Constructions of Multiblock Space–Time Coding Schemes That Achieve the Diversity–Multiplexing Tradeoff

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Abstract**—Constructions of multiblock space–time coding schemes that are optimal with respect to diversity–multiplexing (D-M) tradeoff when coding is applied over any number of fading blocks are presented in this correspondence. The constructions are based on a left-regular representation of elements in some cyclic division algebra. In particular, the main construction applies to the case when the quasi-static fading interval equals the number of transmit antennas, hence the resulting scheme is termed a minimal delay multiblock space–time coding scheme. Constructions corresponding to the cases of nonminimal delay are also provided. As the number of coded blocks approaches infinity, coding schemes derived from the proposed constructions can be used to provide a reliable multiple-input multiple-output (MIMO) communication with vanishing error probability.**

Index Terms**—Cyclic-division algebras, diversity–multiplexing (D-M) tradeoff, fading channels, multiblock space–time codes, multiple–input multiple–output (MIMO) channels, number fields, space–time codes.**

I. Introduction

By deploying multiple antennas at both transmitter and receiver ends, the multiple–input multiple–output (MIMO) technology can significantly increase the ergodic channel capacity as well as improve the link reliability. For example, in a MIMO communication system with n_t transmit and n_r receive antennas, under the quasi-static MIMO Rayleigh block fading channel model, it is known [1] that the ergodic MIMO channel capacity C equals

$$C = \min\{n_t, n_r\} \log_2 \mathrm{SNR} + O(1) \text{ bits/channel use} \tag{1}$$

at high signal-to-noise ratio (SNR) regime.

Coding schemes dedicated to the MIMO systems to achieve higher transmission rate and better link reliability are specifically coined *space–time codes* [2], [3]. Let T denote the quasi-static interval of the quasi-static MIMO Rayleigh block fading channel. An $(n_t \times mT)$ multiblock space–time code $\mathcal{X}$ is a collection of $(n_t \times mT)$ matrices and is used to code messages over m fading blocks, meaning mT channel uses in all. The code $\mathcal{X}$ transmits on average

$$R := \frac{1}{mT} \log_2 |\mathcal{X}| \tag{2}$$

bits per channel use. Let r denote the *normalized rate* of $\mathcal{X}$, also known as the *multiplexing gain* [4], given by

$$r := \frac{R}{\log_2 \mathrm{SNR}}. \tag{3}$$

From (1), it can be seen that to have a reliable MIMO communication, the maximum achievable multiplexing gain equals $\min\{n_t, n_r\}$. Given the multiblock code $\mathcal{X}$ with multiplexing gain r, we say $\mathcal{X}$ achieves *diversity gain* $d(r)$ if at high SNR regime, the codeword error probability of $\mathcal{X}$ is on the order of

$$P_e(r) \doteq \mathrm{SNR}^{-d(r)}. \tag{4}$$

By $\doteq$, we mean the exponential equality defined in [4]. We say the function $f(\mathrm{SNR}) \doteq \mathrm{SNR}^b$ if and only if

$$\lim_{\mathrm{SNR}\to\infty} \frac{\log f(\mathrm{SNR})}{\log \mathrm{SNR}} = b. \tag{5}$$

The notations of $\dot{\geq}$ and $\dot{\leq}$ are defined similarly.

In their ground-breaking paper, Zheng and Tse [4] showed that there exists a fundamental tradeoff between the multiplexing and the diversity gains, referred to as the *diversity–multiplexing* (D-M) tradeoff. For the cases when $T \geq n_t + n_r - 1$ and when the m consecutive fading blocks are statistically independent, the D-M tradeoff asserts that the maximum possible diversity gain $d^*(r)$ for any $(n_t \times mT)$ multiblock space–time coding schemes with multiplexing gain r is a piecewise linear function connecting the points $(k, d^*(k))$, $k = 0, 1, \cdots, \min\{n_t, n_r\}$, and

$$d^*(k) = m(n_t - k)(n_r - k). \tag{6}$$

On the other hand, if $T < n_t + n_r - 1$, only upper and lower bounds on $d^*(r)$ are available in [4].

This remarkable result has spurred a considerable amount of research activities on constructing space–time coding schemes to achieve this optimal tradeoff $d^*(r)$. Much progress has been made in the case of $m = 1$. When $n_t = n_r = T = 2$ and $m = 1$, several D-M optimal (2×2) schemes with tilted-QAM constellations can be found in [5]–[8]. These are the first instances showing that the D-M tradeoff (6) holds even in the case of $T < n_t + n_r - 1$. In particular, the "Golden code" proposed by Belfiore *et al.* [8] is shown to have the best error performance among the aforementioned (2×2) codes. By generalizing the Golden code construction and working on latices with unitary generating matrices, Oggier *et al.* provided in [9] the constructions of (3×3), (4×4), and (5×5) *perfect space–time codes*. El Gamal, Caire, and Damen [10] proposed a construction of $(n_t \times T)$ coding schemes, termed lattice space–time (LAST) codes that are obtained from a nested lattice randomly drawn from an ensemble of lattices having good covering properties. The LAST codes are shown to be D-M optimal for all $T \geq n_r + n_r - 1$ with $m = 1$. By extending an earlier work [11], Kiran and Rajan [12] proposed a construction of $(n_t \times n_t)$ space–time codes that is based on the left-regular representation of elements in a cyclic division algebra (CDA) as square matrices for the cases of $n_t = 2^n, 3 \cdot 2^n$, and 3^n for some positive integer n. The CDA-based codes are known to have a linear dispersion form [13], hence can be decoded using the sphere decoding technique [14]. A sufficient condition for codes to be D-M optimal as well as a general construction of $(n_t \times n_r)$ CDA-based codes satisfying this condition were discovered by Elia *et al.* [15], [16]. In addition, it was shown in [16] that the CDA-based codes are *approximately universal*, a criterion proposed by Tavildar and Viswanath [17]. In [18], Liao and Xia proposed a transformation technique to balance the mean powers at different transmit antennas and introduced a multilayer structure for CDA-based codes. It was shown that the resulting (3×3) code has better performance than the previous ones.

