and activate the research in *global optimal* fuzzy controller design. Further, a tricky unifying of individual matrices into *synthetical* matrices generates a *linear-like* global system representation of a fuzzy system. This *linear-like* representation motivates us to develop the design

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scheme of global optimal fuzzy controller in the way of general linear quadratic (LQ) approach. Moreover, a multistage-decomposition approach is adopted to transform the optimal control problem into an on-going stage-by-stage dynamic issue. This decomposition operation can speed up numerical solution, and keep the global optimality at the same time. The design scheme meets the *necessary and sufficient* condition of global optimum. The derived discrete-time fuzzy control law is theoretically demonstrated to be the best for the entire system to reach the optimal performance index. Finally, the in-depth analysis (controllability, observability, stability, degree of stability and gain margin) in Section IV gives the complete perspective of all facets of the resultant closed-loop fuzzy system; we elicit that this kind of fuzzy controller can stabilize a discrete-time fuzzy system to any prescribed degree of stability; the corresponding closed-loop fuzzy system possesses an infinite gain margin; and the total energy of the system output of the closed-loop fuzzy system is absolutely finite. The design methodology is illustrated by one example.

II. Problem Formulation

In this section, we shall propose an *entire* system representation to maturate the *formulation* of the quadratic optimal fuzzy control problem, and a sound unification of the individual matrices into synthetical matrices to generate a *linear-like* global system representation of a fuzzy system, which helps the derivation of a theoretical design scheme of the quadratic optimal fuzzy controller. We consider a given nonlinear plant described by the so-called T–S type fuzzy model

$$\begin{aligned} &R^i\text{: If } x_n \text{ is } T_{1i}, \dots, x_n \text{ is } T_{ni}, \quad \text{then} \\ &\quad X(k+1) = A_i(k)X(k) + B_i(k)u(k) \\ &\quad Y(k) = C(k)X(k), \quad i = 1, \dots, r \end{aligned} \tag{1}$$

where

R^i	ith rule of the fuzzy model;
$x_1, \dots, x_n$	system states;
$T_{1i}, \dots, T_{ni}$	input fuzzy terms in the ith rule;
$X(k) = [x_1, \dots, x_n]^t \in \Re^n$	state vector;
$Y(k) \in \Re^{n'}$	system output vector;
$u(k) \in \Re^m$	system input (i.e., control output); and sequences $A_i(k), B_i(k)$ and $C(k)$ are, respectively, $n \times n, n \times m$ and $n' \times n$ matrices whose elements are real-valued functions defined on nonnegative real numbers, $\mathcal{N}$.

We, throughout this paper, assume $A_i(k)$ is nonsingular for all k to ensure no *deadbeat* response; in that case, $X(k+1)$ and $u(k)$ cannot define $X(k)$ uniquely, and the poles of the resultant closed-loop system are all located at zero points.

We then assume the desired controller is a rule-based nonlinear fuzzy controller in the form

$$R^i\text{: If } y_1 \text{ is } S_{1i}, \dots, y_{n'} \text{ is } S_{n'i}, \text{ then } u(k) = r_i(k), \quad i = 1, \dots, \delta \tag{2}$$

where

$y_1, \dots, y_{n'}$	elements of output vector $Y(k)$;
$S_{1i}, \dots, S_{n'i}$	input fuzzy terms in the ith control rule;
$u(k)$ or $r_i(k) \in \Re^m$	plant input (i.e., control output) vector.

Then, a *quadratic optimal fuzzy control problem* is described as follows.

Problem 1: Given the rule-based fuzzy system in (1) with $X(k_0) = X_0 \in \Re^n$ and a rule-based nonlinear fuzzy controller in (2), $k \in [k_0, k_1 - 1]$, find a controller, $u^*(\cdot)$, which can minimize the quadratic cost functional

$$J(u(\cdot)) = \sum_{k=k_0}^{k_1-1} [X^t(k)L(k)X(k) + u^t(k)u(k)] + X^t(k_1)QX(k_1) \tag{3}$$

over all possible inputs $u(\cdot)$ of class PC (piecewise continuous), where $L(k)$ and Q are belongs to symmetric positive semidefinite $n \times n$ matrices.

The grounding on distributed fuzzy subsystems and rule-based fuzzy controller brings the researchers in struggle to find out the controller $u^*(\cdot)$, which can achieve global minimum effect under quadratic performance consideration defined on the *entire* fuzzy system and fuzzy controller. In other words, it is a big troublesome challenge to achieve global optimal solution under local-model consideration, and thus so far, this issue has not been attacked directly even that the T–S type fuzzy model has been available for many years. [20] tried to open the deadlock by considering a linear system (instead of fuzzy system) combined with a fuzzy controller. However, the quadratic optimal fuzzy control issue, in fact, remains fully open.

Since each penalty term in the performance index is with regard to the entire fuzzy system and controller, it is realizable to *formulate* the distributed fuzzy subsystems and rule-based fuzzy controller into one equation from the global concept. Therefore, we "fuzzily blend" the well-known T–S type fuzzy model to obtain the following entire fuzzy system formulation

$$\begin{aligned} X(k+1) &= \sum_{i=1}^{r} h_i(X(k))A_i(k)X(k) \\ &\quad + \sum_{i=1}^{r}\sum_{j=1}^{\delta} h_i(X(k))w_j(Y(k))B_i(k)r_j(k) \\ Y(k) &= C(k)X(k) \end{aligned} \tag{4}$$

and the entire controller is

$$u(k) = \sum_{j=1}^{\delta} w_j(Y(k))r_j(k) \tag{5}$$

with $\sum_{i=1}^{r} h_i(X(k)) = 1$ and $\sum_{j=1}^{\delta} w_j(Y(k)) = 1$, where $h_i(X(k))$ and $w_i(Y(k))$ denote, respectively, the normalized firing-strength of the ith rule of the discrete-time fuzzy model and that of the ith fuzzy control rule, i.e., $h_i(X(k)) = \alpha_i(X(k))/\sum_{i=1}^{r}\alpha_i(X(k))$ and $w_i(Y(k)) = \beta_i(Y(k))/\sum_{i=1}^{\delta}\beta_i(Y(k))$ with $\alpha_i(X(k)) =$

$\Pi_{j=1}^{n}\mu_{T_{ji}}(x_j(k))$ and $\beta_i(Y(k)) = \Pi_{j=1}^{n'}\mu_{S_{ji}}(y_j(k))$, where $\mu_{T_{ji}}(x_j(k))$ and $\mu_{S_{ji}}(y_j(k))$ are the membership functions of fuzzy terms T_{ji} and S_{ji}, respectively. Thus, we obtain the formulation of the quadratic optimal fuzzy control problem in Problem 1 as follows.

Problem 2: Given the entire fuzzy system in (4) with the fuzzy controller $u(k)$ in (5) and $X(k_0) = X_0 \in \Re^n, k \in [k_0, k_1 - 1]$, find the optimal control law, $r_i^*(\cdot), i = 1, \ldots, \delta$, to minimize the quadratic cost functional

$$J(r_i(\cdot)) = \sum_{k=k_0}^{k_1-1}\left[X^t(k)L(k)X(k) + \sum_{i=1}^{\delta}\sum_{j=1}^{\delta} w_i(Y(k))w_j(Y(k))r_i^t(k)r_j(k)\right] + X^t(k_1)QX(k_1). \tag{6}$$

This kind of quadratic optimal control problems is, obviously, still too tough for us to engage in. Introducing the following *synthetical* matrices, $H(X(k)), W(Y(k)), A(k), B(k)$ and $R(k)$, can overcome the predicament, where

$$H(X(k)) = [h_1(X(k))I_n \ldots h_r(X(k))I_n]$$
$$W(Y(k)) = [w_1(Y(k))I_m \ldots w_\delta(Y(k))I_m],$$
$$A(k) = \begin{bmatrix} A_1(k) \\ \vdots \\ A_r(k) \end{bmatrix} \quad B(k) = \begin{bmatrix} B_1(k) \\ \vdots \\ B_r(k) \end{bmatrix}$$
$$R(k) = \begin{bmatrix} r_1(k) \\ \vdots \\ r_\delta(k) \end{bmatrix}$$

with I_n and I_m denoting the identity matrices of dimension n and m, respectively. In other words, based on these synthetical notations, Problem 2 can be rewritten as the following final formulation.

Problem 3: Given a nonlinear but *linear-like* fuzzy system

$$\begin{aligned} X(k+1) &= H(X(k))A(k)X(k) \\ &\quad + H(X(k))B(k)W(Y(k))R(k), \\ Y(k) &= C(k)X(k) \end{aligned} \tag{7}$$

with $X(k_0) = X_0 \in \Re^n$, find the optimal synthetical control law, $R^*(\cdot)$, to minimize the quadratic cost functional

$$\begin{aligned} J(R(\cdot)) = \sum_{k=k_0}^{k_1-1} &[X^t(k)L(k)X(k) \\ &+ R^t(k)W^t(Y(k))W(Y(k))R(k)] \\ &+ X^t(k_1)QX(k_1). \end{aligned} \tag{8}$$

This *linear-like* synthetical matrix representation for the entire T–S type fuzzy system materializes the design of the global optimal fuzzy controller in the way of general linear quadratic (LQ) approach, i.e., calculus-of-variation method.

It is important for us to mention here that the process of integrating all distributed fuzzy subsystems into one equation to describe the entire fuzzy system is necessary in order to find out the *global optimal* solution. The proposed *fuzzily blended* entire fuzzy system in (4) provides a practical way to work out the global optimal solution. However, even each fuzzy subsystem in T–S model is linear, the *fuzzily blended* entire fuzzy system in (4) is complicated and highly nonlinear. The further proposed *synthetical matrix* representation of the entire fuzzy system in (7) shall, in the sense of *global optimality*, lower down the order and difficulty of the problem. This kind of global system representation will be the foundation and kernel of the following fuzzy controller design scheme.

III. Discrete-Time Optimal Fuzzy Controller Design

We are going to design the optimal fuzzy controllers for discrete-time fuzzy system with finite-horizon in Section III-A and for that with infinite-horizon in Section III-B. For brevity, we shall not state *discrete-time* obviously in the following work.

A. Finite-Horizon Problem

By describing the fuzzy system from the global concept in Section II, our quadratic optimal fuzzy control problem for the T–S type fuzzy system can be formulated and simplified into Problem 3 in Section II. We shall use the *calculus of variations method* combined with *Lagrange multiplier method* to obtain the necessary and sufficient condition for global optimum. Since the membership functions in the fuzzy controller and fuzzy system are piecewise continuous, it is reasonable to make the following assumption.

Assumption 1: The membership functions of *perturbed extremes* are equivalent to those of *extremes*, i.e., $\mu_{T_{ji}}(x_j^*(k)) = \mu_{T_{ji}}(x_j^*(k) + \epsilon z_j(k)), j = 1, \ldots, n, i = 1, \ldots, r$, and $\mu_{S_{ji}}(y_j^*(k)) = \mu_{T_{ji}}(y_j^*(k) + \epsilon v_j(k)), j = 1, \ldots, n', i = 1, \ldots, \delta$, where ϵ is a very small positive value.

For frequently used membership functions such as bell-shaped, triangular and trapezoid membership functions, this assumption soundly holds. We denote them as nonsharp-profile membership functions. With this assumption, the following theorem gives the necessary and sufficient condition for global optimum.

Theorem 1 (Necessary and Sufficient Condition for Global Optimum): For the fuzzy system in (1) and fuzzy controller in (2) with nonsharp-profile membership functions, the optimal control law is

$$\begin{aligned} R^*(k) = -W^t(Y^*(k))[W(Y^*(k))W^t(Y^*(k))]^{-1} \\ \times B^t(k)H^t(X^*(k))P(k+1) \end{aligned} \tag{9}$$

and the corresponding global minimizer is

$$u^*(k) = -B^t(k)H^t(X^*(k))P(k+1) \tag{10}$$

where $P(k+1)$ satisfies the following nonlinear two-point-boundary-value problem (TPBVP):

$$\begin{bmatrix} X^*(k+1) \\ P(k+1) \end{bmatrix} = \Upsilon(X^*(k), k)\begin{bmatrix} X^*(k) \\ P(k) \end{bmatrix} \tag{11}$$

with $P(k_1) = QX^*(k_1)$ and, by expressing explicit time-dependence with lower index, as shown in the first equation at the bottom of the page, where $X^*(\cdot)$ is the corresponding optimal state trajectory with $X(k_0) = X_0$, and the minimum performance index is $\min_{R_{[k_0,k_1-1]}} J(R(\cdot)) = P^t(k_0)X(k_0)$.

Proof: See the Appendix. □

Solving the nonlinear TPBVP in (11) directly is achievable in conceptual aspect, but is at length in computational aspect. Therefore, searching another circumvent approach to overcoming this difficulty is pressing. A multistage decomposition of optimization scheme, from the essence of the dynamic programming formalism, is used for this purpose [21]. □

Lemma 1 (Multistage Decomposition): A foregoing optimization scheme is a dynamic allocation process or a **successive** multistage decision process. In other words, if we let $k_0 = k_0^1, k_1 = k_1^N, k_0^i = k_1^{i-1}, i = 2,\ldots,N; \triangle k^i = k_1^i - k_0^i, i = 1,\ldots,N$, and define the two equations shown at the bottom of the page, with regard to the state resulting from the previous decision, i.e., $X(k_0^1) = X_0; X(k_0^i) = X^*(k_1^{i-1}), i = 2,\ldots,N$, then

$$\Phi(X(\cdot),u(\cdot)) = \Phi^1(X(\cdot),u(\cdot)) + \cdots + \Phi^N(X(\cdot),u(\cdot)). \quad (12)$$

Hence, we can, by Lemma 1, transform our optimization problem into an on-going stage-by-stage dynamic issue, and thereupon, successively focus on only one stage at a time. The global optimal solution corresponding to each decomposed quadratic optimal fuzzy control problem is as follows.

