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Domination in distance-hereditary graphs☆

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Abstract

The domination problem and its variants have been extensively studied in the literature. In this paper we investigate the domination problem in distance-hereditary graphs. In particular, we give a linear-time algorithm for the domination problem in distance-hereditary graphs by a labeling approach. We actually solve a more general problem, called the L-domination problem, which also includes the total domination problem as a special case. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

All graphs in this paper are finite, undirected, without loops and multiple edges. The distance $d_G(u,v)$ between two vertices u and v of a connected graph G is the minimum length of a u–v path in G. A graph is *distance-hereditary* if each pair of vertices are equidistant in every connected induced subgraph containing them. Properties of and optimization problems in distance-hereditary graphs have been extensively studied during the past two decades. In this paper, we study the domination problem in distance-hereditary graphs. We actually solve a more general problem, called the L-domination problem, which also includes the total domination as a special case.

The concept of domination can be used to model many location problems in operations research. In a graph $G=(V,E)$, a *dominating set* is a subset D of V such that every vertex

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in $V - D$ is adjacent to at least one vertex in D. A dominating set D is *independent*, *connected*, *total* if the subgraph $G[D]$ induced by D has no edges, is connected, and has no isolated vertices. The *domination problem* is one of finding a minimum-sized dominating set of a given graph. The *independent* (*connected*, *total*) *domination problem* is one of finding a minimum-sized independent (connected, total) dominating set of a given graph. The domination problem and its variants have been extensively studied in the literature. For more information, readers are referred to [4,9–11].

It is well-known that the decision version of the domination problem and many of its variants are NP-complete for general graphs. Polynomial-time algorithms exist for the connected domination problem [6], the weighted connected domination problem [19], the connected r-domination problem [2], the domination clique problem [7], and the weighted k-domination and the weighted k-dominating clique problems [20] in distance-hereditary graphs. The main purpose of this paper is to solve the domination problem for distance-hereditary graphs in linear time. We employ a labeling method which is powerful and widely used in many domination problems, see [3,5,12,14,15,17,18]. Our labeling method is mainly for the domination problem although the total domination is a by product. This coincides with the result by Kratsch and Stewart [13] that total domination can be reduced to domination for graph classes closed under adding false twins. We note that Nicolai and Szymczak [16] independently gave a linear-time algorithm for the domination problem in distance-hereditary graphs. Their method in fact solves the r-domination problem on homogeneous graphs, a supper class of distance-hereditary graphs, in $O(|V||E|)$ time.

In Section 2, we define notation and introduce a characterization of distance-hereditary graphs that is useful in our algorithm. In Section 3, we introduce a more general concept, L-domination, which includes domination and total domination as special cases. Section 4 gives reduction lemmas from which a linear-time algorithm for the L-domination problem in distance-hereditary graphs is established. Concluding remarks are made in Section 5.

2. Preliminaries

Suppose $G = (V, E)$ is a graph. For any vertex $v \in V$, the *neighborhood* of v is

$$N_G(v) = \{u \in V\colon uv \in E\}$$

and the *closed neighborhood* of v is $N_G[v] = N_G(v) \cup \{v\}$. For a subset X of V, the *neighborhood* of X is $N_G(X) = \bigcup_{v \in X} N_G(v)$ and the *closed neighborhood* of X is $N_G[X] = \bigcup_{v \in X} N_G[v]$. We use $N(v)$ for $N_G(v)$, $N[v]$ for $N_G[v]$, $N(X)$ for $N_G(X)$, and $N[X]$ for $N_G[X]$ if there is no ambiguity.

Let $G[X]$ denote the subgraph of G induced by $X \subseteq V$ and $G - x$ denote the subgraph induced by $V - \{x\}$ in G. The *degree* of a vertex v is $\deg(v) = |N(v)|$. An *isolated vertex* is a vertex of degree zero and a *leaf* is a vertex of degree one. Two vertices u and v are called *false twins* if $N(u) = N(v)$ and *true twins* if $N[u] = N[v]$. Note that, by the definition, true twins are adjacent and false twins are not.

An ordering $v_1, v_2, \ldots, v_n$ of V is called a *one-vertex-extension ordering* of G if v_i is a vertex of degree at most one or is a twin of some vertex v_j in $G_i = G[V_i]$ for $1 \leqslant i \leqslant n$, where $V_i = \{v_1, v_2, \ldots, v_i\}$. For convenience, we denote $N_{G_i}(x)$ as $N_i(x)$ and $N_{G_i}[x]$ as $N_i[x]$ for all $x \in V_i$.

Theorem 1 (Bandelt and Mulder [1] and Hammer and Maffray [8]). *A graph is distance-hereditary if and only if it has a one-vertex-extension ordering.*

A one-vertex-extension ordering of a distance-hereditary graph can be generated in $O(n + m)$ time, where n is the number of vertices and m is the number of edges (see [8]). In the following section, we assume a one-vertex-extension ordering of a given distance-hereditary graph has been constructed.