While all the aforementioned constructions are D-M optimal, we remark that none of them is capable of providing a reliable MIMO communication [19] due to the nonvanishing error probability. In other words, the error probability $P_e(r)$ achieved by the above schemes is

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bounded away from zero whenever SNR $< \infty$. This is due to that these constructions address only the case of $m = 1$, i.e., the coding is confined within one fading block, and independent fading blocks are coded independently. Therefore, from the D-M tradeoff (6), it follows that in order to achieve a reliable MIMO communication, coding must be applied over multiple fading blocks. In other words, at finite SNR regime, the vanishing error probability $P_e(r)$ can only be approached through lengthening the coding scheme so that $m \gg 1$. This coincides exactly with what we have learned from the conventional SISO communication [19]. Motivated by this, for any set of parameters, m, n_t, n_r, and T, in this correspondence, we will aim at providing explicit constructions of $(n_t \times mT)$ multiblock space–time codes with multiplexing gain r that are D-M optimal (6). Namely, these newly proposed codes will have

$$P_e(r) \doteq \text{SNR}^{-d^*(r)} \longrightarrow 0$$

as m approaches infinity at high SNR regime whenever the transmission rate R is set below the ergodic channel capacity C, or equivalently, the multiplexing gain $r \leq \min\{n_t, n_r\}$.

This correspondence is organized as follows. We will begin with the construction of the $(n_t \times mn_t)$ *minimal delay multiblock space–time coding schemes*[1] for the case $T = n_t$ and for all $m \geq 1$ in Section II. A design example will also be given for illustration. It will be proved in the Appendix that coding schemes derived from this construction are D-M optimal (6), hence are able to provide a reliable MIMO communication as $m \to \infty$. However, it is generally true that the MIMO communication channels are slowly varying and the assumption of independent fading blocks might not hold. In fact, the consecutive fading blocks are expected to be correlated in time, and the degrees of correlations strongly depend upon the conditions of communication environment, such as number of multipaths, Doppler spread, and carrier frequency. In view of this, we will provide in the Appendix a much stronger proof to show that this newly proposed construction is D-M optimal for all kinds of wireless communication channels, including the ones having time correlations, antenna correlations, and having different fading statistics.

In Section II-B, we will generalize the construction to provide nonminimal delay coding schemes when the quasi-static interval $T > n_t$. Furthermore, for the cases of $T \geq mn_t$, it will be seen that codes derived from this generalization might not be efficient in terms of signaling complexity, in the sense that when representing the code in its linear dispersion form [13], each entry of code matrix resulting from this construction is a large linear combination of many signal points drawn from the underlying constellation set. In view of this, a more efficient construction targeting at $T \geq mn_t$ that requires lesser linear combinations will be given in Section II-C. In Section IV, we conclude this correspondence.

II. Constructions of Multiblock Space-Time Codes

Consider a quasi-static MIMO Rayleigh block fading channel with n_t transmit and n_r receive antennas. Throughout this correspondence, we will assume for simplicity that $n_r \geq n_t$ while later in the Appendix it will be straightforward to see that the D-M optimality of the proposed constructions remains to hold even when the number of receiver antennas is less than the number of transmit. Let T be the quasi-static interval and let $\mathcal{X}$ be an $(n_t \times mT)$ multiblock space–time coding scheme that sends coded information over m consecutive, yet statistically independent fading blocks. We will first focus on the case when T equals n_t. The cases of $T > n_t$ will be dealt with later in Sections II-B and II-C.

[1]We were informed that this minimal delay construction was independently discovered by Yang and Belfiore [20], [21] for constructing distributed space–time codes in MIMO amplify-and-forward cooperative channels.

Assuming $X = (X_0, \ldots, X_{m-1}) \in \mathcal{X}$ is the code matrix chosen for transmission, the transmitter actually sends the $(n_t \times n_t)$ submatrix X_i at the ith fading block, $i = 0, 1, \ldots, m-1$. Thus, the received signal matrix corresponding to X_i at the receiver end is modeled as

$$Y_i = \theta H_i X_i + W_i \tag{7}$$

for $i = 0, 1, \ldots, m-1$, where H_i and W_i are, respectively, the $(n_r \times n_t)$ channel and $(n_r \times T)$ noise matrices. Entries of H_i and W_i are modeled as independent identically distributed (i.i.d.), zero mean, circularly symmetric, complex Gaussian random variables with unit variance $\mathbb{CN}(0, 1)$. The parameter θ is chosen to satisfy the following power constraint:

$$\mathbb{E} \sum_{i=0}^{m-1} \|\theta X_i\|_F^2 = m \cdot T \cdot \text{SNR} \tag{8}$$

where by $\|\cdot\|_F$ we mean the Frobenius norm of a matrix.

A. Minimal Delay Construction

Given the desired multiplexing gain r, we first identify the following QAM *base alphabet* [16][2] that is a subset of the Gaussian integer ring and is given by

$$\mathcal{A}(\text{SNR}) = \left\{ a + b\imath : -M + 1 \leq a, b \leq M - 1, a, b \text{ odd integers}, M = \text{SNR}^{\frac{r}{2n_t}} \right\} \tag{9}$$

where $\imath = \sqrt{-1}$.

Next, the construction calls for two number fields [22] L and K that are field extensions of $\mathbb{Q}(\imath)$. The number field L is a cyclic Galois extension of $\mathbb{Q}(\imath)$ with degree

$$n := [L : \mathbb{Q}(\imath)] = mn_t \tag{10}$$

and the number field K is a subfield of L with degree $[K : \mathbb{Q}(\imath)] = m$. We refer the readers to [15] and [16] for a systematic construction of such number field L.

Bearing with the above in mind, let σ be the generator of the Galois group Gal $(L/\mathbb{Q}(\imath))$ and it is clear that σ has order n. Since Gal $(L/\mathbb{Q}(\imath))$ is cyclic, the Galois group Gal (L/K) is also a cyclic group generated by $\zeta = \sigma^m$ whose order equals n_t. Therefore, L is cyclic Galois over K as well and $[L : K] = n_t$. The Galois group for K over $\mathbb{Q}(\imath)$ is the quotient group

$$\text{Gal}\,(K/\mathbb{Q}(\imath)) = \langle\sigma\rangle / \langle\zeta\rangle = \{\sigma^i : i = 0, \ldots, m-1\}. \tag{11}$$

It follows that $\mathfrak{D} = (L/K, \zeta, \gamma)$ is a CDA[3] for some nonnorm element $\gamma \in K^*$. Moreover, $\mathfrak{D}$ can be embedded in an $(n_t \times n_t)$ matrix algebra over L through the left-regular representations of elements in $\mathfrak{D}$ [12]. We have

$$\mathfrak{D} \cong \left\{ D = \begin{bmatrix} x_0 & \gamma\zeta(x_{n_t-1}) & \cdots & \gamma\zeta^{n_t-1}(x_1) \\ x_1 & \zeta(x_0) & \cdots & \gamma\zeta^{n_t-1}(x_2) \\ \vdots & \vdots & \ddots & \vdots \\ x_{n_t-1} & \zeta(x_{n_t-2}) & \cdots & \zeta^{n_t-1}(x_0) \end{bmatrix} : x_i \in L \right\} \tag{12}$$

and it is known [23] that every such $(n_t \times n_t)$ matrix D has determinant in K.