Corollary 1: The optimal control problem for $k \in [k_0^i, k_1^i - 1]$, i.e., for the ith stage, is to find a controller $R^*(\cdot)$ to minimize

$$\begin{aligned} J^i(R(\cdot)) = \sum_{k=k_0^i}^{k_1^i-1} &(X^t(k)L(k)X(k) \\ &+ R^t(k)W^t(Y(k))W(Y(k))R(k)) \\ &+ X^t\left(k_1^i\right)Q^iX\left(k_1^i\right) \end{aligned} \quad (13)$$

where $i = 1,\ldots,N; Q^i$ equals to Q at the Nth stage and is a zero matrix, otherwise; and the related fuzzy system is described by (7) with the initial condition $X(k_0^1) = X_0$ and $X(k_0^i) = X^*(k_1^{i-1})$ for $i = 2,\ldots,N$. Then, the optimal fuzzy control law for the ith stage is $R^*(k)$ in (9), where $P(k)$ satisfies the nonlinear TPBVP in (11) with $P(k_1^i) = Q^iX(k_1^i)$.

We should emphasize that the multistage-decomposition approach in Lemma 1 can transform the optimal control problem into an on-going stage-by-stage dynamic issue. Therefore, the nonlinear TPBVP in Theorem 1 is decomposed into N *segmental* nonlinear TPBVP in Corollary 1, which can be solved by the *collocation method* [7]. This decomposition operation can speed up numerical solution, and keep the global optimality at the same time. Moreover, though the membership functions are dependent on the system state, the state-penalty term $X^t(k)L(k)X(k)$ in the cost functional in (3) can encourage a **smooth** *optimal trajectory* [1]. For a *chosen nonsharp* membership function profile, it is, in concept, reasonable and workable to increase the sampling frequency such that the membership function of the optimal state $X^*(k)$ remains almost invariant during each stage. In other words, we can further adjust the division, i.e., enlarge N, to the extent that $H(X(k))$ and $W(Y(k))$ are almost invariant during the *whole single stage*, and use H_i and W_i to denote those at the ith stage. Then, the optimal control law becomes

$$R^*(k) = -W_i^t\left[W_iW_i^t\right]^{-1}B^t(k)H_i^tP(k+1), \quad k \in [k_0^i, k_1^i - 1] \quad (14)$$

where $P(k+1), k \in [k_0^i, k_1^i - 1]$ satisfies the following *linear* TPBVP, shown in (15), at the bottom of the page, with $H_1 = H(X_0), W_1 = W(Y(k_0)), X(k_0^1) = X_0; H_i = H(X^*(k_1^{i-1})), W_i = W(Y^*(k_1^{i-1})), X(k_0^i) = X^*(k_1^{i-1})$, $\forall i = 2,\ldots,N; P(k_1^i) = Q^iX^*(k_1^i)$, $\forall i = 1,\ldots,N$. The following lemma indicates an efficient way to solve (15).

Lemma 2: Let $P(k) = K(k)X^*(k)$. The TPBVP in (15) is equivalent to one of the following *segmental* recursive Riccati-like equation:

$$\begin{aligned} K(k) = L(k) &+ A^t(k)H_i^tK(k+1)\left[I_n\right. \\ &\left.+ H_iB(k)B^t(k)H_i^tK(k+1)\right]^{-1}H_iA(k) \end{aligned} \quad (16)$$

$$\begin{aligned} K(k) = L(k) &+ A^t(k)H_i^tK(k+1)H_iA(k) \\ &- A^t(k)H_i^tK(k+1)H_iB(k)\left[I_n\right. \\ &\left.+ B^t(k)H_i^tK(k+1)H_iB(k)\right]^{-1} \\ &\times B^t(k)H_i^tK(k+1)H_iA(k) \end{aligned} \quad (17)$$

$$\Upsilon(X_k^*, k) = \begin{bmatrix} H(X_k^*)A_k - H(X_k^*)B_kB_k^tH^t(X_k^*)\left[A_k^tH^t(X_k^*)\right]^{-1}L_k & -H(X_k^*)B_kB_k^tH^t(X_k^*)\left[A_k^tH^t(X_k^*)\right]^{-1} \\ -\left[A_k^tH^t(X_k^*)\right]^{-1}L_k & \left[A_k^tH^t(X_k^*)\right]^{-1} \end{bmatrix}$$

$$\begin{aligned} \Phi(X(\cdot),u(\cdot)) = \min_{u_{[k_0,k_1-1]}} \sum_{k=k_0}^{k_1-1} &[X^t(k)L(k)X(k) \\ &+ u^t(k)u(k)] + X^t(k_1)QX(k_1) \end{aligned}$$

and

$$\Phi^i(X(\cdot),u(\cdot)) = \begin{cases} \min_{u_{[k_0^i,k_1^i-1]}} \sum_{k=k_0^i}^{k_1^i-1}[X^t(k)L(k)X(k) + u^t(k)u(k)], & i = 1,\ldots,N-1, \\ \min_{u_{[k_0^N,k_1-1]}} \sum_{k=k_0^N}^{k_1-1}[X^t(k)L(k)X(k) + u^t(k)u(k)] + X^t(k_1)QX(k_1), & i = N \end{cases}$$

where $k \in [k_0^i, k_1^i - 1], i = 1, \ldots, N,\ k_0^1 = k_0, k_1^N = k_1, K(k_1^i) = Q^i$. The optimal trajectory in (15) becomes

$$X^*(k+1) = \left[I_n + H_iB(k)B^t(k)H_i^tK(k+1)\right]^{-1} \times H_iA(k)X^*(k) \tag{18}$$

the optimal control law turns into

$$R^*(k) = -W_i^t\left[W_iW_i^t\right]^{-1}B^t(k)H_i^tK(k+1) \times \left[I_n + H_iB(k)B^t(k)H_i^tK(k+1)\right]^{-1} \times H_iA(k)X^*(k) \tag{19}$$

and the corresponding global minimizer is

$$u^*(k) = -B^t(k)H_i^tK(k+1) \times \left[I_n + H_iB(k)B^t(k)H_i^tK(k+1)\right]^{-1} \times H_iA(k)X^*(k). \tag{20}$$

Proof: See the Appendix. □

We further define $\bar{N}$ to be the number of stages at which membership functions can be assumed to be invariant during the *whole single stage*. Then, the *backward recursive* Riccati-like equation in (16) or (17) becomes available due to the existence of $\bar{N}$. This avoids the high computational complexity of the collocation method at the expense of *approximate optimality* due to the time-invariant assumption. We can ensure this assumption by checking the following condition at the starting time-step of the i-th stage, saying time-step k_0^i (i.e., time-instant $t_{k_0^i}$)

$$\left\|\frac{dH(X^*(\tau))}{d\tau}\right\|_{\tau=t_{k_0^i}} \le \kappa_{H_1} \tag{21}$$

and then keeping checking the following condition to find the proper length (time steps) of this stage

$$\left\|H(X^*(k)) - H\left(X^*\left(k_0^i\right)\right)\right\| \le \kappa_{H_2}, \quad \forall k \in [k_0^i, k_1^i - 1] \tag{22}$$

where κ_{H_1} and κ_{H_2} are the given tolerance to ensure the almost-invariant criteria. The first inequality in (21) ensures that the membership degrees corresponding to the optimal trajectory $X^*(\cdot)$ at time-step k_0^i does not change in abrupt shape, and also gives a hint that an almost-invariant-membership-function stage from time-step k_0^i is achievable. The second inequality in (22) is to check the almost-invariant criteria for the entire i-th stage, to find out the length (time steps) of the stage, and then, can also provide the information about the value of $\bar{N}$. These two inequalities are used to check the time-invariant criteria in the *dynamic decomposition algorithm* (DDA) in Section V. Now, we summarize the previous derivation in the following assertion. □

Theorem 2 (Multistage Optimization): For the fuzzy system and fuzzy controller represented, respectively, by (1) and (2), let $(X^*(k), R^*(k)), k \in [k_0, k_1 - 1]$, be the optimal solution with respect to $J(R(\cdot))$ in (8), and $(X^{i^*}(k), R^{i^*}(k)), k \in [k_0^i, k_1^i - 1]$, be the ith-stage optimal solution with respect to $J^i(R(\cdot))$ in (13). If $N > \bar{N}$ then

1) $(X^*(k), R^*(k)) = (X^{i^*}(k), R^{i^*}(k))$, for all $k \in [k_0^i, k_1^i - 1], i = 1, \ldots, N$; and $k_0^1 = k_0, k_1^N = k_1, k_0^i = k_1^{i-1}, i = 2, \ldots, N$;
2) for the ith stage, $k \in [k_0^i, k_1^i - 1]$, the optimal control law is

$$R^{i^*}(k) = -W_i^t\left[W_iW_i^t\right]^{-1}B^t(k)H_i^t\pi^i\left(k+1, k_1^i\right)\left[I_n + H_iB(k)B^t(k)H_i^t\pi^i\left(k+1, k_1^i\right)\right]^{-1} \times H_iA(k)X^*(k) \tag{23}$$

and the corresponding global minimizer is

$$u^{i^*}(k) = -B^t(k)H_i^t\pi^i\left(k+1, k_1^i\right)\left[I_n + H_iB(k)B^t(k)H_i^t\pi^i\left(k+1, k_1^i\right)\right]^{-1} \times H_iA(k)X^*(k) \tag{24}$$

where $\pi^i(k+1, k_1^i)$ is the symmetric positive semidefinite solution of the *segmental* recursive Riccati-like equation in (16) or (17); the ith-stage optimal trajectory is

$$X^{i^*}(k+1) = \left[I_n + H_iB(k)B^t(k)H_i^t\pi^i\left(k+1, k_1^i\right)\right]^{-1} \times H_iA(k)X^*(k); \tag{25}$$

3) the minimum performance index is equal to $\sum_{i=1}^N X^{i^{*t}}(k_0^i)\pi^i(k_0^i, k_1^i)X^{i^*}(k_0^i)$.

Proof: This theorem follows the above inference. □

So far, we have solved the optimal fuzzy control problem by finding the optimal solution to the general time-varying case. In the classical linear quadratic optimal control problem, a time-invariant system will give rise to time-invariant linear optimal control law. We are now eager to know if this phenomenon exists in each segmental fuzzy system. Some useful lemmas are demonstrated below in order to develop the design scheme of optimal fuzzy control law regarding to the time-invariant fuzzy system.

Lemma 3: Consider a dynamical system, $X(k+1) = f(X(k), u(k), t)$, with $X(k_0^i) = X_0^i$. Let the pair $(X^*(\cdot), u^*(\cdot))$ be the infinite-horizon optimal solution with the performance index $J(u(\cdot)) = \sum_{k=k_0^i}^{\infty} f_0(X(k), u(k), t)$, and the pair $(\bar{X}^*(\cdot), \bar{u}^*(\cdot))$ be the finite-horizon optimal solution with respect to $\bar{J}(u(\cdot)) = \sum_{k=k_0^i}^{k_1^i-1} f_0(X(k), u(k), t)$, where $f(\cdot,\cdot,\cdot), f_0(\cdot,\cdot,\cdot) \in \mathrm{PC}(\Re^n, \Re^m, \Re)$, a mapping from $\Re^n \times \Re^m \times \Re$ to piecewise-continuous real-valued functions. If $X(k_1^i)$ is a free point, then $(\bar{X}^*(k), \bar{u}^*(k)) = (X^*(k), u^*(k))$ for all $k \in [k_0^i, k_1^i - 1]$.

$$\begin{bmatrix} X^*(k+1) \\ P(k+1) \end{bmatrix} = \begin{bmatrix} H_iA_k - H_iB_kB_k^tH_i^t\left[A_k^tH_i^t\right]^{-1}L_k & -H_iB_kB_k^tH_i^t\left[A_k^tH_i^t\right]^{-1} \\ -\left[A_k^tH_i^t\right]^{-1}L_k & -\left[A_k^tH_i^t\right]^{-1} \end{bmatrix} \times \begin{bmatrix} X^*(k) \\ P(k) \end{bmatrix} \tag{15}$$

Proof: Assume the infinite-horizon optimal solution for some $k \in [k_0^i, k_1^i - 1]$ is not the finite-horizon optimal solution, then

$$\sum_{k=k_0^i}^{k_1^i-1} f_0(\bar{X}^*(k), \bar{u}^*(k), k) < \sum_{k=k_0^i}^{k_1^i-1} f_0(X^*(k), u^*(k), k).$$

If we define a decision sequence $\hat{u}(k)$ as

$$(\hat{X}(k), \hat{u}(k)) = \begin{cases} (\bar{X}^*(k), \bar{u}^*(k)), & k \in \left[k_0^i, k_1^i - 1\right], \\ (X^*(k), u^*(k)), & k \in \left[k_1^i, \infty\right), \end{cases}$$

where $\hat{X}(k)$ denotes the corresponding state trajectory, then we have

$$\begin{aligned} \sum_{k=k_0^i}^{\infty} f_0(\hat{X}(k), \hat{u}(k), k) &= \sum_{k=k_0^i}^{k_1^i-1} f_0(\bar{X}^*(k), \bar{u}^*(k), k) \\ &\quad + \sum_{k=k_1^i}^{\infty} f_0(X^*(k), u^*(k), k) \\ &< \sum_{k=k_0^i}^{\infty} f_0(X^*(k), u^*(k), k). \end{aligned}$$