For our convenience, if v_i and v_j are true twins in G_i with $N_i[v_i] = N_i[v_j] = \{v_i, v_j\}$, we shall view v_i as a leaf adjacent to v_j in G_i. Our algorithm needs the following lemma which is easily seen.

Lemma 2. *Suppose $v_1, v_2, \ldots, v_n$ is a one-vertex-extension ordering of G. If v_i and v_j are twins in G_i, then $v_1, \ldots, v_{j-1}, v_i, v_{j+1}, \ldots, v_{i-1}, v_j, v_{i+1}, \ldots, v_n$ is also a one-vertex-extension of G.*

3. *L*-domination and basic assumptions

In order to solve the domination problem for distance-hereditary graphs, we use a labeling method to design an algorithm in which a one-vertex-extension ordering is used. At each iteration, the algorithm decides if a vertex v_i is to be included in a minimum solution according to its label. It then deletes v_i and updates the label of its twin or its unique neighbor. During the iterations, some information of deleted vertices are still kept in the remaining graph. For this purpose, we introduce the following general concept called L-domination.

Suppose $G = (V, E)$ is a graph whose vertex set V is a subset of a *ground set* V' which is a supper set of V. Furthermore, each vertex v in G is associated with a *regular label* $L(v) = S(v)$ or a *conditional label* $L(v) = S(v) * p(v)$, where $S(v) \subseteq \{\{0\}, \{1\}, \{0, 1\}\}$ and $p(v) \in V' - V$ with the property that $p(x) \neq p(y)$ for any two distinct vertices x and y. An *L-dominating set* of G is a subset $D \subseteq V'$ such that (LD) holds when v has a regular label $L(v) = S(v)$, and (LD) holds or $p(v) \in D$ when v has a conditional label $L(v) = S(v) * p(v)$.

(LD) For each $A \in S(v)$, there exists some $x \in D$ such that $d_G(v, x) \in A$.

In other words, if $L(v) = \{\{0\}\}$, then $v \in D$; if $L(v) = \{\{1\}\}$, then v has a neighbor in D; if $L(v) = \{\{0, 1\}\}$, then either $v \in D$ or v has a neighbor in D; and if $L(v) = \{\{0, 1\}\} * p(v)$, then $p(v) \in D$ or $v \in D$ or v has a neighbor in D; $\ldots$, etc. When $V = V'$ and each vertex has a regular label $\{\{0, 1\}\}$ (respectively, $\{\{1\}\}$), each vertex either is in D or has a neighbor in D (respectively, has a neighbor in D) and so L-domination is just the usual domination (respectively, total domination). The *L-domination number* $\gamma_L(G)$ is the minimum size of an L-dominating set of G. The *L-domination problem* is to determine $\gamma_L(G)$ of a graph G.

When $L(v)$ is a regular (conditional) label, L is (is not) the same as $S(v)$ and so is (is not) considered as a set. In this paper, we always use "$A \in L(v)$" as an abbreviation for "$L(v)$ is a regular label and $A \in L(v)$" and "$L(v) \subseteq S$" for "$L(v)$ is a regular label and $L(v) \subseteq S$".

Lemma 3. *Suppose v is a vertex such that $\{A,B\} \subseteq S(v)$, where A is a proper subset of B. If L' is obtained from L by replacing $S(v)$ with $S(v) - \{B\}$, then L-domination is the same as L'-domination.*

Proof. The lemma follows from the fact that (LD) holds for $A \in S(v)$ implies that (LD) holds for $B \in S(v)$. □

By Lemma 3, we only need consider $\emptyset$, $\{\{0\}\}$, $\{\{1\}\}$, $\{\{0,1\}\}$, $\{\{0\}, \{1\}\}$ for $S(v)$.

Lemma 4. *Suppose vertex v has a conditional label $S(v) * p(v)$ but that neither $S(v) = \{\{1\}\}$ with $N(v) = \emptyset$ nor $S(v) = \{\{0\},\{1\}\}$. If L' is obtained from L by relabeling v with the regular label $S(v)$, then $\gamma_L(G) = \gamma_{L'}(G)$.*

Proof. Any L'-dominating set of G is clearly an L-dominating set of G. So, $\gamma_L(G) \leqslant \gamma_{L'}(G)$. Conversely, suppose D is an L-dominating set of G. If $p(v) \notin D$, then D is also an L'-dominating set of G. If $p(v) \in D$, then $D' = (D - \{p(v)\}) \cup \{v\}$ (when $\{1\} \notin S(v)$) or $D' = (D - \{p(v)\}) \cup \{v'\}$ (when $S(v) = \{\{1\}\}$ and $v' \in N(v) \neq \emptyset$) is an L'-dominating set of G such that $|D'| \leqslant |D|$. So, $\gamma_{L'}(G) \leqslant \gamma_L(G)$. The lemma then follows. □