[2]For brevity, here we only provide the constructions of codes with QAM base alphabet. The ones for the HEX base alphabet [16] can be obtained in a similar fashion.

[3]For readers not familiar with the subject of CDA, we refer them to [12] and [11] for a nice introduction.

For simplicity, here we restrict ourselves to the case of $\gamma \in \mathbb{Z}[\imath]$ while, in general, the construction can be generalized to take unit modulus $\gamma \in \mathcal{O}_K$ to yield the multiblock version of perfect space–time codes [24], where $\mathcal{O}_K$ is the ring of algebraic integers in K.

Let $\tilde{\mathcal{X}}$ be an $(n_t \times n_t)$ space–time coding scheme

$$\tilde{\mathcal{X}} := \left\{ \begin{bmatrix} x_0 & \gamma\zeta(x_{n_t-1}) & \cdots & \gamma\zeta^{n_t-1}(x_1) \\ x_1 & \zeta(x_0) & \cdots & \gamma\zeta^{n_t-1}(x_2) \\ \vdots & \vdots & \ddots & \vdots \\ x_{n_t-1} & \zeta(x_{n_t-2}) & \cdots & \zeta^{n_t-1}(x_0) \end{bmatrix} : x_i = \sum_{j=1}^{n} a_{i,j} e_j, a_{i,j} \in \mathcal{A}(\text{SNR}) \right\} \quad (13)$$

where $\mathcal{B} := \{e_1, e_2, \ldots, e_n\}$ is an integral basis for L over $\mathbb{Q}(\imath)$. Then, the proposed $(n_t \times mn_t)$ multiblock space–time coding scheme $\mathcal{X}$ is given by

$$\mathcal{X} := \left\{ X = \left(\tilde{X}, \sigma(\tilde{X}), \ldots, \sigma^{m-1}(\tilde{X})\right) : \tilde{X} \in \tilde{\mathcal{X}} \right\}. \quad (14)$$

In other words, if $\tilde{X}$ was the code matrix chosen from $\tilde{\mathcal{X}}$ for transmission, then the transmitter actually sends $\sigma^i(\tilde{X})$ during the ith fading block, $i = 0, 1, \ldots, (m-1)$. One direct consequence of the above construction is the following.

Proposition 1: Let $\tilde{\mathcal{X}}$ and $\mathcal{X}$ be defined as above; then for every nonzero codeword $(X_0, \ldots, X_{m-1}) \in \mathcal{X}$, we have

$$\left| \prod_{i=0}^{m-1} \det(X_i) \right| \geq 1. \quad (15)$$

Proof: First, note that $X_i = \sigma^i(\tilde{X})$ for some nonzero $\tilde{X} \in \tilde{\mathcal{X}}$. As the nonnorm element γ lies in $\mathbb{Z}[\imath]$, the entries of $\tilde{X}$ are in $\mathcal{O}_L$, the ring of algebraic integers in L, i.e., the integral closure of $\mathbb{Z}$ in L. It then follows from [16] and [23] that $0 \neq \det(\tilde{X}) \in \mathcal{O}_K$, where $\mathcal{O}_K$ is the ring of algebraic integers in K. Now the proof is complete after noting

$$\begin{aligned} \prod_{i=0}^{m-1} \det(X_i) &= \prod_{i=0}^{m-1} \det(\sigma^i(\tilde{X})) \\ &= \prod_{i=0}^{m-1} \sigma^i(\det(\tilde{X})) \\ &= N_{K/\mathbb{Q}(\imath)}\left(\det(\tilde{X})\right) \in \mathbb{Z}[\imath] \end{aligned}$$

where $N_{K/\mathbb{Q}(\imath)}(a)$ denotes the algebraic norm of a from K to $\mathbb{Q}(\imath)$. $\square$

The above property should be regarded as the *generalized nonvanishing determinant* property. To see this, setting $m = 1$ in Proposition 1 yields $|\det(X)| \geq 1$ for every nonzero code matrix $X \in \mathcal{X}$, and we recover the nonvanishing determinant criterion stated in [8], [12], and [16]. Furthermore, as

$$|\mathcal{X}| = |\tilde{\mathcal{X}}| = |\mathcal{A}(\text{SNR})|^{n_t n} = \text{SNR}^{rn}$$

the $(n_t \times mn_t)$ multiblock space–time coding scheme $\mathcal{X}$ achieves transmission rate $R = r \log_2 \text{SNR}$ bits per channel use and has full rate in terms of the size of $\mathcal{A}(\text{SNR})$. To ensure that $\mathcal{X}$ satisfies the power constraint (8), the parameter θ should be set at

$$\theta^2 \doteq \text{SNR}^{1-\frac{r}{n_t}} \quad (16)$$

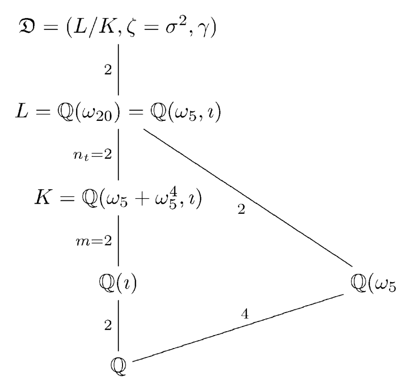
due to $\sum_{i=0}^{m-1} \mathbb{E}\|\theta X_i\|_F^2 \doteq \theta^2 \text{SNR}^{\frac{r}{n_t}}$.