This conflicts with the notion that $(X^*(k), u^*(k))$ is the infinite-horizon optimal solution. Thus, $(\bar{X}^*(k), \bar{u}^*(k)) = (X^*(k), u^*(k))$ holds for all $k \in [k_0^i, k_1^i - 1]$ positively. □

From the proof in Theorem 1, we know, for all $k \in [k_0^i, k_1^i - 1]$, the optimal performance index has the following relation:

$$\min_{u_{[k, k_1^i-1]}} \sum_{l=k}^{k_1^i-1} [X^t(l)L(l)X(l) + u^t(l)u(l)] + X^t\left(k_1^i\right) Q^i X\left(k_1^i\right) = X^{i^{*^t}}(k)\pi^i\left(k, k_1^i\right) X^{i^*}(k). \quad (26)$$

It is obvious that $\pi^i(k, k_1^i)$ is monotonic increasing with k_1^i. On the other hand, completely controllable system guarantees the infinite-horizon performance index to be finite. Hence, $\pi^i(k, k_1^i)$ is bounded above for all k_1^i. In other words, there exists $\bar{\pi}^i(k)$ such that $\lim_{k_1^i \to \infty} \pi^i(k, k_1^i) = \bar{\pi}^i(k)$, where $\bar{\pi}^i(k)$ satisfies the recursive Riccati-like equation in (16) or (17). Moreover, for the case of constant L, A and $B, \bar{\pi}^i(k)$ becomes independent of k, and accordingly, we can regard it as the solution of the steady state version of (16) or (17), i.e., $\lim_{k_1^i \to \infty} \pi^i(k, k_1^i) = \pi_\infty^i$. Furthermore, we adopt two proposed lemmas [21] below to link the controllability and observability of the fuzzy subsystem to those of the entire fuzzy system, and then a simple criteria for a more implementable and concise optimal solution is available.

Lemma 4 (Controllability): (A_i, B_i) is completely controllable (c.c.) for all $i = 1, \ldots, r$ if and only if $\text{rank}[\lambda I_n - H(X(k))A \quad H(X(k))B] = n$, for all $k \in [k_0, k_1 - 1]$ and $\lambda \in \sigma(H(X(k))A)$, where $\sigma(H(X(k))A)$ denotes the spectrum of $H(X(k))A$.

Lemma 5 (Observability): (A_i, C) is completely observable (c.o.) for all $i = 1, \ldots, r$ if and only if

$$\text{rank}\begin{bmatrix} \lambda I_n - H(X(k))A \\ C \end{bmatrix} = n, \quad \forall \lambda \in \sigma(H(X(k))A).$$

Now, the aforementioned inference can be summarized into the following lemma.

Lemma 6: For each segmental dynamical fuzzy system

$$\begin{aligned} X(k+1) &= H_iAX(k) + H_iBW_iR(k), \\ Y(k) &= CX(k) \end{aligned} \quad (27)$$

with $L = C^tC$ and $X(k_0^i)$ known. If (A_i, B_i) is c.c. and (A_i, C) is c.o., $\forall i = 1, \ldots, r$, then

1) there exists an unique $n \times n$ symmetric positive semidefinite solution, π_∞^i, of the discrete-time algebraic Riccati-like equation

$$K = L + A^tH_i^tK\left[I_n + H_iBB^tH_i^tK\right]^{-1} H_iA \quad (28)$$

$$K = L + A^tH_i^tKH_iA - A^tH_i^tKH_iB\left[I_n + B^tH_i^tKH_iB\right]^{-1} B^tH_i^tKH_iA; \quad (29)$$

2) the asymptotically optimal control law is

$$R_\infty^{i^*}(k) = -W_i^t\left[W_iW_i^t\right]^{-1} B^tH_i^t\pi_\infty^i\left[I_n + H_iBB^tH_i^t\pi_\infty^i\right]^{-1} H_iAX_\infty^{i^*}(k), \quad k \in \left[k_0^i, \infty\right) \quad (30)$$

which minimizes

$$J_\infty^i(R(\cdot)) = \sum_{k=k_0^i}^{\infty} \left[X^t(k)LX(k) + R^t(k)W_i^tW_iR(k)\right]; \quad (31)$$

3) and the optimal closed-loop fuzzy system

$$X_\infty^{i^*}(k+1) = \left[I_n + H_iBB^tH_i^t\pi_\infty^i\right]^{-1} H_iAX_\infty^{i^*}(k), \quad k \in [k_0^i, \infty) \quad (32)$$

is asymptotically and exponentially stable;

4) the minimum performance index is equal to $\sum_{i=1}^N X^{i^{*^t}}(k_0^i)\pi_\infty^i X^{i^*}(k_0^i)$.

Proof: We know, from Lemmas 4 and 5, (A_i, B_i) is c.c., $\forall i = 1, \ldots, r$, if and only if $\text{rank}[\lambda I_n - H(X(k))A \quad H(X(k))B] = n, \forall \lambda \in \sigma(H(X(k))A)$, and accordingly, $\text{rank}[\lambda I_n - H_iA \quad H_iB] = n, \forall \lambda \in \sigma(H_iA), i = 1, \ldots, N; (A_i, C)$ is c.o., $\forall i = 1, \ldots, r$, if and only if $\text{rank}[\begin{smallmatrix} \lambda I_n - H(X(k))A \\ C \end{smallmatrix}] = n, \forall \lambda \in \sigma(H(X(k))A)$, which ensures $\text{rank}[\begin{smallmatrix} \lambda I_n - H_iA \\ C \end{smallmatrix}] = n, \forall \lambda \in \sigma(H_iA)$. Therefore, (A_i, B_i) c.c. and (A_i, C) c.o., $\forall i = 1, \ldots, r$, guarantee (H_iA, H_iB) c.c. and (H_iA, C) c.o., respectively. Then, by the classical discrete-time linear quadratic optimal control theorem [2], we have the optimal solution for the segmental fuzzy system in (27). □

Then, a more implementable and important theorem for the time-invariant fuzzy system can be extracted on the ground of the aforementioned Lemmas and Theorem 2, which concerns the time-varying fuzzy subsystem.

Theorem 3 (Time-Invariant Multistage Optimization): Consider the time-invariant fuzzy system and fuzzy controller described, respectively, by (1) and (2) with $L = C^tC$. Let $(X^*(k), R^*(k)), k \in [k_0, k_1 - 1]$, denote the optimal solution with respect to $J(R(\cdot))$ in (8), $(X^{i^*}(k), R^{i^*}(k)), k \in [k_0^i, k_1^i - 1]$, denote the ith-stage optimal solution with respect to $J^i(R(\cdot))$ in (13), and $(X_\infty^{i^*}(k), R_\infty^{i^*}(k)), k \in [k_0^i, k_1^i - 1]$, be the ith-stage asymptotically optimal solution with respect to $J_\infty^i(R(\cdot))$ in (31). If $N > \bar{N}, (A_i, B_i)$ is c.c. and (A_i, C) is c.o., for all $i = 1, \ldots, r$, then

1) See (33), shown at the bottom of the page, where $k_0^i = k_1^{i-1}, i = 2, \ldots, N; k_0^1 = k_0$;
2) for the ith stage, $i = 1, \ldots, N-1$, the optimal control law is $R_\infty^{i^*}(\cdot)$ in (30), and the optimal trajectory is $X_\infty^{i^*}(\cdot)$ in (32), where π_∞^i is the unique symmetric positive semidefinite solution of the discrete-time algebraic Riccati-like equation in (28) or (29);
3) as for the last stage, the Nth stage, the optimal control law is $R^{i^*}(\cdot)$ in (23), and the optimal trajectory is $X^{i^*}(\cdot)$ in (25), where $\pi^i(k, k_1^i)$ is the symmetric positive semidefinite solution of the segmental recursive Riccati equation in (16) or (17);
4) the minimum performance index is

$$\min_{R_{[k_0, k_1-1]}} J(R(\cdot)) = \sum_{i=1}^{N-1} \left[X_\infty^{i^{*^t}}\left(k_0^i\right) \pi_\infty^i X_\infty^{i^*}\left(k_0^i\right) \right] + X^{N^{*^t}}\left(k_0^N\right) \pi^N\left(k_0^N, k_1\right) X^{N^*}\left(k_0^N\right).$$

Proof: (1) Based on Lemma 1, the whole optimization is decomposed into an N-stage decision process with, at each stage, the initial state resulting from the decision of its previous stage. Now, our optimal fuzzy control problem, Problem 3, can be attacked in the following two issues, with both regarding to the same dynamical fuzzy system described by (7) except that the initial stage is $X(k_0^i)$ and the time interval is $[k_0^i, k_1^i - 1]$ for the ith stage

$$\text{(a)} \min_{R_{[k_0^N, k_1-1]}} \sum_{k=k_0^N}^{k_1-1} [X^t(k)L(k)X(k) + R^t(k)W^t(Y(k))W(Y(k))R(k)] + X^t(k_1)QX(k_1), \quad (34)$$

$$\text{(b)} \min_{R_{[k_0^i, k_1^i-1]}} \sum_{k=k_0^i}^{k_1^i-1} [X^t(k)L(k)X(k) + R^t(k)W^t(Y(k))W(Y(k))R(k)], \quad i = 1, \ldots, N-1. \quad (35)$$

Furthermore, by Lemma 3, the optimal solution with respect to (35) can be regarded as the one with respect to

$$\min_{R_{[k_0^i, \infty]}} \sum_{k=k_0^i}^{\infty} [X^t(k)L(k)X(k) + R^t(k)W^t(Y(k))W(Y(k))R(k)], \quad i = 1, \ldots, N-1. \quad (36)$$

Notice that this equivalence only exists in period $[k_0^i, k_1^i - 1]$. Therefore, we, hereinafter, can pay attention only to (36) for the time interval $[k_0^i, k_1^i - 1], i = 1, \ldots, N-1$, and to (34) for the time interval $[k_0^N, k_1]$.

(2) For $N > \bar{N}, H(X(k))$ and $W(Y(k))$ in the dynamical fuzzy system described by (7) can be replaced, respectively, by constant matrices H_i and W_i for the ith stage. Therefore, the whole fuzzy system in (7) can be rewritten as a linear system represented by (27). The Nth-stage optimal solution indeed follows the optimal solution in Theorem 2. As for the other stages, we know, from the proof of Lemmas 6, (A_i, B_i) c.c. and (A_i, C) c.o., $\forall i = 1, \ldots, r$, guarantee, respectively, (H_iA, H_iB) c.c. and (H_iA, C) c.o., $\forall i = 1, \ldots, N$, where r and N are, respectively, the number of rules of the fuzzy system in (1), and the number of stages of the process described by the dynamical fuzzy system in (27). Hence, we can obtain the optimal solution for the first $N-1$ stages via Lemma 6. □

So, for the first $N-1$ stages, a time-invariant fuzzy system can still give rise to the time-invariant linear optimal fuzzy control law.

B. Infinite-Horizon Problem

The purpose of this section is to design the optimal fuzzy controller concerning the infinite-horizon problem, which is the case that the operating time goes to infinity or is much larger than the time-constant of the dynamic system. It is critical to notice the problem: Does the minimal performance index finitely exists? We introduce the concept proposed by Macki and Strauss (1982): *If the linearized system of a nonlinear system with respect to (w.r.t) some state $X_o \in \Re^n$ is c.c., then X_o is an interior point of the controllable set (the set of all initial points which can be steered to the target)*. Now, the linearized system of the fuzzy system in (7) with respect to point X_o is

$$X(k+1) = H(X_o)A(k)X(k) + H(X_o)B(k)u(k). \quad (37)$$

Therefore, to ensure that our problem is solvable, it is necessary that the pair $(H(X_o)A(\cdot), H(X_o)B(\cdot))$ is controllable at all time and for all $X_o \in \Re^n$. We can now find out the design scheme of the infinite-horizon optimal fuzzy controller.

Theorem 4 (Multistage Optimization): For the fuzzy system and fuzzy controller described by (1) and (2), respectively, let

$$(X^*(k), R^*(k)) = \begin{cases} \left(X_\infty^{i^*}(k), R_\infty^{i^*}(k)\right), & \forall k \in \left[k_0^i, k_1^i - 1\right], \quad i = 1, \ldots, N-1, \\ \left(X^{N^*}(k), R^{N^*}(k)\right), & \forall k \in [k_0^N, k_1 - 1], \end{cases} \quad (33)$$

$(X^*_\infty(k), R^*_\infty(k)), k \in [k_0, \infty)$, be the optimal solution with respect to

$$J_\infty(R(\cdot)) = \sum_{k=k_0}^{\infty} [X^t(k)L(k)X(k) + R^t(k)W^t(Y(k))W(Y(k))R(k)] \tag{38}$$

and $(X^{i^*}(k), R^{i^*}(k)), k \in [k_0^i, k_1^i - 1]$, be the ith-stage optimal solution with respect to (39), as shown at he bottom of the page. If the linearized fuzzy system in (37) is controllable, and there exists $\bar{N}$ such that if $N > \bar{N}$, then

1) $(X^*_\infty(k), R^*_\infty(k)) = (X^{i^*}(k), R^{i^*}(k))$, $k \in [k_0^i, k_1^i - 1], i = 1, \ldots, N$; where $k_0^i = k_1^{i-1}, i = 2, \ldots, N; k_0^1 = k_0, k_1^N = \infty$;
2) for the ith stage, $k \in [k_0^i, k_1^i - 1], i = 1, \ldots, N$, the optimal control law, the corresponding global minimizer, the optimal trajectory and the minimum performance index satisfy the same corresponding equations in Theorem 2, except that $k_1^N = \infty$ and $Q^i = 0$ for all $i = 1, \ldots, N$.