According to Lemma 4, the only possible conditional labels are $\{\{1\}\} * p(v)$ with $N(v) = \emptyset$ and $\{\{0\},\{1\}\} * p(v)$. We shall assume that originally the graph has no isolated vertex and so has no conditional label $\{\{1\}\} * p(v)$. Our lemmas in this paper will ensure that there is no $\{\{1\}\} * p(v)$ to be created. Hence we in fact assume that the only conditional label is $\{\{0\},\{1\}\} * p(v)$. Thus, for each vertex v, we may assume that

$$L(v) \text{ is one of the following}: \emptyset,\ \{\{0\}\},\ \{\{1\}\},\ \{\{0,1\}\},\ \{\{0\},\{1\}\},\ \text{and } \{\{0\},\{1\}\} * p(v).$$

4. The algorithm

Now, we are ready to develop reduction lemmas as the basis of the algorithm for the L-domination problem in distance-hereditary graphs. Lemmas 5–11 are also valid for a general graph G.

Lemma 5. *Suppose* $L(x)=\{\{0\},\{1\}\}*p(x)$ *and* $L(y)=\{\{0\},\{1\}\}*p(y)$ *for an edge* xy. *If* L' *is obtained from* L *by relabeling* x *and* y *with* $\{\{0\}\}$, *then* $\gamma_L(G)=\gamma_{L'}(G)$.

Proof. Any L'-dominating set of G is clearly an L-dominating of G. So, $\gamma_L(G)\leqslant\gamma_{L'}(G)$. Conversely, suppose D is an L-dominating set of G. By the definition of L-domination, either $x\in D$ or $p(x)\in D$; also, either $y\in D$ or $p(y)\in D$. In any case, $D'=(D-\{p(x),p(y)\})\cup\{x,y\}$ is an L'-dominating set of G such that $|D'|\leqslant|D|$. So, $\gamma_{L'}(G)\leqslant\gamma_L(G)$. The lemma then follows. □

Throughout this paper, by deleting/adding sets A from/to a regular (conditional) label $S(v)$ $(S(v)*p(v))$ we mean relabeling v with the regular label $S(v)-\{A\}/S(v)\cup\{A\}$. After adding A to $S(v)$, by Lemma 3, we can then delete any proper superset B of A from $S(v)\cup\{A\}$. Note that when A is added to a conditional label $\{\{0\},\{1\}\}*p(x)$, the resulting label is always the regular label $\{\{0\},\{1\}\}$.

Lemma 6. *Suppose* xy *is an edge and* x *has a regular label* $L(x)$ *containing* $\{0\}$. *If* L' *is obtained from* L *by deleting* $\{1\}$ *and* $\{0,1\}$ *from* $L(y)$, *then* $\gamma_L(G)=\gamma_{L'}(G)$.

Proof. Since $\{0\}\in L(x)=L'(x)$, $x\in D$ for any L-dominating or L'-dominating set D of G. The condition (LD) for vertex y and set $A=\{1\}$ or $\{0,1\}$ is then redundant. So, any L'-dominating set of G is also an L-dominating set of G. Thus, $\gamma_L(G)\leqslant\gamma_{L'}(G)$. Conversely, suppose D is an L-dominating set of G. If $p(y)\notin D$, then D is also an L'-dominating set of G. If $p(y)\in D$, then $D'=(D-\{p(y)\})\cup\{y\}$ is an L'-dominating set of G such that $|D'|\leqslant|D|$. So, $\gamma_{L'}(G)\leqslant\gamma_L(G)$. The lemma then follows. □

Lemma 7. *If* $L(x)=\{\{0\}\}$ *and* $L(z)$ *is a regular label containing neither* $\{1\}$ *nor* $\{0,1\}$ *for all* $z\in N(x)$, *then* $\gamma_L(G)=\gamma_L(G-x)+1$.

Proof. If D is an L-dominating set of $G-x$, then $D\cup\{x\}$ is clearly an L-dominating set of G. So, $\gamma_L(G)\leqslant\gamma_L(G-x)+1$. Conversely, suppose D is an L-dominating set of G. Since $\{0\}\in L(x)$, we have $x\in D$. Also, since $L(z)$ is a regular label containing neither $\{1\}$ nor $\{0,1\}$ for all $z\in N(x)$, $D'=D-\{x\}$ is an L'-dominating set of $G-x$ such that $|D'|\leqslant|D|-1$. So, $\gamma_L(G-x)\leqslant\gamma_L(G)-1$. The lemma then follows. □

Lemma 8. *If* x *and* y *are two distinct vertices such that* $N(x)\subseteq N(y)$ *and* $L(x)=\emptyset$, *then* $\gamma_L(G)=\gamma_L(G-x)$.