All in all, the proposed coding scheme $\mathcal{X}$ is full-rate, has a signal constellation that is a linear combination of points in $\mathcal{A}(\text{SNR})$, and satisfies the "generalized" nonvanishing determinant property. The following theorem shows that $\mathcal{X}$ is in fact optimal with respect to the D-M tradeoff. The proof to this theorem is relegated to the Appendix. Furthermore, by adopting techniques from [20], it will be shown that the proposed construction satisfies the approximately universal property, meaning that this code is D-M optimal for any kinds of fading distributions, including the time-correlated channels.

Theorem 2: Let $\mathcal{X}$ be the $(n_t \times mn_t)$ multiblock space–time coding scheme defined as in (14); then, $\mathcal{X}$ is optimal with respect to the D-M tradeoff (6), i.e., it achieves simultaneously multiplexing gain r and diversity gain $d^*(r)$. $\square$

In the following, we give an example construction of multiblock $(2 \times 2m)$ space–time codes for better understanding of this construction.

Example 1: We wish to construct a multiblock $(2 \times 2m)$ space–time code with $n_t = T = 2$ to code information over m independently faded blocks. Our construction calls for the number field L that is a degree-2-m cyclic Galois extension over $\mathbb{Q}(\imath)$ and the field K that is a subfield of L with degree m over $\mathbb{Q}(\imath)$. For instance, say $m = 2$. We use methods described in [15] and [16] to construct such fields, as shown in the following diagram, where $\omega_{20} = \exp(\imath 2\pi/20)$ and $\omega_5 = \exp(\imath 2\pi/5)$:



The Galois group Gal $(L/\mathbb{Q}(\imath))$ is generated by $\sigma : \omega_5 \mapsto \omega_5^2$, hence the Galois group Gal (L/K) is

$$\text{Gal}\,(L/K) = \langle \zeta = \sigma^2 \rangle = \{\sigma_1, \sigma_4\} \quad (17)$$

where by σ_i we mean the automorphism $\sigma_i : \omega_5 \mapsto \omega_5^i$. Furthermore, it can be verified that $\gamma = \imath$ is a valid nonnorm element for the cyclic division algebra $\mathfrak{D} = (L/K, \zeta, \gamma = \imath)$. Noting that $\mathcal{B} = \{1, \omega_5, \omega_5^2, \omega_5^3\}$ is an integral basis for L over $\mathbb{Q}(\imath)$, the resulting $(n_t \times n_t)$ and $(n_t \times 2n_t)$ space–time coding schemes are given, respectively, by

$$\tilde{\mathcal{X}} = \left\{ \tilde{X} = \begin{bmatrix} \sum_{i=0}^{3} a_i \omega_5^i & \imath \sum_{i=0}^{3} b_i \omega_5^{4i} \\ \sum_{i=0}^{3} b_i \omega_5^i & \sum_{i=0}^{3} a_i \omega_5^{4i} \end{bmatrix} : a_i, b_i \in \mathcal{A}(\text{SNR}) \right\} \quad (18)$$

$$\begin{aligned} \mathcal{X} = \Bigg\{ \Bigg(X_0 &= \begin{bmatrix} \sum_{i=0}^{3} a_i \omega_5^i & \imath \sum_{i=0}^{3} b_i \omega_5^{4i} \\ \sum_{i=0}^{3} b_i \omega_5^i & \sum_{i=0}^{3} a_i \omega_5^{4i} \end{bmatrix}, \\ X_1 &= \begin{bmatrix} \sum_{i=0}^{3} a_i \omega_5^{2i} & \imath \sum_{i=0}^{3} b_i \omega_5^{3i} \\ \sum_{i=0}^{3} b_i \omega_5^{2i} & \sum_{i=0}^{3} a_i \omega_5^{3i} \end{bmatrix} \Bigg) : \\ & a_i, b_i \in \mathcal{A}(\text{SNR}) \Bigg\} \end{aligned} \quad (19)$$

for the QAM base alphabet $\mathcal{A}(\text{SNR}) \subset \mathbb{Z}[\imath]$ of size $\text{SNR}^{\frac{r}{2}}$ with $0 \leq r \leq 2$ given in (9). It can be easily verified that

$$\prod_{j=0}^{1} \det\left(\begin{bmatrix} \sum_{i=0}^{3} a_i\omega_5^{2^j \cdot i} & \imath\sum_{i=0}^{3} b_i\omega_5^{2^{j+2}\cdot i} \\ \sum_{i=0}^{3} b_i\omega_5^{2^j\cdot i} & \sum_{i=0}^{3} a_i\omega_5^{2^{j+2}\cdot i}\end{bmatrix}\right) = N_{K/\mathbb{Q}(\imath)}\left(\det\left(\begin{bmatrix} \sum_{i=0}^{3} a_i\omega_5^{i} & \imath\sum_{i=0}^{3} b_i\omega_5^{4i} \\ \sum_{i=0}^{3} b_i\omega_5^{i} & \sum_{i=0}^{3} a_i\omega_5^{4i}\end{bmatrix}\right)\right)$$

lies in $\mathbb{Z}[\imath]$ for all $a_i, b_i \in \mathbb{Z}[\imath]$, as claimed in Proposition 1. Theorem 2 then asserts that the codeword error performance of $\tilde{\mathcal{X}}$ at high SNR regime is on the order of

$$P_e(r) \doteq \text{SNR}^{-d^*(r)} \tag{20}$$

where the optimal tradeoff $d^*(r)$ is given by the piecewise-linear function connecting the points $(k, d^*(k))$, and $d^*(k) = 2(n_r - k)(2 - k)$ for $k = 0, 1, 2$. □

Here we remark that the code $\mathcal{X}$ in the above example can also be derived from the constructions provided in [21], and was used for the purpose of distributed space–time coding.

B. Nonminimal Delay Multiblock Construction for $T > N_t$.

In this section, we will extend the minimal delay multiblock construction in Theorem 2 to the case when $T > n_t$. To this end, we set the base alphabet $\mathcal{A}(\text{SNR})$ as

$$\mathcal{A}(\text{SNR}) = \Big\{a + b\imath : -M + 1 \leq a, b \leq M - 1, a, b \text{ odd integers}, \ M = \text{SNR}^{\frac{r}{2T}}\Big\}. \tag{21}$$

Comparing to the earlier construction (9), this time we only need a smaller QAM constellation $\mathcal{A}(\text{SNR})$ to begin with. Next, let L be a number field that is cyclic Galois over $\mathbb{Q}(\imath)$ with $[L : \mathbb{Q}(\imath)] = mT$ and let K be a subfield of L with $[K : \mathbb{Q}(\imath)] = m$. Let σ be the generator of the Galois group $\text{Gal}\,(L/\mathbb{Q}(\imath))$; then, we have $\text{Gal}\,(L/K) = \langle\zeta = \sigma^m\rangle$, which is a cyclic group generated by ζ with order T. Again, the Galois group of K over $\mathbb{Q}(\imath)$ is the quotient group $\langle\sigma\rangle/\langle\zeta\rangle$.