Proof: This theorem obviously holds with Theorem 2. For the Nth stage, the controllable criterion can ensure the existence of the limit value of $\pi^N(k, k_1)$; i.e., $\bar{\pi}^N(k) = \lim_{k_1 \to \infty} \pi^N(k, k_1)$ exists for all $k \in [k_0^N, k_1]$, and $\bar{\pi}^N(k)$ is still the symmetric positive semidefinite solution of the segmental recursive Riccati-like equation in (16) or (17). □

For the time-invariant case, the pair $(H(X_o)A, H(X_o)B)$ c.c. is equivalent to $\text{rank}[\lambda I_n - H(X_o)A, \quad H(X_o)B] = n, \forall \lambda \in \sigma(H(X_o)A)$, and this condition, by Lemma 4, can be satisfied if (A_i, B_i) is c.c., for all $i = 1, \ldots, r$. So, we need the following assumption as the prerequisite for the optimal controller design in the time-invariant infinite-horizon case.

Assumption 2: (A_i, B_i) is c.c., for all $i = 1, \ldots, r$.

Theorem 5 (Time-Invariant Multistage Optimization): Consider the time-invariant fuzzy system and fuzzy controller described, respectively, by (1) and (2) with $L = C^tC$. If there exists $\bar{N}$ such that if $N > \bar{N}, (A_i, B_i)$ is c.c. and (A_i, C) is c.o., for all $i = 1, \ldots, r$, then

1) For each stage

$$(X^*_\infty(k), R^*_\infty(k)) = \left(X^{i^*}_\infty(k), R^{i^*}_\infty(k)\right), \quad \forall k \in \left[k_0^i, k_1^i - 1\right], \quad k_0^1 = k_0, \quad k_1^N = \infty \tag{40}$$

where $R^{i^*}_\infty(k)$ is the ith-stage asymptotically optimal control law in (30), and $X^{i^*}_\infty(k)$ is the corresponding asymptotically optimal trajectory in (32), where π^i_∞ is the unique symmetric positive semidefinite solution of the discrete-time algebraic Riccati-like equation in (28) or (29);

2) the minimum performance index is

$$\min_{R_{[k_0,\infty)}} J_\infty(R(\cdot)) = \sum_{i=1}^{N} \left[X^{i^{*^t}}_\infty \left(k_0^i\right) \pi^i_\infty X^{i^*}_\infty \left(k_0^i\right)\right].$$

Proof: This Theorem obviously holds according to Theorem 3. □

IV. Stability and Gain Margin

In this section, we shall show that the designed control law can not only asymptotically and exponentially stabilize the fuzzy system, but also form a closed-loop fuzzy system with any desired degree of stability. We also concern with the range of the feedback gain, *gain margin*, to which we can increase under the stability consideration.

A. Global Stability

As remarked earlier, the whole optimal trajectory is decomposed into N segments, and more, if each fuzzy subsystem in (1) is well-behaved (c.c. and c.o.) and $N > \bar{N}$, then each segment can be described by its corresponding asymptotically optimal trajectory during the same period of this segment; i.e.,

$$X^*_\infty(k) \equiv X^{i^*}(k) \equiv X^{i^*}_\infty(k), \qquad \forall k \in \left[k_0^i, k_1^i - 1\right] \tag{41}$$

where $i = 1, \ldots, N$ and $k_0^1 = k_0, k_1^N = k_1, k_0^i = k_1^{i-1}, i = 2, \ldots, N$. That is, the behavior of the closed-loop fuzzy system can be captured by the corresponding asymptotic behavior of these N segments.

Theorem 6: For the time-invariant fuzzy system and fuzzy controller described, respectively, in (1) and (2) with $L = C^tC$. If there exists $\bar{N}$ such that if $N > \bar{N}, (A_i, B_i)$ is c.c., and (A_i, C) is c.o. for $i = 1, \ldots, r$, then

1) the optimal closed-loop fuzzy system

$$X^*_\infty(k+1) = \left[I_n + H_iBB^tH_i^t\pi^i_\infty\right]^{-1} \times H_iAX^*_\infty(k), \quad k \in \left[k_0^i, k_1^i - 1\right] \tag{42}$$

where $i = 1, \ldots, N, k_0^1 = k_0, k_1^N = \infty, k_0^i = k_1^{i-1}, i = 2, \ldots, N$, is exponentially stable;
2) the total energy of system output is finite, i.e., $\sum_{k=k_0}^{\infty} \|Y^*(k)\|^2 < \infty$.

Proof: (1) Recall that $X^{i^*}_\infty(\cdot)$ is the ith-stage asymptotically optimal trajectory of the quadratic optimal control problem, i.e., minimizing the performance index $J^i_\infty(R(\cdot))$ in (31) with respect to the dynamical fuzzy system in (27). Moreover, (A_i, B_i) c.c. and (A_i, C) c.o., $\forall i = 1, \ldots, r$, guarantees, from part (2) in the proof of Theorem 3, (H_iA, H_iB) c.c. and

$$J^i(R(\cdot)) = \begin{cases} \sum_{k=k_0^N}^{\infty} (X^t(k)L(k)X(k) + R^t(k)W^t(Y(k))W(Y(k))R(k)), & \forall k \in [k_0^N, \infty), \quad i = N, \\ \sum_{k=k_0^i}^{k_1^i - 1} (X^t(k)L(k)X(k) + R^t(k)W^t(Y(k))W(Y(k))R(k)), & \forall k \in [k_0^i, k_1^i - 1], \quad \text{otherwise}. \end{cases} \tag{39}$$

(H_iA, C) c.o., $\forall i = 1, \ldots, N$. Hence, we know, from Lemma 6, the ith-stage asymptotically optimal trajectory

$$X_\infty^{i^*}(k+1) = \left[I_n + H_iBB^tH_i^t\pi_\infty^i\right]^{-1} \times H_iAX_\infty^{i^*}(k), \quad k \in \left[k_0^i, \infty\right)$$

is asymptotically and exponentially stable, i.e., $\sigma([I_n + H_iBB^tH_i^t\pi_\infty^i]^{-1}H_iA) \subset D(0,1)$, where $D(0,1)$ denotes the open unit disc in the complex plane. Hence, via (41), the optimal trajectory described by (42) is asymptotically and exponentially stable since all eigenvalues of the system matrix characterizing the dynamical behavior of each segment lie inside the unit circle.

(2) From (7)

$$\begin{aligned}\sum_{k=k_0}^{\infty} \|Y^*(k)\|^2 &= \sum_{k=k_0}^{\infty} |CX_\infty^*(k)\|^2 \\ &\le \sum_{k=k_0}^{\infty} \|C\|^2 \|X_\infty^*(k)\|^2 \\ &= \sum_{i=1}^{N} \sum_{k=k_0^i}^{k_1^i-1} \|C\|^2 \left\|X_\infty^{i^*}(k)\right\|^2. \end{aligned} \qquad (43)$$

From (1) in the proof, the ith-stage asymptotically optimal trajectory $X_\infty^{i^*}(\cdot)$ is exponentially stable. The term, *exponentially stable*, means *uniformly asymptotically stable* in the stability concept [3], which means that for all $X(k_0^i) \in \Re^n$ and $k_0^i \in \mathcal{N}, X_\infty^{i^*}(k)$ satisfies the following two properties.

a) The range of mapping from k to $X_\infty^{i^*}(k)$ is bounded on $k \ge k_0^i$ uniformly, i.e.,

$$\exists q < \infty \text{ s.t. } \left\|X_\infty^{i^*}(k)\right\| < q, \quad \forall k \ge k_0^i.$$

b) The range of mapping from k to $X_\infty^{i^*}(k)$ tends to zero as $k \longrightarrow \infty$ uniformly, i.e.,

$$\forall \epsilon > 0, \quad \exists \text{ an integer } T(\epsilon) > 0 \quad \text{s.t. } \left\|X_\infty^{i^*}(k)\right\| \le \epsilon, \quad \forall k \ge T(\epsilon).$$

Assume $T(\epsilon)$ is located in the N_o-th stage, i.e., $k_0^{N_o} \le T(\epsilon) \le k_1^{N_o}$. Then, (43) becomes

$$\begin{aligned}\sum_{k=k_0}^{\infty} \|Y^*(k)\|^2 \le & \sum_{i=1}^{N_o-1} \sum_{k=k_0^i}^{k_1^i} \|C\|^2 \left\|X_\infty^{i^*}(k)\right\|^2 \\ &+ \sum_{k=k_0^{N_o}+1}^{T(\epsilon)-1} \|C\|^2 \left\|X_\infty^{i^*}(k)\right\|^2 \\ &+ \sum_{k=T(\epsilon)}^{k_1^{N_o}} \|C\|^2 \left\|X_\infty^{i^*}(k)\right\|^2 \\ &+ \sum_{i=N_o+1}^{N} \sum_{k=k_0^i+1}^{k_1^i} \|C\|^2 \left\|X_\infty^{i^*}(k)\right\|^2 \end{aligned} \qquad (44)$$

where $k_1^N = \infty$. We know that the first two terms are finite and the others are infinitesimal, and thereupon, the total energy of system output is absolutely finite. □

The stability of the closed-loop fuzzy system in time-varying case can still be ensured if the corresponding asymptotically optimal trajectory of each segment is exponentially stable.

B. Stabilization to any Desired Degree

Before investigating further, we demonstrate the importance of the resultant closed-loop fuzzy system with a degree of stability of at least some prescribed constant $\alpha, \alpha > 1$, which means that the state approaches zero at least by the rate of α^{-k} or the poles of the resultant closed-loop fuzzy system are all constrained to lie inside a circle with the radius of $1/\alpha$. Let $X(k) = \hat{X}(k)\alpha^k, u(k) = \hat{u}(k)\alpha^k$, the performance index $J_\infty(R(\cdot))$ in (38) with $u(t) = W(Y(t))R(t)$ can be rewritten as $J_\infty(R(\cdot)) = \sum_{k_0}^{\infty} \alpha^{2k}[\hat{X}^t(t)L\hat{X}(t) + \hat{u}^t(k)\hat{u}(k)]$. To ensure that the optimal value of $J_\infty(R(\cdot))$ is finite, the item should approach zero as k approaches infinity, and hence, $\hat{X}(\cdot)$ should decay faster than α^{-k} as k approaches infinity. This is equivalent to requiring the modified closed-loop fuzzy system to have a degree of stability of at least α. Of course, the larger the desired degree of stability is, the more stable the closed-loop fuzzy system is. However, a high degree of stability may only be achieved at the expense of excessive control energy consumption.

Lemma 7: For a system $R_d = [A, B, C]$: $X(k+1) = AX(k) + Bu(k), Y(k) = CX(k)$, where A, B and C are $n \times n, n \times m$ and $n' \times n$ matrices, (A, B) c.c. is equivalent to $(\alpha A, \alpha B)$ c.c., and (A, C) c.o. is equivalent to $(\alpha A, C)$ c.o., for any complex value α.

Proof: (1) (A, B) is c.c. if and only if $\text{rank}[B \ AB \ldots A^{n-1}B] = n$. Let $\hat{B} \triangleq \alpha^{-k}B$. Then we have $\text{rank}[\hat{B} \ A\hat{B} \ldots A^{n-1}\hat{B}] = \text{rank}[B \ AB \ldots A^{n-1}B]$, which means (A, B) is c.c. if and only if $(A, \hat{B})$ is c.c. Similar operation, except $\hat{C} \triangleq \alpha^kC$, can be adopted to show that (A, C) is c.o. if and only if $(A, \hat{C})$ is c.o. Now, consider two systems: $X(k+1) = AX(k) + \hat{B}u(k), Y(k) = \hat{C}X(k)$ and $\hat{X}(k+1) = \alpha A\hat{X}(k) + \alpha Bu(k), Y(k) = C\hat{X}(k)$. Obviously, they are related by a nonsingular linear transformation α^k. Therefore, $(A, \hat{B})$ is c.c. if and only if $(\alpha A, \alpha B)$ is c.c. and $(A, \hat{C})$ is c.o. if and only if $(\alpha A, C)$ is c.o., for any complex value α. From (1) and (2), we conclude that Lemma 7 holds. □

Theorem 7: Consider the time-invariant fuzzy system and fuzzy controller described, respectively, by (1) and (2) with $L = C^tC$. There exists $\bar{N}$ such that if $N > \bar{N}, (A_i, B_i)$ is c.c., and (A_i, C) is c.o. for all $i = 1, \ldots, r$, then the fuzzy system can be stabilized to any desired degree of stability; i.e., all the poles of the resultant closed-loop fuzzy system are located inside a circle with the radius of $1/\alpha$, where $\alpha > 1$.