Proof. Since $L(x)=\emptyset$, any L-dominating set of $G-x$ is an L-dominating set of G. So, $\gamma_L(G)\leqslant\gamma_L(G-x)$. Conversely, suppose D is an L-dominating set of G. If $x\notin D$, then D is also an L-dominating set of $G-x$. If $x\in D$, since $N(x)\subseteq N(y)$ and $L(x)=\emptyset$, $D'=(D-\{x\})\cup\{y\}$ is an L-dominating set of $G-x$ such that $|D'|\leqslant|D|$. So, $\gamma_L(G-x)\leqslant\gamma_L(G)$. The lemma then follows. □

Lemma 9. *Suppose x is a leaf adjacent to y.*

1. *Suppose $L(x)=\{\{0,1\}\}$ and $L(y)=\{\{1\}\}$. If L' is obtained from L by relabeling y with $\{\{0\},\{1\}\}*x$, then $\gamma_L(G)=\gamma_{L'}(G-x)$.*
2. *Suppose $\{1\}\in L(x)$ or $L(x)=\{\{0,1\}\}$* with *$L(y)\neq\{\{1\}\}$. If L' is obtained from L by adding $\{0\}$* to *$L(y)$, then $\gamma_L(G)=\gamma_{L'}(G)$.*
3. *If $L(x)=\{\{0\},\{1\}\}*p(x)$ and $L(y)\subseteq\{\{0,1\}\}$, then $\gamma_L(G)=\gamma_L(G-x)+1$.*
4. *Suppose $L(x)=\{\{0\},\{1\}\}*p(x)$ and $L(y)=\{\{1\}\}$. If L' is obtained from L by relabeling y with $\{\{0,1\}\}$, then $\gamma_L(G)=\gamma_{L'}(G-x)+1$.*

Proof. (1) For any L'-dominating set D' of $G-x$, either $x=p(y)\in D'$ or $\{y,z\}\subseteq D'$ for some $z\in N(y)-\{x\}$. In either case, D' is an L-dominating set of G. So, $\gamma_L(G)\leqslant\gamma_{L'}(G-x)$. Conversely, suppose D is an L-dominating set of G. Also, either $x\in D$ or $\{y,z\}\subseteq D$ for some $z\in N(y)-\{x\}$. Hence D is an L'-dominating set of $G-x$. So, $\gamma_{L'}(G-x)\leqslant\gamma_L(G)$. (1) of the lemma then follows.

(2) Any L'-dominating set D' of G is also an L-dominating set of G, since we only add $\{0\}$ to $L(y)$. So, $\gamma_L(G)\leqslant\gamma_{L'}(G)$. Conversely, suppose D is an L-dominating set of G. Consider the following two cases.

Case 1. $y\in D$. It is easy to see that D is also an L'-dominating set of D if $L(y)$ is a regular label or $L(y)$ is a conditional label with D containing a neighbor of y. Suppose $L(y)$ is a conditional label but D does not contain a neighbor of y. By definition, $p(y)\in D$. Hence, $D'=(D-\{p(y)\})\cup\{x\}$ is an L'-dominating set of G such that $|D'|\leqslant|D|$.

Case 2. $y\notin D$. In this case, we have that $\{1\}\notin L(x)$ and then $L(x)=\{\{0,1\}\}$. By the assumption, $L(y)\neq\{\{1\}\}$. Thus, $L(y)=\emptyset$ or $\{\{0,1\}\}$ or $\{\{0\},\{1\}\}$ or $\{\{0\},\{1\}\}*p(y)$. Since x is only adjacent to y, $x\in D$ and so $D'=(D-\{x\})\cup\{y\}$ is an L'-dominating set of G such that $|D'|\leqslant|D|$.

In any case, $\gamma_{L'}(G)\leqslant\gamma_L(G)$. (2) of the lemma then follows.

(3) For any L-dominating set D' of $G-x$, $D'\cup\{p(x)\}$ is clearly an L-dominating set of G. So, $\gamma_L(G)\leqslant\gamma_L(G-x)+1$. Conversely, suppose D is an L-dominating set of G. There are three possible cases: $p(x)\in D$ but $x\notin D$, $p(x)\in D$ and $x\in D$, $p(x)\notin D$ but $\{x,y\}\subseteq D$. For these cases, $D'=D-\{p(x)\}$, $D'=(D-\{p(x),x\})\cup\{y\}$, $D'=D-\{x\}$, respectively, are L'-dominating sets of $G-x$ such that $|D'|\leqslant|D|-1$. So, $\gamma_L(G-x)\leqslant\gamma_L(G)-1$. (3) of the lemma then follows.