Now let $\tilde{\mathcal{X}}$ be a $(T \times T)$ space–time coding scheme obtained by the left-regular representation of elements of the cyclic division algebra $\mathfrak{D} = (L/K, \zeta, \gamma)$ for some nonnorm element $\gamma \in \mathbb{Z}[\imath]$, and by restricting $a_{i,j}$ to be in the set $\mathcal{A}(\text{SNR})$ [cf. (13)]. Removing any, but in a fixed fashion, $(T - n_t)$ rows of the matrices in $\tilde{\mathcal{X}}$ gives the resulting $(n_t \times T)$ space–time coding scheme $\hat{\mathcal{X}}$. Thus, we have the following construction.

Theorem 3: Let $\tilde{\mathcal{X}}$ and $\hat{\mathcal{X}}$ be defined as above; then, the $(n_t \times mT)$ multiblock space–time coding scheme $\mathcal{X}$

$$\mathcal{X} = \Big\{X = \Big(\hat{X}, \sigma(\hat{X}), \ldots, \sigma^{m-1}(\hat{X})\Big) : \hat{X} \in \hat{\mathcal{X}}\Big\} \tag{22}$$

is optimal with respect to the D-M tradeoff. □

C. Stacking Construction of Nonminimal Multiblock Codes When $T \geq mn_t$

In the previous sections, we have provided constructions of $(n_t \times mT)$ multiblock space–time codes that are D-M optimal for any number of transmit antennas n_t, any number of blocks m, and for $T \geq n_t$. Here we wish to give an alternative construction when the quasi-static interval $T \geq mn_t$, meaning the channel is extremely slowly varying. It will be seen that this alternative construction requires much lesser signaling complexity than that resulting from Theorem 3.

Let $\mathcal{A}(\text{SNR})$ be a QAM base alphabet given by

$$\mathcal{A}(\text{SNR}) = \Big\{a + b\imath : -M + 1 \leq a, b \leq M - 1, a, b \text{ odd integers}, \ M = \text{SNR}^{\frac{rm}{2T}}\Big\} \tag{23}$$

and let E be a number field that is cyclic Galois over $\mathbb{Q}(\imath)$ with degree T; then, $\mathfrak{D} = (E/\mathbb{Q}(\imath), \sigma, \gamma)$ is a cyclic division algebra, where σ is the generator of the Galois group $\text{Gal}\,(E/\mathbb{Q}(\imath))$ and $\gamma \in \mathbb{Z}[\imath]$ is some nonnorm element. Let $\tilde{\mathcal{X}}$ be the $(T \times T)$ space–time coding scheme obtained by left-regular representation of elements in $\mathfrak{D}$ and by restricting $a_{i,j}$ to the set $\mathcal{A}(\text{SNR})$ [cf. (13)]. Remove any fixed $(T - mn_t)$ rows from code matrices in $\tilde{\mathcal{X}}$, and let $\hat{\mathcal{X}}$ denote the resulting $(mn_t \times T)$ space–time coding scheme. By rearranging the rows of the $(mn_t \times T)$ code matrices in $\hat{\mathcal{X}}$, we offer the following construction, termed *stacking construction*. Specifically, given any $(mn_t \times T)$ code matrix $\hat{X} \in \hat{\mathcal{X}}$, the construction first vertically partitions $\hat{X}$ into m submatrices, each of size $(n_t \times T)$. Say $X_0, X_i, \ldots, X_{m-1}$ are the resulting m submatrices; then, the stacking construction will put these m submatrices side-by-side to yield the desired code matrix of size $(n_t \times mT)$. It turns out that such $(n_t \times mT)$ code is D-M optimal.

Theorem 4: Assuming $T \geq mn_t$, let $\hat{\mathcal{X}}$ be defined as above; then, the $(n_t \times mT)$ multiblock space–time coding scheme $\mathcal{X}$

$$\mathcal{X} = \left\{(X_0, X_1, \ldots, X_{m-1}) : \hat{X} = \begin{bmatrix} X_0 \\ \vdots \\ X_{m-1}\end{bmatrix} \in \hat{\mathcal{X}}\right\} \tag{24}$$

is D-M optimal, where the submatrices X_i are of size $(n_t \times T)$. □

To see the advantage of the above construction, recall that in Theorem 3 the construction of $(n_t \times mT)$ multiblock codes with $T \geq mn_t$ calls for a number field L with $[L : \mathbb{Q}(\imath)] = mT$. It in fact means that entries of the resulting code matrix are linear combinations of mT points drawn from $\mathcal{A}(\text{SNR})$. However, in the alternative construction of Theorem 4, only T linear combinations are required, hence it has much lower signaling complexity.

III. Simulation Results

In this section, we will provide some simulation results of the multiblock code given in Example 1 for a MIMO system with $n_t = 2$ transmit and $n_r = 2$ receive antennas at the transmission rate of 4 bits per channel use. The results are shown in Fig. 1 and all the codes used in simulation are normalized to satisfy the power constraint (8). First, we consider the case when the channel quasi-static interval T equals 2 and the channel varies independently for every consecutive fading block. Using sphere decoding [14], the performance result of the (2×4) multiblock code $\mathcal{X}$ given in (19) is shown in solid line in Fig. 1, and it is seen that $\mathcal{X}$ achieves the codeword error probability of 10^{-4} at SNR = 17 dB. Comparing with $\mathcal{X}$, the Golden code [8] that is the known best code for the (2×2) MIMO system requires SNR = 23 dB to achieve the same codeword error probability. The gain of 6 dB in SNR for the multiblock code $\mathcal{X}$ is due to the fact that the Golden code was originally designed to code information within one fading block only, not across consecutive fading blocks. On the other hand, it is true that for the very slowly varying fading channels, the consecutive fading blocks could be almost the same. Thus, in Fig. 1, we have also considered the case when the two consecutive fading blocks are identical. For such channel, the performance result of $\mathcal{X}$ is shown in dash line in Fig. 1, and it can be seen that the code $\mathcal{X}$ achieves 10^{-4} at SNR = 20.2 dB. Clearly, the degradation in performance is due to the lesser degrees of freedom in channel variation. However, even in this case, the multiblock code $\mathcal{X}$ still shows an excellence performance and gains in SNR for about 2.8 dB, compared to that of the Golden code.