Proof: (1) As we know, for $N > \bar{N}$ and well-behaved fuzzy subsystems, the behavior of $X_\infty^*(\cdot)$ is fully described by $X_\infty^{i^*}(\cdot), i = 1, \ldots, N$. Hence, we now pay attention to such quadratic optimal control problem: minimizing the performance index in (31) with respect to the linear time-invariant fuzzy system in (27). Let Ai, Bi and u denote, respectively,

H_iA, H_iB and W_iR. Then, we have the following optimization problem:

$$\min \cdot \sum_{k=k_0^i}^{\infty} [X^t(k)LX(k) + u^t(k)u(k)] \text{ w.r.t.} \times \begin{cases} X(k+1) = AiX(k) + Biu(k) \\ Y(k) = CX(k). \end{cases} \quad (45)$$

Let $\hat{X}(k) = \alpha^k X(k), \hat{Y}(k) = \alpha^k Y(k)$ and $\hat{u}(k) = \alpha^k u(k)$. Equation (45) can be rewritten as

$$\min \cdot \sum_{k=k_0^i}^{\infty} \alpha^{-2k}[\hat{X}^t(k)L\hat{X}(k) + \hat{u}^t(k)\hat{u}(k)] \text{ w.r.t.} \begin{cases} \hat{X}(k+1) = \alpha Ai\hat{X}(k) + \alpha Bi\hat{u}(k), \\ \hat{Y}(k) = C\hat{X}(k). \end{cases} \quad (46)$$

(2) From Lemma 7, we know that (Ai, B) c.c. and (Ai, C) c.o., $\forall i = 1, \ldots, r$, if and only if $(\alpha Ai, \alpha Bi)$ c.c. and $(\alpha Ai, C)$ c.o., $\forall i = 1, \ldots, N$. Hence, following Lemma 6, the global minimizer for the modified fuzzy system in the above is $\hat{u}_{\infty}^{i^*}(k) = -Bi^t\hat{\pi}_{\infty}^i(\alpha)[\alpha^{-2}I_n + BiBi^t\hat{\pi}_{\infty}^i(\alpha)]^{-1}AiX_{\infty}^{i^*}(k), i = 1, \ldots, N$, where $\hat{\pi}_{\infty}^i(\alpha)$ is the positive–semidefinite solution of the modified discrete-time algebraic Riccati-like equation

$$K(\alpha) = L + Ai^tK(\alpha)[\alpha^{-2}I_n + BiBi^tK(\alpha)]^{-1}Ai \quad (47)$$

$$K(\alpha) = L + \alpha^2Ai^tK(\alpha)Ai - \alpha^2Ai^tK(\alpha)Bi[\alpha^{-2}I_n + Bi^tK(\alpha)Bi]^{-1}Bi^tK(\alpha)Ai \quad (48)$$

and the modified fuzzy system is asymptotically stable, i.e., $\hat{X}_{\infty}^{i^*}(k) \to 0, \forall i = 1, \ldots, N$, as $k \to \infty$. Then, $X_{\infty}^{i^*}(k), \forall i = 1, \ldots, N$, decays faster than α^{-k} as $k \to \infty$ since $\hat{X}_{\infty}^{i^*}(k) = \alpha^k X_{\infty}^{i^*}(k)$. Via (41), $X_{\infty}^*(\cdot)$ will approach zero at least by the rate of α^{-k}, i.e., the poles of the resultant closed-loop fuzzy system are all located inside the circle of radius $1/\alpha$. □

C. Gain Margin

In the remainder of Section IV, we examine another characteristic, *gain margin*, of the resulting closed-loop fuzzy system. For the time-invariant well-behaved fuzzy subsystems, if $N > \bar{N}$, then $X_{\infty}^*(k)$ is coincident with $X_{\infty}^{i^*}(k)$, for all $k \in [k_0^i, k_1^i - 1]$, where $i = 1, \ldots, N$. Therefore, we can only discuss the asymptotic behavior of the dynamical fuzzy system of each stage, and then turns it into the behavior of the entire dynamical fuzzy system via (41). From Lemma 6, the designed ith-stage *asymptotically* global minimizer is

$$u_{\infty}^{i^*}(k) = -B^tH_i^t\pi_{\infty}^i\left[I_n + H_iBB^tH_i^t\pi_{\infty}^i\right]^{-1} \times H_iAX_{\infty}^{i^*}(k), \quad k \in \left[k_0^i, \infty\right). \quad (49)$$

In order to measure the gain margin, we consider the following corresponding controller:

$$u(k) = -\left[\beta\left(B^tH_i^t\pi_{\infty}^i\left[I_n + H_iBB^tH_i^t\pi_{\infty}^i\right]^{-1}H_iA\right)\right] X(k), \quad \beta \geq 1, \quad k \in [k_0^i, \infty). \quad (50)$$

The gain margin of *the ith-stage closed-loop fuzzy system is defined as the amount by which β can be increased until the system becomes unstable* [2]. Now, let $v(k) \triangleq u(k)/\beta$, and then we have

$$J_{\infty}^i(u(\cdot)) = \sum_{k=k_0^i}^{\infty}(X^t(k)LX(k) + u^t(k)u(k)) = \sum_{k=k_0^i}^{\infty}(X^t(k)LX(k) + \beta^2v^t(k)v(k)), \quad \beta \geq 1. \quad (51)$$

We further consider

$$\hat{J}_{\infty}^i(v(\cdot)) = \sum_{k=k_0^i}^{\infty}(qX^t(k)LX(k) + v^t(k)v(k)), \quad q > 0. \quad (52)$$

Notice that $J_{\infty}^i(u(\cdot)) = \beta^2\hat{J}_{\infty}^i(v(\cdot))$ and $q = 1/\beta^2$. Comparing (52) to (51), we find that the larger the β is, the smaller the q is, which means that when q goes to zero, the gain margin of the ith-stage closed-loop fuzzy system becomes infinite.

The following theorem shows that the resultant closed-loop fuzzy system possesses an infinite gain margin. We shall first show that the closed-loop fuzzy system for $X_{\infty}^{i^*}(k), k \in [k_0^i, k_1^i]$ possesses an infinite gain margin, and then, via (41), concludes that the gain margin of the resultant closed-loop fuzzy system is infinite.

Lemma 8: For the fuzzy system in (27), if (H_iA, H_iB) is c.c., (H_iA, C) is c.o., and $\hat{\pi}_{\infty}^i(q)$ is the positive semidefinite solution of one of the following modified discrete-time algebraic Riccati-like equation

$$K(q) = qL + A^tH_i^tK(q)\left[I_n + H_iBB^tH_i^tK(q)\right]^{-1}H_iA, \quad (53)$$

$$K(q) = qL + A^tH_i^tK(q)H_iA - A^tH_i^tK(q)H_iB\left[I_n + B^tH_i^tK(q)H_iB\right]^{-1}B^tH_i^tK(q)H_iA, \quad (54)$$

where $K(q)$ is the dependent variable of the algebraic equation, then $\lim_{q\to 0}\hat{\pi}_{\infty}^i(q)$ exists and is equal to $\hat{\pi}_{\infty}^i(0)$, which is the symmetric positive–semidefinite solution of discrete-time Riccati-like equation

$$K(0) = A^tH_i^tK(0)\left[I_n + H_iBB^tH_i^tK(0)\right]^{-1}H_iA. \quad (55)$$

Proof: Denote H_iA and H_iB by Ai and Bi to simplify notation.

(1) We now consider the optimal solution for minimizing

$$\hat{J}_{\infty}^i(u(\cdot)) = \sum_{k=k_0^i}^{\infty}(qX^t(k)LX(k) + u^t(k)u(k)), \quad \forall q > 0.$$

It is realizable to include q into the state penalty matrix L. From Lemma 6, for any $q > 0$, the global minimizer is

$$\hat{u}_{\infty}^i(k) = -Bi^t\hat{\pi}_{\infty}^i(q)\left[I_n + BiBi^t\hat{\pi}_{\infty}^i(q)\right]^{-1} \times Ai\hat{X}_{\infty}^{i^*}(k), \quad \beta \geq 1, \quad k \in \left[k_0^i, \infty\right)$$

where $\hat{\pi}^i_\infty(q)$ is the symmetric positive semidefinite solution of the modified discrete-time algebraic Riccati-like equation in (53) or (54), and the corresponding closed-loop system

$$\hat{X}^{i^*}_\infty(k+1) = \left[I_n + BiBi^t\hat{\pi}^i_\infty(q)\right]^{-1} Ai\hat{X}^{i^*}_\infty(k), \quad k \in \left[k^i_0, \infty\right) \tag{56}$$

is exponentially stable; i.e., the radius of spectrum of system matrix, $\rho\{[I_n + BiBi^t\hat{\pi}^i_\infty(q)]^{-1}Ai\}$, is less than 1.

(2) Now, we will make sure if the limit value of $\hat{\pi}^i_\infty(q)$ exists and is equal to $\hat{\pi}^i_\infty(0)$. For simplifying notation, we use K_q and K^+_q to denote $\hat{\pi}^i_\infty(q)$ and $\hat{\pi}^i_\infty(q+\epsilon)$, where $\hat{\pi}^i_\infty(q+\epsilon)$ is the symmetric positive–semidefinite solution of the following equation:

$$K^+_q = (q+\epsilon)L + Ai^t\left[I_n + K^+_q BiBi^t\right]^{-1} K^+_q Ai. \tag{57}$$

Define $\delta K_q \triangleq K^+_q - K_q$, then

$$\delta K_q = \epsilon L + Ai^t\left\{K^+_q\left[I_n + BiBi^tK^+_q\right]^{-1} - \left[I_n + K_qBiBi^t\right]^{-1}K_q\right\}Ai. \tag{58}$$

Let $\bar{A}$ and $\bar{A}_+$ denote, respectively, $[I_n + BiBi^tK_q]^{-1}Ai$ and $[I_n + BiBi^tK^+_q]^{-1}Ai$, then

$$\delta K_q = \epsilon L + \bar{A}^t\delta K_q\bar{A}_+. \tag{59}$$

Let $Z_q = (\partial K_q/\partial q) = \lim_{\epsilon\to 0}((K^+_q - K_q)/\epsilon)$, then we obtain a discrete-time Lyapunov-like equation

$$Z_q = L + \bar{A}^tZ_q\bar{A}_+. \tag{60}$$

From (1), we know $\rho(\bar{A}), \rho(\bar{A}_+) < 1$, and accordingly, the unique solution is

$$Z_q = \sum_{k=k^i_0}^{\infty}(\bar{A}^k)^tL\left(\bar{A}^k_+\right) > 0. \tag{61}$$

In other words, $x^tZ_qx = (\partial/\partial q)x^tK_qx > 0$ for all $x \in \Re^n$. Hence, the function x^tK_qx is monotonic decreasing as $q \to 0$, and bounded below by 0; i.e., $\lim_{q\to 0} x^tK_qx$ constantly exists for all $x \in \Re^n$. We can pick special x's to let $\lim_{q\to 0} K_q = K_0$, i.e., $\lim_{q\to 0}\hat{\pi}^i_\infty(q) = \hat{\pi}^i_\infty(0)$. □

Theorem 8: Consider the time-invariant fuzzy system and fuzzy controller described, respectively, by (1) and (2) with $L = C^tC$. If $N > \bar{N}, (A_i, B_i)$ is c.c., (A_i, C) is c.o., for all $i = 1, \ldots, r$, and $\lim_{\beta\to\infty}(\beta-1)\rho(H_iBB^tH^t_i\hat{\pi}^i_\infty(1/\beta^2)) < 2$, then the global minimizer in (49) generates a closed-loop fuzzy system with an infinite gain margin; i.e., the modified closed-loop fuzzy system $\{I_n - \beta H_iBB^tH^t_i\hat{\pi}^i_\infty(q)[I_n + H_iBB^tH^t_i\hat{\pi}^i_\infty(q)]^{-1}\}H_iA$, $k \in [k^i_0, k^i_1 - 1], i = 1, \ldots, N$, is always stable for any $\beta \geq 1$, where $q = 1/\beta^2$ and $\hat{\pi}^i_\infty(q)$ is the positive–semidefinite solution of (53) or (54).

Proof: (1) Denote H_iA and H_iB by Ai and Bi to simplify notation. From (1) of the proof in Lemma 8, we know $|\lambda_1| < 1, \forall\lambda_1 \in \sigma([I_n + BiBi^t\hat{\pi}^i_\infty(q)]^{-1}Ai)$, and the asymptotically behavior of the enlarged closed-loop fuzzy system for each segment is

$$X^{i^*}_\infty(k+1) = \left\{I_n - \beta BiBi^t\hat{\pi}^i_\infty(q)\left[I_n + BiBi^t\hat{\pi}^i_\infty(q)\right]^{-1}\right\}AiX^{i^*}_\infty(k), \quad k \in \left[k^i_0, \infty\right). \tag{62}$$

By (41), the modified closed-loop fuzzy system is

$$X^*_\infty(k+1) = \left(I_n - \beta BiBi^t\hat{\pi}^i_\infty(q)\left[I_n + BiBi^t\hat{\pi}^i_\infty(q)\right]^{-1}\right) \times AiX^*_\infty(k), \quad k \in \left[k^i_0, \infty\right) \tag{63}$$

where $i = 1, \ldots, N$, and $k^1_0 = k_0, k^N_1 = \infty, k^i_0 = k^{i-1}_1, i = 2, \ldots, N$. We will make sure $\rho\{(I_n - \beta BiBi^t\hat{\pi}^i_\infty(1/\beta^2)[I_n + BiBi^t\hat{\pi}^i_\infty(1/\beta^2)]^{-1})Ai\} < 1$ for all $\beta \geq 1$.