(4) Suppose D' is an L'-dominating set of $G-x$. If $y\notin D'$, then $z\in D'$ for some $z\in N(y)-\{x\}$ and so $D'\cup\{p(x)\}$ is an L-dominating set of G. If $y\in D'$, then $D'\cup\{x\}$ is clearly an L-dominating set of G. So, $\gamma_L(G)\leqslant\gamma_{L'}(G-x)+1$. Conversely, suppose D is an L-dominating set of G. There are three possible cases: $p(x)\in D$ but $x\notin D$, $p(x)\in D$ and $x\in D$, $p(x)\notin D$ but $\{x,y\}\subseteq D$. For these cases, $D'=D-\{p(x)\}$, $D'=(D-\{p(x),x\})\cup\{y\}$, $D'=D-\{x\}$, respectively, are L'-dominating sets of $G-x$ such that $|D'|\leqslant|D|-1$. So, $\gamma_{L'}(G-x)\leqslant\gamma_L(G)-1$. (3) of the lemma then follows. □

Lemma 10. *Suppose x and y are false twins such that $N(x)=N(y)\neq\emptyset$.*

(1) *If* $L(x)=\{\{0\},\{1\}\}*p(x)$ *and* $L(y)=\{\{0\},\{1\}\}*p(y)$, *then* $\gamma_L(G)=\gamma_L(G-x)+1$.
(2) *Suppose* $L(x)$ *is a regular label containing* $\{1\}$ *or* $\{0,1\}$ *and* $L(y)\neq\emptyset$. *If* L' *is obtained from* L *by deleting* $\{1\}$ *and* $\{0,1\}$ *from* $L(x)$ *and adding* $\{1\}$ *to* $L(y)$, *then* $\gamma_L(G)=\gamma_{L'}(G)$.

Proof. (1) For any L-dominating set D' of $G-x$, $D'\cup\{p(x)\}$ is clearly an L-dominating set of G. So, $\gamma_L(G)\leqslant\gamma_L(G-x)+1$. Conversely, suppose D is an L-dominating set of G. If $x\notin D$, then $p(x)\in D$ and so $D'=D-\{p(x)\}$ is an L-dominating set of $G-x$ such that $|D'|=|D|-1$. If $y\notin D$, then $p(y)\in D$ and so $D'=D-\{p(y)\}$ is an L-dominating set of $G-y$ such that $|D'|=|D|-1$. Since x and y are false twins, if either $x\notin D$ or $y\notin D$, then we can interchange the roles of x and y to say that $G-x$ has an L-dominating set D'' of size $|D''|=|D|-1$. So, we may assume $x\in D$ and $y\in D$. In this case, $D'=D-\{x,p(x)\}$ is an L-dominating set of $G-x$ such that $|D'|\leqslant|D|-1$. So, $\gamma_L(G-x)\leqslant\gamma_L(G)-1$. (1) of the lemma then follows.

(2) For any L'-dominating set D' of G, since $L'(y)$ is a regular label containing $\{1\}$, $z\in D'$ for some $z\in N(y)=N(x)$ and so D' is an L-dominating set of G. So, $\gamma_L(G)\leqslant\gamma_{L'}(G)$. Conversely, suppose D is an L-dominating set of G. Consider the following two cases.

Case 1. $D\cap N(x)=D\cap N(y)\neq\emptyset$. It is easy to verify that D is also an L'-dominating set of G if $L(y)$ is a regular label or $L(y)$ is a conditional label with $y\in D$. Suppose $L(y)$ is a conditional label and $y\notin D$. Then $p(y)\in D$ and hence $D'=(D-\{p(y)\})\cup\{y\}$ is an L'-dominating set of G such that $|D'|=|D|$.

Case 2. $D\cap N(x)=D\cap N(y)=\emptyset$. In this case, $\{1\}\notin L(x)$ and so $L(x)=\{\{0,1\}\}$ and $x\in D$. Similarly, $L(y)=\{\{0,1\}\}$ or $\{\{0\}\}$ or $\{\{0\},\{1\}\}$ or $\{\{0\},\{1\}\}*p(y)$, and so $y\in D$ or $p(y)\in D$. Choose a vertex $z\in N(x)=N(y)$. Then $D'=(D-\{x\})\cup\{z\}$ or $D'=(D-\{x,p(y)\})\cup\{z,y\}$ is an L'-dominating set of G such that $|D'|\leqslant|D|$.