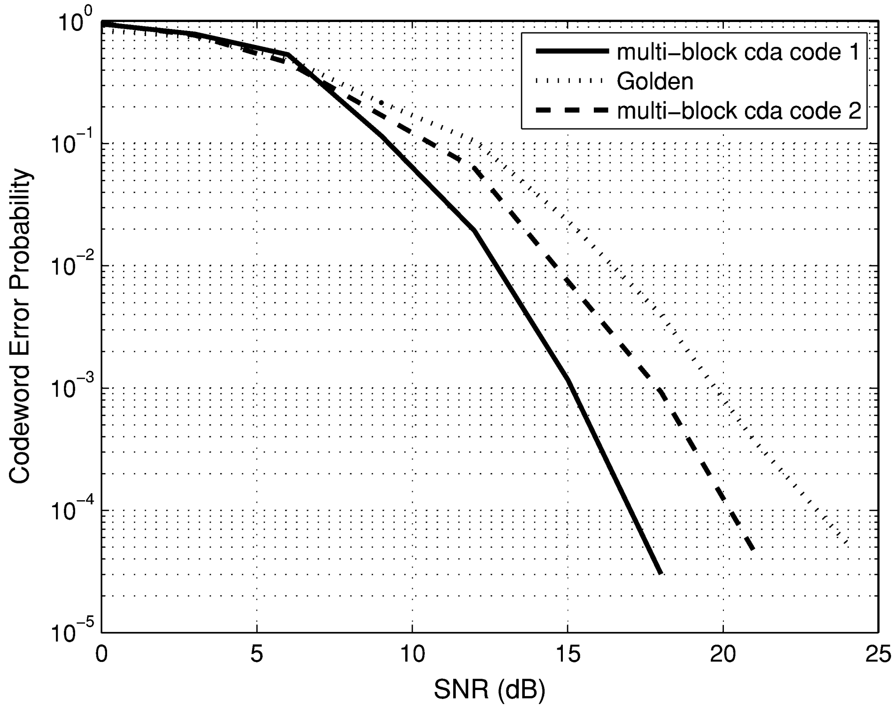


Fig. 1. Performance simulations of the multiblock code $\mathcal{X}$ given in Example 1 and the Golden code for the (2×2) MIMO system.

IV. Conclusion

In this correspondence, we had presented systematic constructions of multiblock space–time codes that can be used to encode and transmit coded information over quasi-static MIMO fading channels. The constructions are based on a left-regular representation of elements in a cyclic division algebra whose center is a field extension of the quadratic number field of finite degree. Constructions of codes with minimal or nonminimal delays were both given. In particular, when the MIMO channel is extremely slowly varying, an alternative construction was also given to reduce the number of linear combinations required for encoding, and to yield codes with lower signaling complexity. We had proved that all the constructions proposed in this correspondence are optimal in terms of D-M tradeoff, and can be used to provide a reliable MIMO communication with vanishing error probability when the number of coded blocks m is sufficiently large. Furthermore, we had given a stronger proof showing that codes resulting from the proposed constructions are approximately universal and can cope with situations when the fading coefficients are correlated in time, are correlated among different antenna, and/or are of different kinds of statistics.

Appendix

A. Proof of Theorem 2

Recall that given $X = (X_0, X_1, \ldots, X_{m-1}) \in \mathcal{X}$, the code matrix transmitted over m fading blocks, the $(n_r \times n_t)$ received signal matrix at the ith fading block is

$$Y_i = \theta H_i X_i + W_i$$

where $\theta^2 = \text{SNR}^{1-\frac{r}{n_t}}$ is given in (16), and where we have set $T = n_t$. The goal here is to show the codeword error probability of $\mathcal{X}$ is on the order of $P_e(r) \doteq \text{SNR}^{-d^*(r)}$. We will adopt some techniques from [16] and [20]. For any distinct pair of code matrices $X \neq X' \in \mathcal{X}$ with $X = (X_0, \ldots, X_{m-1})$ and $X' = (X'_0, \ldots, X'_{m-1})$, let $\lambda_{i,1} \leq \lambda_{i,2} \leq \cdots \leq \lambda_{i,n_t}$ and $\ell_{i,1} \geq \ell_{i,2} \geq \cdots \geq \ell_{i,n_t}$ be, respectively, the ordered eigenvalues of the matrices $H_i^\dagger H_i$ and $\Delta X_i^\dagger \Delta X_i$, where $\Delta X_i = X_i - X'_i$. Then, by using the mismatch bound [25], [16], it can be shown that the squared Euclidean distance between the noise-free received signal matrices corresponding, respectively, to X and X' is

$$\begin{aligned} d_E^2(X, X') &:= \sum_{i=0}^{m-1} \|\theta H_i X_i - \theta H_i X'_i\|_F^2 \\ &\geq \theta^2 \sum_{i=0}^{m-1} \sum_{j=1}^{n_t} \lambda_{i,j} \ell_{i,j}. \end{aligned} \tag{25}$$

Moreover, we may reorder and reindex the values

$$(\lambda_{0,1}, \ldots, \lambda_{0,n_t}, \ldots, \lambda_{m-1,n_t})$$

as a nondecreasing sequence

$$\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$$

and similarly, $(\ell_{0,1}, \ldots, \ell_{0,n_t}, \ldots, \ell_{m-1,n_t})$ as a nonincreasing sequence

$$\ell_1 \geq \ell_2 \geq \cdots \geq \ell_n$$

where $n = mn_t$. Now with the above reordering, $d_E^2(X, X')$ can be further lower bounded by

$$\begin{aligned} d_E^2(X, X') &\geq \theta^2 \sum_{i=1}^{n} \lambda_i \ell_i \geq \theta^2 \sum_{i=n-k+1}^{n} \lambda_i \ell_i \\ &\dot{\geq} \theta^2 \left(\prod_{i=n-k+1}^{n} \lambda_i \ell_i \right)^{\frac{1}{k}} \end{aligned} \tag{26}$$

for $k = 1, 2, \ldots, n$, where the last inequality follows from the arithmetic–geometric mean inequality. In particular, by making use of