(2) For any eigenpair, (λ_1, v_1), of $[I_n + BiBi^t\hat{\pi}^i_\infty(q)]^{-1}Ai$, we know $[I_n + BiBi^t\hat{\pi}^i_\infty(q)]^{-1}Aiv_1 = \lambda_1v_1$. By (53), we have $(\hat{\pi}^i_\infty(q) - qL)v_1 = \lambda_1Ai^t\hat{\pi}^i_\infty(q)v_1$. Hence, $|\hat{\pi}^i_\infty(q) - qL - \lambda_1Ai^t\hat{\pi}^i_\infty(q)| = 0$. Therefore, for all $\lambda_1 \in \sigma([I_n + BiBi^t\hat{\pi}^i_\infty(q)]^{-1}Ai)$, λ_1 is also the eigenvalue of $(\hat{\pi}^i_\infty(q) - qL)(Ai^t\hat{\pi}^i_\infty(q))^{-1}$, which is equivalent to $Ai^t\hat{\pi}^i_\infty(q)[I_n + BiBi^t\hat{\pi}^i_\infty(q)]^{-1}Ai(Ai^t\hat{\pi}^i_\infty(q))^{-1}$. To ensure this, $Ai^t\hat{\pi}^i_\infty(q)$ commutes with $[I_n + BiBi^t\hat{\pi}^i_\infty(q)]^{-1}Ai$ obviously, i.e.,

$$\begin{aligned} &Ai^t\hat{\pi}^i_\infty(q)\cdot\left[I_n + BiBi^t\hat{\pi}^i_\infty(q)\right]^{-1}Ai \\ &\quad = \left[I_n + BiBi^t\hat{\pi}^i_\infty(q)\right]^{-1}Ai\cdot Ai^t\hat{\pi}^i_\infty(q), \\ &AiAi^t\hat{\pi}^i_\infty(q)\cdot\left[I_n + BiBi^t\hat{\pi}^i_\infty(q)\right]^{-1}\cdot Ai \\ &\quad = Ai\cdot\left[I_n + BiBi^t\hat{\pi}^i_\infty(q)\right]^{-1}\cdot AiAi^t\hat{\pi}^i_\infty(q) \end{aligned}$$

and then, Ai commutes with $[I_n + BiBi^t\hat{\pi}^i_\infty(q)]^{-1}$ or, more precisely, with $BiBi^t\hat{\pi}^i_\infty(q)$.

(3) The above analysis shows that $[I_n + (1-\beta)BiBi^t\hat{\pi}^i_\infty(q)]$ commutes with $[I_n + BiBi^t\hat{\pi}^i_\infty(q)]^{-1}Ai$. Recall that if A and B are commutative operators, then $\rho(AB) \leq \rho(A)\rho(B)$. So

$$\begin{aligned} &\rho\left\{\left(I_n - \beta BiBi^t\hat{\pi}^i_\infty(q)\left[I_n + BiBi^t\hat{\pi}^i_\infty(q)\right]^{-1}\right)Ai\right\} \\ &\quad = \rho\left\{\left[I_n + (1-\beta)BiBi^t\hat{\pi}^i_\infty(q)\right] \times \left[I_n + BiBi^t\hat{\pi}^i_\infty(q)\right]^{-1}Ai\right\} \\ &\quad \leq \rho\left\{I_n + (1-\beta)BiBi^t\hat{\pi}^i_\infty(q)\right\} \cdot\rho\left\{\left[I_n + BiBi^t\hat{\pi}^i_\infty(q)\right]^{-1}Ai\right\} \\ &\quad < \rho\left\{I_n - (\beta-1)BiBi^t\hat{\pi}^i_\infty(q)\right\} < 1, \\ &\qquad \forall q = 1/\beta^2 \geq 0, \end{aligned} \tag{64}$$

since $1 > 1 - (\beta-1)\lambda(BiBi^t\hat{\pi}^i_\infty(1/\beta^2)) > 1 - \lim_{\beta\to\infty}(\beta-1)\lambda(BiBi^t\hat{\pi}^i_\infty(1/\beta^2)) - 1$. Therefore, since the spectrum of system matrix, which characterizes the dynamical behavior of each segment of the modified optimal trajectory, is always located in the unit disc of complex space, the resultant closed-loop fuzzy system possesses infinite gain margin. □

V. Practical Application

In this section, we propose an algorithm to implement the theorems in Section IV, and consider an optimal backing up control of a computer simulated trunk-trailer to illustrate the proposed optimal fuzzy control scheme.

A. Dynamic Decomposition Algorithm

We shall propose a procedure to check the two inequalities in (21) and (22), which can ensure the almost-invariant-membership-function criteria during a whole single stage. Now, denoting the time-dependence as a lower index (i.e., X_τ^* for $X^*(\tau)$) and substituting k_0^i by k (i.e., time-instant $t_{k_0^i}$ by t_k) for notation simplification, we can rewrite (21) as follows:

$$\left\|\frac{dH(X^*(\tau))}{d\tau}\right\|_{\tau=t_k} = \left\|\left[\nabla^t_{X_\tau^*} h_1(X_\tau^*) \cdot \frac{dX_\tau^*}{d\tau}\cdot I_n, \ldots, \nabla^t_{X_\tau^*} h_r(X_\tau^*)\cdot \frac{dX_\tau^*}{d\tau}\cdot I_n\right]\right\|_{\tau=t_k} \tag{65}$$

where $\nabla^t_{X_\tau^*} h_i(X_\tau^*) = [(dh_i(X_\tau^*)/dx_1), \ldots, (dh_i(X_\tau^*)/dx_n)], i = 1, \ldots, r$, and $(dX_\tau^*/d\tau)|_{\tau=t_k} = ((X^*(k^+) - X^*(k))/\kappa_T)$, where k_T denotes a very short time skip. Substituting $P_k = K_k X_k^*$ into the TPBVP in Theorem 1, we have

$$\begin{aligned} K_l &= L_l + A_l^t H^t(X_l^*) K_{l+1}\left[I_n + H(X_l^*)B_l B_l^t H(X_l^*)K_{l+1}\right]^{-1} H(X_l)A_l \\ K_{k_1} &= Q \end{aligned} \tag{66}$$

where $l \in [k_0, k_1 - 1]$. Though the entire backward recursive Riccati equation in the above is unavailable in practice, the relationship between two time-steps is always available. In other words, at time-step k, we have

$$K_k = L_k + A_k^t H^t(X_k^*)K_{k^+}\left[I_n + H(X_k^*)B_k B_k^t H(X_k^*)K_{k^+}\right]^{-1} H(X_k)A_k. \tag{67}$$

And, according to Lemma 3, the finite-horizon optimal solution for the free-end problem is the same as the optimal solution of the infinite-horizon issue. Therefore, the solution, π_k, for (67) is also the solution, $\bar{\pi}_k$, of the following asymptotic Riccati-like equation:

$$\bar{K}_k = L_k + A_k^t H^t(X_k^*)\bar{K}_k\left[I_n + H(X_k^*)B_k B_k^t H(X_k^*)\bar{K}_k\right]^{-1} H(X_k)A_k. \tag{68}$$

Also, we have

$$X_{k^+}^* = \left[I_n + H(X_k^*)B_k B_k^t H(X_k^*)\bar{\pi}_k\right]^{-1} H(X_k^*)A_k X_k^*. \tag{69}$$

Therefore, we obtain (70), shown at the bottom of the page. Hence, via (65) and (70), we can check the inequality in (21) for any time-step (time-instant), and the *existence* of $\bar{N}$ is guaranteed if the inequality holds at the starting time step of every stage.

We then propose the following *dynamic decomposition algorithm* to check the two inequalities in (21) and (22), and to find the proper number of time steps in each stage N_i and also the value of $\bar{N}$ to ensure that the membership functions are almost invariant during a whole stage.

Algorithm DDA: Dynamic Decomposition Algorithm

Input: the initial chosen membership functions; initial state $X(k_0)$; sampling period T_s; maximum number of design trials n_t.

Output: optimal controller $u^*(\cdot)$; optimal trajectory $X^*(\cdot)$; value of N_i; value of $\bar{N}$ ($\bar{N}$ being initialized as $\bar{N} = 0$).

Step 0: (set threshold parameters) Set the default values of k_T, κ_{H1} and κ_{H2}.

Step 1: (initial check)

IF ($\|(dH(X^*(\tau))/d\tau)\|_{\tau=t_{k_0}} \le \kappa_{H_1}$), THEN {go to **Step 2**}

ELSE {choose a more smooth membership function and go back to **Step 1**, or break after n_t times of failing trials.}

END

Step 2: (k^i denoting the time step in the ith stage, i.e., $k^i \in [k_0^i, k_1^i - 1]$)

(a) Find out the solution $\bar{\pi}_{k^i}$ of (68) with the membership function $H(X^*_{k_0^i})$.

(b) Calculate $u^*_{k^i}$ and $X^*_{k^i+1}$ by

$$\begin{aligned} u^*_{k^i} = &-B^t_{k^i} H^t\left(X^*_{k_0^i}\right)\bar{\pi}_{k^i}\left[I_n + H\left(X^*_{k_0^i}\right) \times B_{k^i}B^t_{k^i}H^t\left(X^*_{k_0^i}\right)\bar{\pi}_{k^i}\right]^{-1} \\ &\times H\left(X^*_{k_0^i}\right)A_{k^i}X^*_{k^i} \end{aligned} \tag{71}$$

$$\begin{aligned} X^*_{k^i+1} = &\left[I_n + H\left(X^*_{k_0^i}\right)B_{k^i}B^t_{k^i}H^t\left(X^*_{k_0^i}\right)\bar{\pi}_{k^i}\right]^{-1} \\ &\times H\left(X^*_{k_0^i}\right)A_{k^i}X^*_{k^i}. \end{aligned} \tag{72}$$

(c) IF ($\|H(X^*_{k^i+1}) - H(X^*_{k_0^i})\| \le \kappa_{H_2}$) THEN {Go to (a)}

END

(d) IF ($k^i = k_1 - 1$) THEN {$N_i = k_1 - k_0^i$; Stop}

END

Step 3: (find the starting point of the next stage, i.e., the starting time step k_0^{i+1} and the corresponding time instant $t_{k_0^{i+1}}$)

IF ($\|(dH(X^*(\tau))/d\tau)\|_{\tau=t_{k^i+1}} \le \kappa_{H_1}$) THEN {$k_0^{i+1} = k^i + 1; t_{k_0^{i+1}} = t_{k^i+1}; N_i = k_0^{i+1} - k_0^i - 1$;

$\bar{N} = \bar{N} + 1$; jump to **Step 2**.}

ELSE IF ($\|(dH(X^*(\tau))/d\tau\|_{\tau=t_{k^i}} \le \kappa_{H_1}$) THEN {$k_0^{i+1} = k^i; t_{k_0^{i+1}} = t_{k^i}; N_i = k_0^{i+1} - k_0^i - 1$;

$\bar{N} = \bar{N} + 1$; jump to **Step 2**}

ELSE {decrease T_s to get finer division (if it is adjustable) or choose another membership function and jump to **Step 1**, or break after n_t times of failing trials.}

END.

For the time-invariant finite-horizon (except the Nth stage) or infinite-horizon problem, the estimated optimal solutions, $u^*_{k^i}$ in (71) and $X^*_{k^i}$ in (72), are also the optimal solutions $u^{i^*}_\infty(k)$ in (49) and $X^{i^*}_\infty(k)$ in (32), where the estimated $\bar{\pi}_{k^i}$ equals to π^i_∞

$$\left.\frac{dX_\tau^*}{d\tau}\right|_{\tau=t_k} = \frac{\left(\left[I_n + H(X_k^*)B_k B_k^t H(X_k^*)\bar{\pi}_k\right]^{-1} H(X_k^*)A_k - I_n\right)X_k^*}{\kappa_T}. \tag{70}$$