In any case, $\gamma_{L'}(G)\leqslant\gamma_L(G)$ (2) of the lemma then follows. □

Lemma 11. *Suppose* x *and* y *are true twins such that* $N[x]=N[y]\neq\{x,y\}$.
(1) *If* $L(x)=\emptyset$ *and* $L(y)\neq\{\{1\}\}$, *then* $\gamma_L(G)=\gamma_L(G-x)$.
(2) *Suppose* $L(x)=\{\{1\}\}$ *and* $L(y)=\emptyset$. *If* L' *is obtained from* L *by relabeling* y *with* $\{\{0,1\}\}$, *then* $\gamma_L(G)=\gamma_{L'}(G-x)$.
(3) *Suppose* $L(x)=\{\{0,1\}\}$ *or* $\{\{1\}\}$ *and* $L(y)=\{\{0\},\{1\}\}*p(y)$. *If* L' *is obtained from* L *by relabeling* y *with* $\{\{0\},\{1\}\}$, *then* $\gamma_L(G)=\gamma_{L'}(G-x)$.
(4) *If* $L(x)$ *and* $L(y)$ *are both* $\{\{0,1\}\}$ *or* $\{\{1\}\}$ *except that* $L(x)=\{\{0,1\}\}$ *and* $L(y)=\{\{1\}\}$, *then* $\gamma_L(G)=\gamma_L(G-x)$.

Proof. (1) Since $L(x)=\emptyset$, any L-dominating set of $G-x$ is an L-dominating set of G. So, $\gamma_L(G)\leqslant\gamma_L(G-x)$. Conversely, suppose D is an L-dominating set of G. If $x\notin D$, then D is also an L-dominating set of $G-x$. Now suppose $x\in D$. If $y\in D$, then $D'=(D-\{x\})\cup\{z\}$ is an L-dominating set of $G-x$ such that $|D'|\leqslant|D|$, where $z\in N(y)-\{x\}$. Suppose $y\notin D$. Then, either $L(y)$ is a conditional label with $p(y)\in D$ or $L(y)$ is a regular label with $\{0\}\notin L(y)$. By assumption, $\{1\}\notin L(y)$ if $L(y)$ is a regular label. Thus $D' = (D - \{x\}) \cup \{y\}$ is an L-dominating set of $G - x$ such that $|D'| \leqslant |D|$. So, $\gamma_L(G - x) \leqslant \gamma_L(G)$. (1) of the lemma then follows.

(2) For any L'-dominating set D' of $G-x$, since $L'(y) = \{\{0,1\}\}$, we have $D' \cap N(x) = D' \cap (N[y] - \{x\}) \neq \emptyset$ and so D' is also an L-dominating set of G. So, $\gamma_L(G) \leqslant \gamma_{L'}(G-x)$. Conversely, suppose D is an L-dominating set of G. Since $L(x) = \{\{1\}\}$, we have $D \cap N(x) \neq \emptyset$. Then $D' = D$ (when $x \notin D$) or $D' = (D - \{x\}) \cup \{y\}$ (when $x \in D$) is an L'-dominating set of $G - x$ with $|D'| \leqslant |D|$. So, $\gamma_{L'}(G - x) \leqslant \gamma_L(G)$. (2) of the lemma then follows.

(3) For any L'-dominating set D' of $G - x$, since $\{0\} \in L'(y)$, $y \in D'$ and so D' is an L-dominating set of G. So, $\gamma_L(G) \leqslant \gamma_{L'}(G - x)$. Conversely, suppose D is an L-dominating set of G. Then either $z_1 = p(y) \in D$ with some $z_2 \in D \cap N[x]$, or $p(y) \notin D$ with some $z_1 = y$, $z_2 \in D \cap N(y)$. In any case, $D' = (D - \{z_1, z_2\}) \cup \{y, z\}$ is an L'-dominating set of $G - x$ such that $|D'| \leqslant |D|$ for some $z \in N(y) - \{x\}$. So, $\gamma_{L'}(G - x) \leqslant \gamma_L(G)$. (3) of the lemma then follows.

(4) For any L-dominating set D' of $G - x$, since $L(y) = \{\{0,1\}\}$ or $\{\{1\}\}$, D' contains some vertex in $N[y] - \{x\}$ and so D' is also an L-dominating set of G. So, $\gamma_L(G) \leqslant \gamma_L(G - x)$. Conversely, suppose D is an L-dominating set of G. If $x \notin D$, then D is also an L-dominating set of $G - x$. Suppose $x \in D$. Consider the following three cases.

Case 1. $y \in D$. In this case $D' = (D - \{x\}) \cup \{z\}$ is an L-dominating set of $G - x$ with $|D'| \leqslant |D|$ for some $z \in N(y) - \{x\}$.

Case 2. $y \notin D$ and $D \cap (N(y) - \{x\}) \neq \emptyset$. In this case, $D' = (D - \{x\}) \cup \{y\}$ is an L-dominating set of $G - x$ with $|D'| \leqslant |D|$.

Case 3. $D \cap (N[y] - \{x\}) = \emptyset$. In this case, $L(x) = \{\{0,1\}\}$ and $L(y) = \{\{0,1\}\}$. Thus, $D' = (D - \{x\}) \cup \{y\}$ is an L-dominating set of $G - x$ with $|D'| \leqslant |D|$.