Proposition 1 and again by the arithmetic–geometric mean inequality, the serial product of ℓ_i can be lower bounded by

$$\prod_{i=n-k+1}^{n} \ell_i = \frac{\prod_{i=0}^{m-1} \det(\triangle X_i \triangle X_i^\dagger)}{\prod_{i=1}^{n-k} \ell_i} \geq \frac{1}{\prod_{i=1}^{n-k} \ell_i} \geq \left(\frac{\sum_{i=1}^{n-k} \ell_i}{n-k}\right)^{-(n-k)} \dot{\geq} \|\triangle X\|_F^{-2(n-k)} \doteq \text{SNR}^{-\frac{r(n-k)}{n_t}}. \quad (27)$$

As $\lambda_1 > 0$ with Probability 1, define

$$\alpha_i := -\log_{\text{SNR}} \lambda_i \quad \text{and} \quad \underline{\alpha} := [\alpha_1, \ldots, \alpha_n]^t. \quad (28)$$

Then, substituting (27) into (26) yields

$$d_E^2(X, X') \dot{\geq} \theta^2 \left(\prod_{i=n-k+1}^{n} \lambda_i\right)^{\frac{1}{k}} \|\triangle X\|_F^{-2\frac{(n-k)}{k}} = \text{SNR}^{\delta_k(\underline{\alpha})} := d_{E,k}^2(\underline{\alpha}) \quad (29)$$

which is independent of the choices of X and X', and

$$\delta_k(\underline{\alpha}) := 1 - \frac{r}{n_t} - \frac{1}{k}\sum_{i=n-k+1}^{n} \alpha_i - \frac{r(n-k)}{n_t\, k} = 1 - \frac{rm}{k} - \frac{1}{k}\sum_{i=n-k+1}^{n} \alpha_i \quad (30)$$

for $k = 1, 2, \ldots, n$. Thus, the codeword error probability given $\underline{\alpha}$ can be upper bounded by

$$P_e(r|\underline{\alpha}) \leq \Pr\left\{\sum_{i=0}^{m-1} \|W_i\|_F^2 \geq \frac{d_{E,k}^2(\underline{\alpha})}{4}\right\} = \exp\left(-\frac{d_{E,k}^2(\underline{\alpha})}{4}\right) \sum_{t=0}^{n_r mT-1} \frac{\left(d_{E,k}^2(\underline{\alpha})/4\right)^t}{t!} := P_k(\underline{\alpha}) \quad (31)$$

and it should be noted that $P_k(\underline{\alpha}) \doteq 0$ if $\delta_k(\underline{\alpha}) > 0$. Since $P_k(\underline{\alpha}) \leq 1$, it follows that

$$P_e(r) \leq \min_k \{\mathbb{E}_{\underline{\alpha}} P_k(\underline{\alpha})\} \leq \Pr\{\underline{\alpha} : \delta_k(\underline{\alpha}) \leq 0, k = 1, \ldots, n\}.$$

Define

$$\alpha_{i,j} := -\log_{\text{SNR}} \lambda_{i,j}. \quad (32)$$

Now bearing in mind that $\alpha_{i,1} \geq \alpha_{i,2} \cdots \geq \alpha_{i,n_t}$ and $\alpha_1 \geq \alpha_2 \cdots \geq \alpha_n$, by arguing similarly as [20] and [26], it can be shown that

$$\{\underline{\alpha} : \delta_k(\underline{\alpha}) \leq 0, k = 1, \ldots, n\} = \left\{\underline{\alpha} : \sum_{i=0}^{m-1}\sum_{j=1}^{n_t} (1 - \alpha_{i,j})^+ \leq rm\right\} \quad (33)$$

where $(x)^+ := \max\{0, x\}$. A proof similar to (33) can also be found in [27]. Note that the right-hand side of (33) equals the channel outage probability, that is, we have

$$P_e(r) \leq \Pr\left\{\underline{\alpha} : \sum_{i=0}^{m-1}\sum_{j=1}^{n_t} (1 - \alpha_{i,j})^+ \leq mr\right\} \doteq \Pr\left\{\frac{1}{m}\sum_{i=0}^{m-1} \log\left(I_{n_t} + \text{SNR} H_i^\dagger H_i\right) \leq \text{SNR}^r\right\}$$

where I_{n_t} is the identity matrix of size n_t. With the outage bound from [4], we have proved that the code $\mathcal{X}$ is approximately universal and is D-M optimal for any kinds of fading distributions, including the time-correlated channels. In particular, for quasi-static Rayleigh fading channel with independent fading blocks, the probability density function of $\underline{\alpha}$ can be derived from Wishart distribution (see [4] and [26]), and it can be shown without using (33) that the diversity gain $d(r)$ achieved by $\mathcal{X}$ equals $d^*(r)$ defined in (6). The missing details can be found in the conference version [26] of this correspondence.

B. Proof of Theorem 3

By construction, the size of $\mathcal{X}$ equals

$$|\mathcal{X}| = |\tilde{\mathcal{X}}| = |\mathcal{A}(\text{SNR})|^{mT\cdot T} \doteq \text{SNR}^{rmT}$$

since there is a one-to-one correspondence between matrices in $\hat{\mathcal{X}}$ and $\tilde{\mathcal{X}}$. This follows from the fact that the difference between every distinct pair of matrices in $\tilde{\mathcal{X}}$ has full rank T. Thus, $\mathcal{X}$ achieves multiplexing gain at value r. To ensure the power constraint (8), this time we will set the parameter θ at

$$\theta^2 \doteq \text{SNR}^{1-\frac{r}{T}}. \quad (34)$$

Furthermore, we may assume without loss of generality that $\hat{\mathcal{X}}$ is obtained by removing the last $(T - n_t)$ rows of code matrices in $\tilde{\mathcal{X}}$. Suppose that $X = (\hat{X}, \ldots, \sigma^{m-1}(\hat{X})) \in \mathcal{X}$ was transmitted, and that $\tilde{X} \in \tilde{\mathcal{X}}$ is the corresponding $(T \times T)$ code matrix. The received signal matrix at the ith fading block can be written as

$$Y_i = \theta H_i \left[\sigma^i(\hat{X})\right] + W_i = \theta \tilde{H}_i \left[\sigma^i(\tilde{X})\right] + W_i$$

for $i = 0, \ldots, m-1$, where $\tilde{H}_i$ is the equivalent $(n_r \times T)$ ith fading channel matrix given by

$$\tilde{H}_i := \begin{bmatrix} H_i & \mathbf{0}_{n_r \times (T-n_t)} \end{bmatrix}.$$