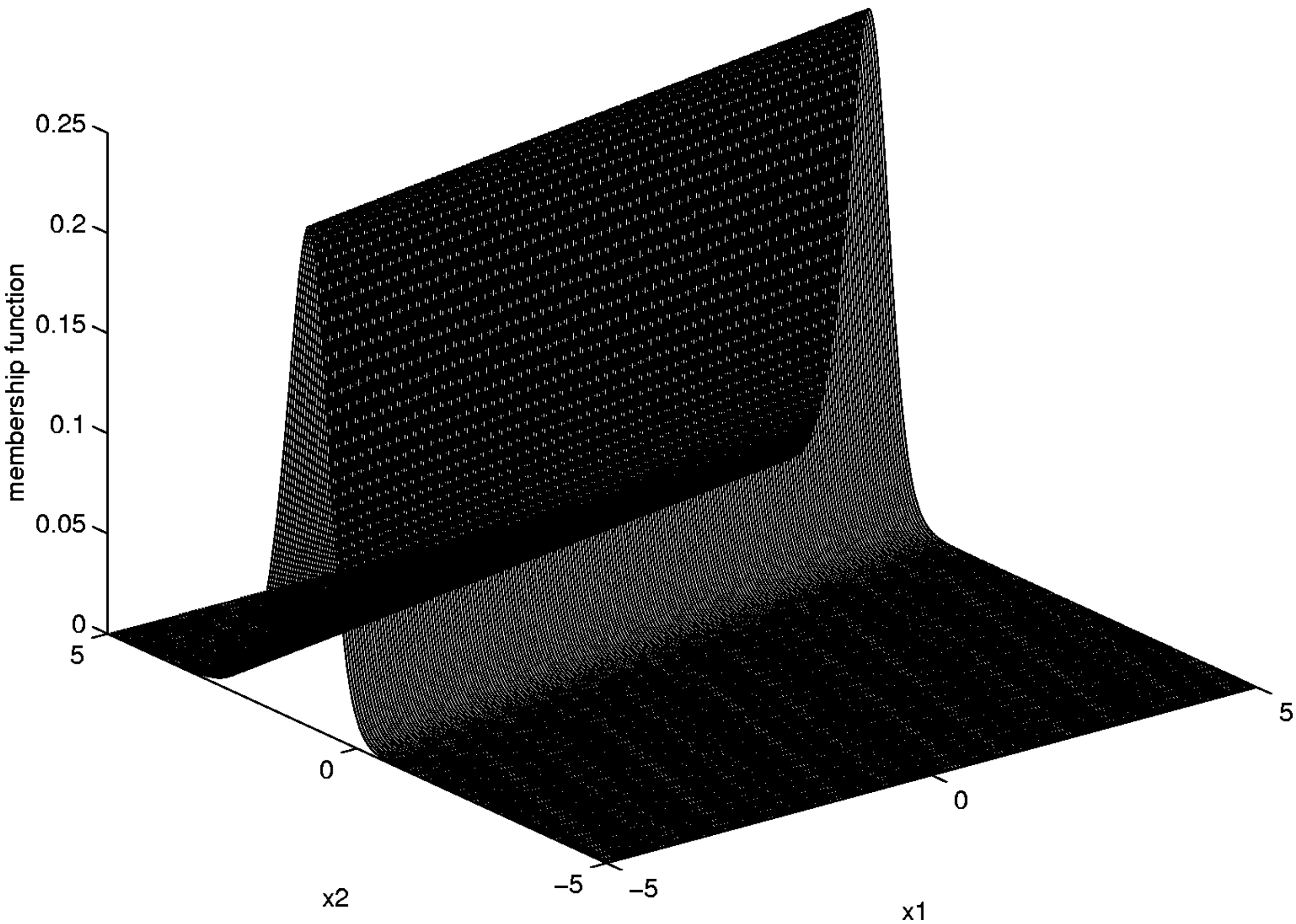

Fig. 1. Profile of the chosen membership function $\alpha_1(k)$ in (73).

in (28) or (29). As for the other case, we can obtain the optimal solutions $X^{i^*}(k)$ in (25) and $u^{i^*}(k)$ via (20) with the aid of the estimated N_i and $\pi^i(k, k_1^i)$ in (16) or (17).

B. Numerical Simulations

In this section, we consider an optimal backing up control of a computer simulated truck-trailer to illustrate the proposed optimal fuzzy control scheme and its theoretic aspect. Tanaka and Sano [6] described a computer simulated truck-trailer with the mathematical model

$$\begin{aligned} x_1(k+1) &= (1 - v \cdot t'/L')x_1(k) + v \cdot t'/l \cdot u(k) \\ x_2(k+1) &= x_2(k) + v \cdot t'/L' \cdot x_1(k) \\ x_3(k+1) &= x_3(k) + v \cdot t' \cdot \sin(x_2(k) \\ &\qquad + v \cdot t'/2L' \cdot x_1(k)) \end{aligned}$$

where

l	length of truck;
L'	length of trailer;
t'	sampling time;
v	constant speed of the backward movement.

Then, they used the following fuzzy model to represent the truck-trailer system:

$$\begin{aligned} &R^1\text{: If } z(k) \equiv x_2(k) + v \cdot t'/\{2L'\} \cdot x_1(k) \text{ is about } 0, \\ &\quad \text{then } X(k+1) = A_1 X(k) + B_1 u(k) \\ &R^2\text{: If } z(k) \equiv x_2(k) + v \cdot t'/\{2L'\} \\ &\qquad \cdot x_1(k) \text{ is about } \pi \text{ or } -\pi, \\ &\quad \text{then } X(k+1) = A_2 X(k) + B_2 u(k) \end{aligned}$$

and the system output is $Y(k) = CX(k)$ with $C = [0 \;\; 0 \;\; 1], l = 2.8, L' = 5.5, v = -1.0, t' = 2.0$ and $X(k) = [x_1(k) x_2(k) x_3(k)]^t$, where

$$A_1 = \begin{bmatrix} 1.3636 & 0 & 0 \\ -0.3636 & 1.0 & 0 \\ 0.0120 & -2.0 & 1.0 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 1.3636 & 0 & 0 \\ -0.3636 & 1.0 & 0 \\ 0 & -0.0064 & 1.0 \end{bmatrix}$$

$$B_1 = B_2 = \begin{bmatrix} -0.7143 \\ 0 \\ 0 \end{bmatrix}.$$

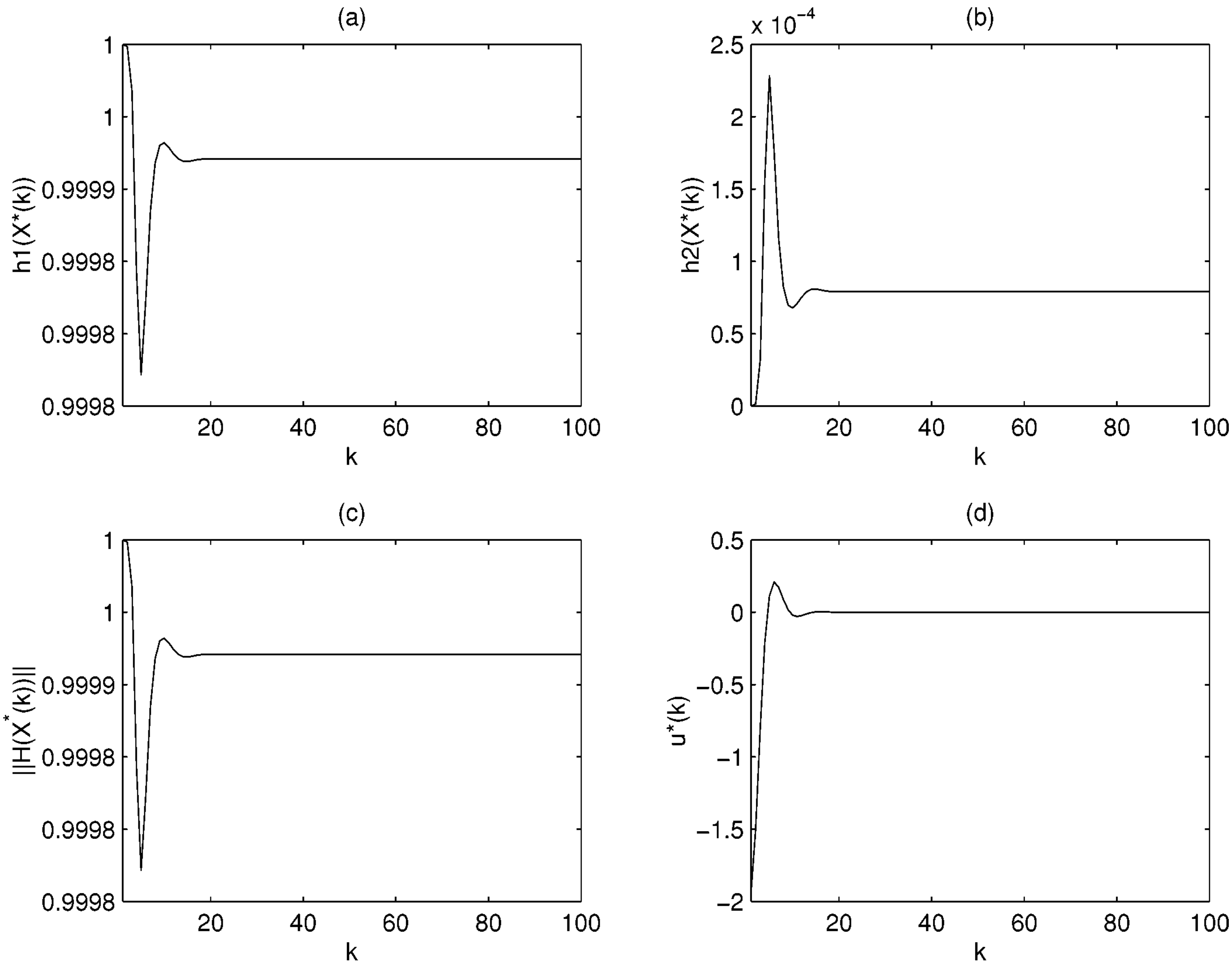

Fig. 2. (a) and (b) The normalized firing strengths, $h_1(X^*(k))$ and $h_2(X^*(k))$, corresponding to the optimal trajectory. (c) The value of the norm of $H(X^*(k))$. (d) The outputs of the optimal fuzzy controller $(X_0 = [-\pi/2, -3\pi/4, -10]^t)$.

We further assume our fuzzy controller is

$$R^1\text{: If } z(k) \equiv x_2(k) + v \cdot t'/\{2L'\} \cdot x_1(k) \text{ is about } 0, \text{then } u(k) = r_1(k)$$
$$R^2\text{: If } z(k) \equiv x_2(k) + v \cdot t'/\{2L'\} \cdot x_1(k) \text{ is about } \pi \text{ or } -\pi, \text{then } u(k) = r_2(k).$$

With the chosen membership functions shown in Fig. 1 [6], the firing-strength is

$$h_1(X(k)) = \alpha_1(k) = \left(1 - \frac{1}{1+e^{-3(z(k)-\pi/2)}}\right) \cdot \left(\frac{1}{1+e^{-3(z(k)+\pi/2)}}\right) \quad (73)$$
$$h_2(X(k)) = \alpha_2(k) = 1 - \alpha_1(k) \quad (74)$$

which, in this case, are also the normalized firing-strengths of the rules for fuzzy system and controller, i.e., $w_i(Y(k)) = h_i(X(k)), i = 1, 2$. Therefore, the *linear-like* dynamical fuzzy system representation for the nonlinear truck-trailer system can be described by (7) with $A = [\begin{smallmatrix} A_1 \\ A_2 \end{smallmatrix}], B = [\begin{smallmatrix} B_1 \\ B_2 \end{smallmatrix}], R = [\begin{smallmatrix} r_1 \\ r_2 \end{smallmatrix}], H(X(k)) = [h_1(X(k)) \;\; h_2(X(k))]$ and $W(Y(k)) = [w_1(Y(k)) \;\; w_2(Y(k))]$.

For the finite-horizon free-end optimal control problem, the performance index is

$$J(R(\cdot)) = \sum_{k=0}^{99} [X^t(k)LX(k) + R^t(k)W(k)^t W(k)R(k)] + X^t(100)QX(100) \quad (75)$$

where $L = C^t C$ and $Q = I_3$. As for the infinite-horizon case, the performance index is

$$J_\infty(R(\cdot)) = \sum_{k=0}^{\infty} [X^t(k)LX(k) + R^t(k)W(k)^t W(k)R(k)]. \quad (76)$$

Though the fuzzy subsystem is unstable (the spectrum of system matrix $\sigma(A_i) = \{1, 1, 1.36\}, i = 1, 2$), it is time-invariant and well-behaved; i.e., the fuzzy subsystem is c.c. and