In any case, $\gamma_L(G - x) \leqslant \gamma_L(G)$. (4) of the lemma then follows. $\square$

Based on these lemmas, we present the following algorithm for the L-domination problem in distance-hereditary graphs.

Algorithm LD-dh. Find the L-domination number of a distance-hereditary graph.
Input. A distance-hereditary graph $G = (V, E)$ with label $L(v)$ for all $v \in V$ and a one-vertex-extension ordering $v_1, v_2, \ldots, v_n$.
Output. $\gamma_L(G)$.

Method.

$\gamma \leftarrow 0$; /$*$ as a short notation for $\gamma_L(G)$ $*$/
update labels according to Lemmas 3 and 4;
for $i = n$ **to** 1 **step** -1 **do**
{ $x = v_i$;
Case 1. x is an isolated vertex in G_i, i.e., $N_i(x) = \emptyset$.

if $\{1\} \in L(x)$ **then STOP** since there is no feasible solution **else** $\gamma \leftarrow \gamma + 1$;
Case 2. x is a leaf adjacent to y in G_i.

2.1 **if** $N[x] = N[y] = \{x, y\}$ and $L(x) = \emptyset$ **then** interchange x and y by applying Lemma 2;
2.2 **if** $L(x)$ and $L(y)$ are conditional labels **then** $L(x) \leftarrow L(y) \leftarrow \{\{0\}\}$;
2.3 **if** $\{1\} \in L(x)$ or $L(x) = \{\{0, 1\}\}$ with $L(y) \neq \{\{1\}\}$ **then** add $\{0\}$ to $L(y)$;
2.4 **if** $L(y)$ contains $\{0\}$ **then** delete $\{1\}$ and $\{0, 1\}$ from $L(x)$;
2.5 **if** $N[x] = N[y] = \{x, y\}$ and $L(x) = \emptyset$ **then** interchange x and y by applying Lemma 2;
2.6 **if** $L(x) = \{\{0\}\}$ **then** delete $\{1\}$ and $\{0, 1\}$ from $L(y)$ and $\gamma \leftarrow \gamma + 1$;
2.7 **if** $L(x) = \{\{0, 1\}\}$ and $L(y) = \{\{1\}\}$ **then** relabel y with $\{\{0\}, \{1\}\} * x$;
2.8 **if** $L(x)$ is a conditional label and $L(y) \subseteq \{\{0, 1\}\}$ **then** $\gamma \leftarrow \gamma + 1$;
2.9 **if** $L(x)$ is a conditional label and $L(y) = \{\{1\}\}$ **then** {relabel y with $\{\{0, 1\}\}$; $\gamma \leftarrow \gamma + 1$; }

Case 3. x and y are false twins in G_i with $N_i(x) = N_i(y) \neq \emptyset$.
3.1 **if** $L(x)$ and $L(y)$ are conditional labels **then** $\gamma \leftarrow \gamma + 1$ **else**
3.2 {assume $L(x) = \emptyset$ or $L(x)$ is a regular label with $L(y) \neq \emptyset$ by interchanging x and y if necessary;
3.3 **if** $L(x)$ contains $\{1\}$ or $\{0, 1\}$ **then** delete $\{1\}$ and $\{0, 1\}$ from $L(x)$ and add $\{1\}$ to $L(y)$;
3.4 **if** $L(x) = \{\{0\}\}$ **then** delete $\{1\}$ and $\{0, 1\}$ from $L(z)$ for all $z \in N(x)$ and $\gamma \leftarrow \gamma + 1$;}

Case 4. x and y are true twins in G_i with $N_i[x] = N_i[y] \neq \{x, y\}$.
4.1 **if** $L(x)$ and $L(y)$ are conditional labels **then** $L(x) \leftarrow L(y) \leftarrow \{\{0\}\}$;
4.2 **if** $\{0\} \in L(x)$ **then** delete $\{1\}$ and $\{0, 1\}$ from $L(y)$;
if $\{0\} \in L(y)$ **then** delete $\{1\}$ and $\{0, 1\}$ from $L(x)$;
if $\{0\} \in L(x)$ **then** delete $\{1\}$ and $\{0, 1\}$ from $L(y)$;
4.3 assume one of the following cases holds by interchanging x and y if necessary;
Case 4.1. $L(x) = \{\{0\}\}$: delete $\{1\}$ and $\{0, 1\}$ from $L(z)$ for all $z \in N(x)$ and $\gamma \leftarrow \gamma + 1$;
Case 4.2. $L(x) = \emptyset$ and $L(y) \neq \{\{1\}\}$: do nothing;
Case 4.3. $L(x) = \{\{1\}\}$ and $L(y) = \emptyset$: relabel y with $\{\{0, 1\}\}$;
Case 4.4. $L(x) = \{\{0, 1\}\}$ or $\{\{1\}\}$ and $L(y) = \{\{0\}, \{1\}\} * p(y)$: relabel y with $\{\{0\}, \{1\}\}$;
Case 4.5. $L(x)$ and $L(y)$ are $\{\{0, 1\}\}$ or $\{\{1\}\}$ except $L(x) = \{\{0, 1\}\}$ and $L(y) = \{\{1\}\}$: do nothing;
}

Theorem 12. *Algorithm LD-dh solves the L-domination problem for distance-hereditary graphs in linear time.*

Proof. The algorithm certainly runs in linear time. To see that it solves the L-domination problem for distance-hereditary graphs, we need only show that it covers all possible cases for various values of $L(x)$ and $L(y)$ at every iteration by indicating the corresponding lemma(s) each line of the algorithm uses.