$\mathbf{0}_{n_r \times (T-n_t)}$ denotes the $(n_r \times (T - n_t))$ all-zero matrix. Now for any $X \neq X' \in \mathcal{X}, X = (\hat{X}, \ldots, \sigma^{m-1}(\hat{X}))$ and $X' = (\hat{X}', \ldots, \sigma^{m-1}(\hat{X}'))$, let $\tilde{X}$ (resp., $\tilde{X}'$) be the code matrix in $\tilde{\mathcal{X}}$ that is associated with $\hat{X}$ (resp., $\hat{X}'$). Arguing similarly as in part A of the Appendix, the squared Euclidean distance $d_E^2(X, X')$ is lower bounded by

$$d_E^2(X, X') = \sum_{i=0}^{m-1} \|\theta \tilde{H}_i \sigma^i(\tilde{X} - \tilde{X}')\|_F^2 \dot{\geq} \theta^2 \left(\prod_{i=mn_t-k+1}^{mn_t} \lambda_i \tilde{\ell}_{m(T-n_t)+i}\right)^{\frac{1}{k}} \quad (35)$$

for $k = 1, 2, \ldots, mn_t$. $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{mn_t}$ and $\tilde{\ell}_1 \geq \tilde{\ell}_2 \geq \cdots \geq \tilde{\ell}_{mT}$ are, respectively, the ordered nonzero eigenvalues of the set of matrices $\{H_i^\dagger H_i\}$ and $\{[\sigma^i(\triangle\tilde{X})][\sigma^i(\triangle\tilde{X})]^\dagger\}$, where $\triangle\tilde{X} = \tilde{X} - \tilde{X}'$. Furthermore, a similar argument as in part A of the Appendix shows that

$$\prod_{i=mn_t-k+1}^{mn_t} \tilde{\ell}_{m(T-n_t)+i} = \prod_{i=mT-k+1}^{mT} \tilde{\ell}_i \dot{\geq} \text{SNR}^{-\frac{r}{T}(mT-k)}. \quad (36)$$

Substituting (36) into (35) yields the $\delta_k(\underline{\alpha})$ defined as in (29)

$$\delta_k(\underline{\alpha}) = 1 - \frac{r}{T} - \frac{1}{k}\sum_{i=mn_t-k+1}^{mn_t} \alpha_i - \frac{r(mT-k)}{kT} = 1 - \frac{rm}{k} - \frac{1}{k}\sum_{i=mn_t-k+1}^{mn_t} \alpha_i$$

for $k = 1, \ldots, mn_t$, where $\underline{\alpha}$ is defined as in (28). By arguing in exactly the same way as in part A of the Appendix, we see that the code $\mathcal{X}$ is approximately universal and that the diversity gain achieved by $\mathcal{X}$ equals $d^*(r)$ given in (6) for the case of independent block fading. The proof is now complete.

C. Proof of Theorem 4

For simplicity, here we only prove the case of $T = mn_t$, and the case of $T > mn_t$ can be done by extending the arguments in part B of the Appendix. First, to satisfy the power constraint (8), the parameter θ is set at

$$\theta^2 \doteq \mathrm{SNR}^{1-\frac{r}{n_t}}. \tag{37}$$

Next, given the transmitted code matrix $X = (X_0, X_1, \ldots, X_{m-1}) \in \mathcal{X}$, we rearrange the received signal matrices $Y_0, \ldots, Y_{m-1}$ as the following $(mn_r \times T)$ matrix:

$$\begin{aligned}\tilde{Y} &:= \begin{bmatrix} Y_0 \\ \vdots \\ Y_{m-1} \end{bmatrix} \\ &= \theta \begin{bmatrix} H_0 & & \\ & \ddots & \\ & & H_{m-1} \end{bmatrix} \begin{bmatrix} X_0 \\ \vdots \\ X_{m-1} \end{bmatrix} + \begin{bmatrix} W_0 \\ \vdots \\ W_{m-1} \end{bmatrix} \\ &= \theta \tilde{H} \tilde{X} + \tilde{W} \end{aligned} \tag{38}$$

where $\tilde{H}$ is the $(mn_r \times mn_t)$ block-diagonal channel matrix and where $\tilde{W}$ is an $(mn_r \times T)$ noise matrix. It should be noted that the matrix $\tilde{X} \in \tilde{\mathcal{X}}$ by construction. Thus, a similar argument from part A of the Appendix shows that for every $X \neq X' \in \mathcal{X}$, $d_E^2(X, X')$ can be lower bounded by

$$d_E^2(X, X') \dot{\geq} \theta^2 \left(\prod_{i=T-k+1}^{T} \lambda_i \tilde{\ell}_i \right)^{1/k}$$

for $k = 1, 2, \ldots, T$, where $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_T$ and $\tilde{\ell}_1 \geq \tilde{\ell}_2 \geq \cdots \geq \tilde{\ell}_T$ are, respectively, the ordered nonzero eigenvalues of the matrices $\tilde{H}^\dagger \tilde{H}$ and $\triangle\tilde{X} \triangle\tilde{X}^\dagger \cdot \triangle\tilde{X} = \tilde{X} - \tilde{X}'$ and $\tilde{X}'$ is the code matrix obtained by rearranging X' as (38). In particular, we have

$$\prod_{i=T-k+1}^{T} \tilde{\ell}_i \dot{\geq} \|\triangle\tilde{X}\|_F^{-2(T-k)} \doteq \mathrm{SNR}^{-\frac{r(T-k)}{n_t}}.$$

Moreover, the exponent $\delta_k(\underline{\alpha})$ defined in (29) now equals

$$\begin{aligned}\delta_k(\underline{\alpha}) &= 1 - \frac{r}{n_t} - \frac{1}{k} \sum_{i=T-k+1}^{T} \alpha_i - \frac{r(T-k)}{kn_t} \\ &= 1 - \frac{rm}{k} - \frac{1}{k} \sum_{i=T-k+1}^{T} \alpha_i.\end{aligned}$$

Now by the same arguments as in part A of the Appendix, it can be shown that $\mathcal{X}$ is approximately universal and achieves diversity gain $d^*(r)$ defined in (6) for the case of independent block fading.

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