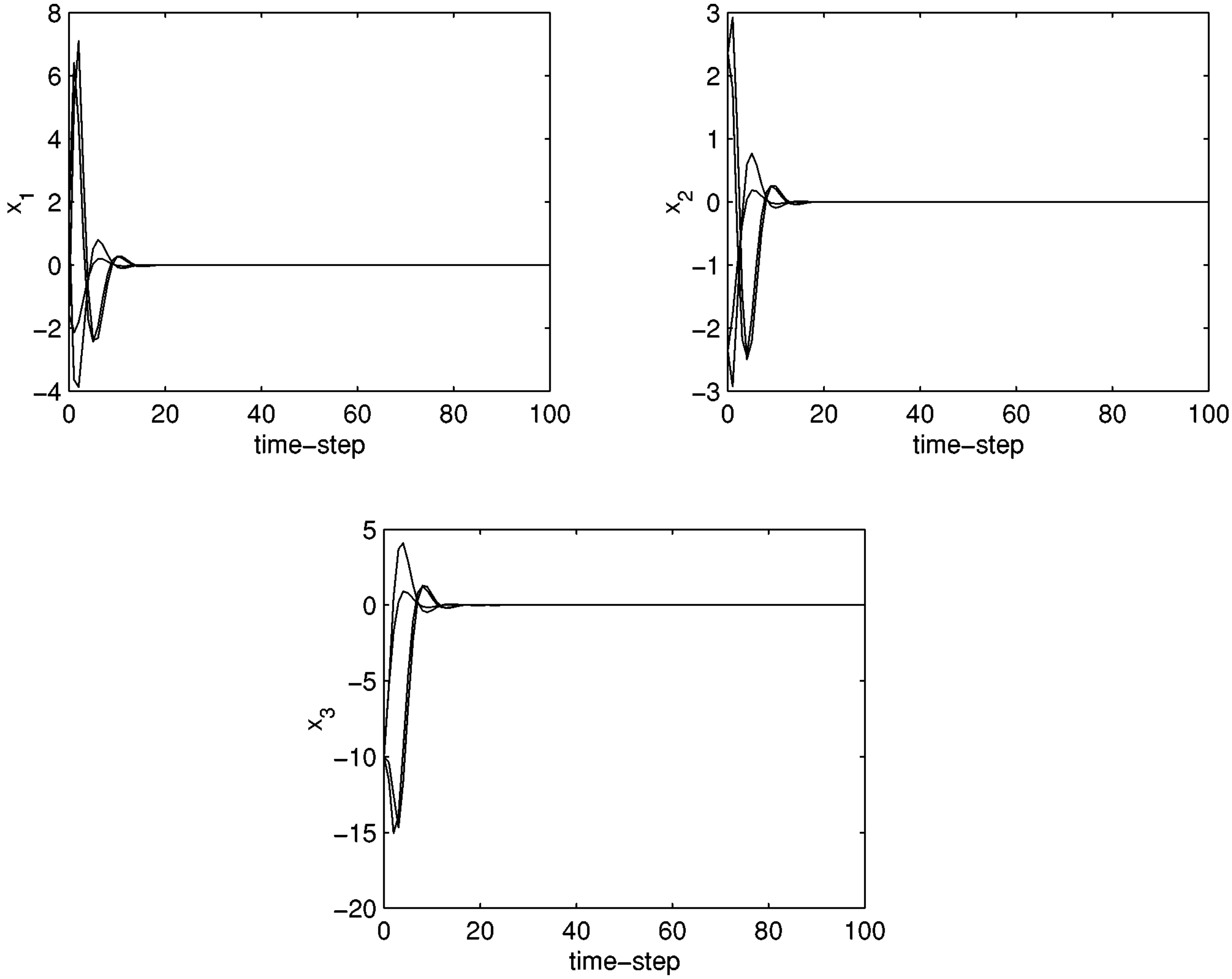

Fig. 3. State responses of the discrete-time fuzzy system with the designed optimal fuzzy controller in the finite-horizon quadratic optimal control problem of Section V at the four initial conditions: $X(0) = (-\pi/2, -3\pi/4, -10)^t, (-\pi/2, 3\pi/4, -10)^t, (\pi/2, -3\pi/4, -10)^t$ and $(\pi/2, 3\pi/4, -10)^t$.

c.o. ($\mathrm{rank}[\lambda I_3 - A_i B_i] = \mathrm{rank}[\lambda I_3 - A_i C] = 3$, for all $\lambda \in \sigma(A_i)$). Moreover, the asymptotic Riccati-like equation in (68) becomes

$$\bar{K}_k = L + A^t H^t(X_k^*)\bar{K}_k[I_n + H(X_k^*)BB^t H(X_k^*)\bar{K}_k]^{-1} H(X_k)A. \quad (77)$$

Therefore, the steps a) and b) in **Step 2** in Algorithm DDA can be simplified as

a) find out the constant solution $\bar{\pi}$ of (77) with the membership function $H(X^*_{k_0^i})$;

b) calculate $u^*_{k^i}$ and $X^*_{k^i+1}$ by

$$u^*_{k^i} = -B^t H^t\left(X^*_{k_0^i}\right)\bar{\pi}\left[I_n + H\left(X^*_{k_0^i}\right)BB^t H^t\left(X^*_{k_0^i}\right)\bar{\pi}\right]^{-1} \times H\left(X^*_{k_0^i}\right)AX^*_{k^i} \quad (78)$$

$$X^*_{k^i+1} = \left[I_n + H\left(X^*_{k_0^i}\right)BB^t H^t\left(X^*_{k_0^i}\right)\bar{\pi}\right]^{-1} \times H\left(X^*_{k_0^i}\right)AX^*_{k^i}. \quad (79)$$

Since the chosen membership functions are smooth Gaussian functions (see Fig. 1), we can efficiently obtain the optimal fuzzy controller with the aid of the DDA algorithm for determining appropriate segmentation under the almost-invariant-membership-function criteria. For the initial state $X_0 = [-\pi/2, -3\pi/4, -10]^t$, the individual normalized firing strengths for the optimal trajectory (i.e., $h_1(X^*(k))$ and $h_2(X^*(k))$), and also the value of the norm of their synthetical matrix (i.e., $\|H(X^*(k))\|$) are shown in Figs. 2(a), (b) and (c), respectively; the outputs of the designed optimal fuzzy controller are shown in Fig. 2(d). The state responses of the resultant closed-loop fuzzy system in the finite-horizon case is shown in Fig. 3, which reveals that the designed optimal fuzzy controller can promptly push the simulated truck-trailer system from various initial states to and stay at the desired state. Hence, the finite-state-trajectory penalty vanishes and Theorems 3 and 5 are coincident. Our simulation results also show that the state responses of the resultant closed-loop fuzzy system in the infinite-horizon case are the same as those in Fig. 3.

VI. Conclusion

The entire fuzzy system representation was proposed to formulate the quadratic optimal fuzzy control problem, and further, the unification of the individual matrices into synthetical matrices was proposed to generate a *linear-like* global system representation of discrete-time fuzzy systems. Based on this representation, the design scheme of global discrete-time optimal fuzzy controllers was derived theoretically. Furthermore, a multistage decomposition of optimization scheme was proposed to design the global optimal fuzzy controller more efficiently and keep the global optimality at the same time. Grounding on this efficient design scheme, several fascinating characteristics have been shown to exist in the resultant closed-loop discrete-time fuzzy system.

Overall, the fuzzily-blended entire fuzzy system is considered to formulate the quadratic optimal fuzzy control problem, and the global optimal effect can then be achieved even though the chosen system model is composed of distributed rule-based fuzzy subsystems. This formation sheds light on the deadlock of the research of quadratic optimal fuzzy control. Moreover, the proposed *linear-like* synthetical matrix representation and the systematic design procedures might activate a new research direction in the quadratic optimal fuzzy control. Furthermore, the proposed in-depth analysis on the degree of stability and gain margin can provide the researchers with complete perspective of all facets of the resultant closed-loop fuzzy system. Simulation results have manifested that the designed optimal fuzzy controllers can effectively drive the fuzzy system to the target points in short time.

Appendix A

Proof of Theorem 1: (1) Define

$$\Phi(X(\cdot),R(\cdot)) \triangleq \sum_{l=k}^{k_1-1}[X^t(l)L(l)X(l)+R^t(l)W^t(Y(l)) \times W(Y(l))R(l)] + X^t(k_1)QX(k_1), \quad k\in[k_0,k_1-1]$$

where $X(k)=X^*(k)$ is the initial state at time k. By Lagrange multiplier method, we turn the optimization problem into the problem of minimizing

$$\bar{\Phi}(X(\cdot),R(\cdot)) = \Phi(X(\cdot),R(\cdot)) - 2\sum_{l=k}^{k_1-1}P^t(l+1)[X(l+1) - H(X(l))A(l)X(l) - H(X(l))B(l)W(Y(l))R(l)] \tag{80}$$

where $P(l+1)\in\Re^n$ is the Langrange multiplier vector. Now, we assume the optimal solutions $(X^*(\cdot),Y^*(\cdot),R^*(\cdot))$ exist, and, according to the calculus of variations method, let $X(l)=X^*(l)+\epsilon Z(l), Y(l)=Y^*(l)+\epsilon\nu(l)$, $R(l)=R^*(l)+\epsilon V(l), l\in[k,k_1-1]$, where $V(l)\in\Re^{m\delta}$ is the perturbation vector with respect to $R(l)$, and $Z(k)=0$ since the initial state at time k is $X(k)=X^*(k)$. To simplify notations, we shall omit the explicit time- and state-dependence; e.g., we write H for $H(X(l)), X$ for $X(l),\ldots$, and use $X_{k_1}, Z_{k_1}, Z_{l+1}$ and P_{l+1} to denote, respectively, $X(k_1), Z(k_1), Z(l+1)$ and $P(l+1)$ in the following derivation. Then, substituting these variables into (80) and assuming Assumption 1 holds, we have

$$\bar{\Phi}(X(\cdot),R(\cdot)) = \bar{\Phi}(X^*(\cdot),R^*(\cdot)) + \epsilon^2\Phi(Z(\cdot),V(\cdot)) + 2\epsilon\sum_{l=k}^{k_1-1}\{Z^tLX^* + V^tW^tWR^* - P_{l+1}^t[Z_{l+1} - HAZ - HBWV]\} + 2\epsilon Z_{k_1}^tQX_{k_1}^*.$$

We know that a minimum of $\bar{\Phi}(X(\cdot),R(\cdot))$ requires

$$\left.\frac{\partial\bar{\Phi}(X(\cdot),R(\cdot))}{\partial\epsilon}\right|_{\epsilon=0} = 0, \quad \left.\frac{\partial^2\bar{\Phi}(X(\cdot),R(\cdot))}{\partial\epsilon^2}\right|_{\epsilon=0} > 0.$$

The second criteria holds positively since

$$\frac{\partial^2\bar{\Phi}(X(\cdot),R(\cdot))}{\partial\epsilon^2} = 2\left\{Z_{k_1}^tQZ_{k_1} + \sum_{l=k}^{k_1-1}[Z^tLZ + V^tW^tWV]\right\} > 0.$$

Hence, the necessary and sufficient condition for optimality is

$$\sum_{l=k}^{k_1-1}\{Z^tLX^* - P_{l+1}^tZ_{l+1} + P_{l+1}^tHAZ + V^tW^tWR^* + P_{l+1}^tHBWV\} + Z_{k_1}^tQX_{k_1}^* = 0. \tag{81}$$

Via the fact $\sum_{l=k}^{k_1-1}P_{l+1}^tZ_{l+1} = \sum_{l=k}^{k_1-1}[P^tZ] + P_{k_1}^tZ_{k_1} - P_k^tZ_k$ and $Z_k=0$, We have

$$\sum_{l=k}^{k_1-1}V^t[W^tWR^* + W^tB^tH^tP_{l+1}] + \sum_{l=k}^{k_1-1}Z^t[LX^* + A^tH^tP_{l+1} - P] + Z_{k_1}^k[QX_{k_1}^* - P(k_1)] = 0. \tag{82}$$

Since $Z(\cdot)$ and $V(\cdot)$ are independent, we obtain the global minimizer $u^*(k)$ in (10), and the corresponding optimal control law $R^*(k)$ in (9), where $P(k)$ and the optimal trajectory $X^*(k)$ satisfies (11) with $X(k_0)=X_0$ and $P(k_1)=QX_{k_1}^*$.

(3) Now, we step for finding the minimum performance index:

$$\Phi(X(\cdot),R(\cdot)) = \sum_{l=k}^{k_1-1}\left\{X^tLX + u^tu - P_{l+1}^t[X_{l+1} - HAX - HBu]\right\}.$$

From (82), we know $P^t_{l+1}HA = P^t - X^{*^t}L$ and $P^t_{l+1}HB = -u^{*^t}$, and accordingly

$$\begin{aligned}\min_{R_{[k,k_1-1]}} J(R(\cdot)) &= \sum_{l=k}^{k_1-1}\{P^t(l)X(l) - P^t(l+1)X(l+1)]\\ &\quad + X^t(k_1)QX(k_1)\\ &= P^t(k)X(k).\end{aligned}$$

Hence, $J(R(\cdot)) = P^t(k_0)X(k_0)$. This completes the proof. □

Proof of Lemma 2: (1) Since $P(k) = K(k)X^*(k)$ and $P(k_1^i) = Q^iX^*(k_1^i)$, we obtain $K(k_1^i) = Q^i$. On the other hand, (15) gives

$$P(k) = L(k)X^*(k) + A^t(k)H_i^tP(k+1) \quad (83)$$

$$\begin{aligned}X^*(k+1) &= H_iA(k)X^*(k)\\ &\quad - H_iB(k)B^t(k)H_i^tP(k+1). \quad (84)\end{aligned}$$

Substituting $P(k+1) = K(k+1)X^*(k+1)$ into (84), we obtain the optimal trajectory $X^*(k)$ in (18), and accordingly, we rewritten (83) as

$$\begin{aligned}&\Big\{K(k) - L(k) - A^t(k)H_i^tK(k+1)\\ &\quad\times \left[H_i^tI_n + H_iB(k)B^t(k)H_i^tK(k+1)\right]^{-1}\\ &\quad H_iA(k)\Big\}X^*(k) = 0.\end{aligned}$$

To ensure existence of the above equality no matter what $X^*(k)$ is, (16) holds positively.

(2) Through standard matrix manipulations, $\{[H_iB(k)B^t(k)H_i^tK(k+1)]^{-1} + I_n\}^{-1} = H_iB(k)[I_n + B^t(k)H_i^tK(k+1)H_iB(k)]^{-1}B^t(k)H_i^tK(k+1)$. Therefore, (17) becomes

$$\begin{aligned}K(k) &= L(k) + A^t(k)H_i^tK(k+1)\\ &\qquad\times\left\{\left[H_iB(k)B^t(k)H_i^tK(k+1)\right]^{-1} + I_n\right\}^{-1}H_iA(k)\\ &= L(k) + A^t(k)H_i^tK(k+1)\left[I_n\right.\\ &\qquad\left. + H_iB(k)B^t(k)H_i^tK(k+1)\right]^{-1}H_iA(k).\end{aligned}$$

Moreover, substituting $P(k+1) = K(k+1)X^*(k+1)$ and $X^*(k+1)$ in (18) into (9), we obtain $R^*(k)$ in (19) and then $u^*(k)$ in (20). This completes the proof. □

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