Case 1. x is an isolated vertex in G_i. The correctness follows from definition.

Case 2. x is a leaf adjacent to y in G_i. If $L(x)$ and $L(y)$ are conditional labels, by Lemma 5, we can change them to $\{\{0\}\}$ as in line 2.2. Now, either $L(x)$ or $L(y)$ is regular. If $\{1\} \in L(x)$ or $L(x) = \{\{0,1\}\}$ with $L(y) \neq \{\{1\}\}$, then Lemma 9 (2) is applied to add $\{0\}$ to $L(y)$ as in line 2.3. Then Lemma 6 is applied to remove $\{1\}$ and $\{0,1\}$ from $L(x)$ when $\{0\} \in L(y)$ as in line 2.4. Immediately after line 2.4 is executed, we have that $L(x)=\emptyset$, or $L(x)=\{\{0\}\}$, or $L(x)=\{\{0,1\}\}$ with $L(y)=\{\{1\}\}$, or $L(x)$ is a conditional label with $\{0\} \notin L(y)$. After the special treatments in lines 2.1 and 2.5, it is impossible that $N[x]=N[y]=\{x,y\}$ and $L(x)=\emptyset$ and $L(y) \neq \emptyset$. Therefore, after line 2.5, there are six possibilities: (1) $N[x]=N[y]=\{x,y\}$ and $L(x)=N(y)=\emptyset$, (2) $N(y)$ contains some $y' \neq x$ and $L(x)=\emptyset$, (3) $L(x)=\{\{0\}\}$, (4) $L(x)=\{\{0,1\}\}$ and $L(y)=\{\{1\}\}$, (5) $L(x)$ is a conditional label and $L(y) \subseteq \{\{0,1\}\}$, (6) $L(x)$ is a conditional label and $L(y)=\{\{1\}\}$. For the first case, by definition, we need to only delete x from G. For the second case, since $N(x) \subseteq N(y')$, we also delete x from G by applying Lemma 8. The other four cases are handled in lines 2.6 to 2.9 by using Lemmas 6, 7, 9 (1), 9 (3), 9 (4), respectively.

Case 3. x and y are false twins in G_i with $N_i(x)=N_i(y) \neq \emptyset$. By Lemma 10 (1), line 3.1 handles the case when $L(x)$ and $L(y)$ are conditional labels. By Lemma 2, we can interchange x and y, if necessary, to assume $L(x)=\emptyset$ or $L(x)$ is a regular label with $L(y) \neq \emptyset$, as shown in line 3.2. By Lemma 10 (2), we can remove $\{1\}$ and $\{0,1\}$ from $L(x)$, if necessary, as in line 3.3. Now, either $L(x)=\emptyset$ or $L(x)=\{\{0\}\}$. If $L(x)=\{\{0\}\}$, then Lemmas 6 and 7 are applied as in line 3.4. If $L(x)=\emptyset$, then Lemma 8 is applied and D remains unchanged.

Case 4. x and y are true twins in G_i with $N_i[x]=N_i[y] \neq \{x,y\}$. If $L(x)$ and $L(y)$ are conditional labels, by Lemma 5, we can change them to $\{\{0\}\}$ as in line 4.1. If $\{0\} \in L(x)$ or $\{0\} \in L(y)$, then we can apply Lemma 6 to delete $\{1\}$ and $\{0,1\}$ from $L(y)$ or $L(x)$, respectively, as in line 4.2. Lemma 6 is applied three times to handle the case of $L(x)=\{\{0\},\{1\}\} * p(x)$ and $\{0\} \in L(y)$. Then, by applying Lemma 2 to interchange x and y if necessary, there are five possibilities, which are handled as in Cases 4.1–4.5 by using Lemmas 7 and 11. □

5. Concluding remarks

The weighted connected domination problem is solvable in linear time for distance-hereditary graphs [19]. In this paper, we give a linear-time algorithm for the domination and the total domination problems in distance-hereditary graphs. On the other hand, the time complexity of the independent domination problem in distance-hereditary graphs is still unknown. It is desirable study the time complexity of the independent domination, the weighted domination, and the weighted total domination problems in distance-hereditary graphs.

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For further reading

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