OPTIMAL CONSECUTIVE-k-OUT-OF-n: G CYCLE FOR $n \leq 2k + 1$*

DING-ZHU DU†, FRANK K. HWANG‡, XIAOHUA JIA§, AND HUNG Q. NGO¶

Abstract. A cyclic consecutive-k-out-of-n: G system consists of n components lying on a cycle. Those components are exchangeable but may have different working probabilities. The system works if and only if there are k consecutive components at work. What is the optimal assignment of components for maximizing the reliability of the system? Does the optimal assignment depend on the working probability values of components? For $k \leq n \leq 2k + 1$, Zuo and Kuo in 1990 proposed a solution independent from the working probability values of components, called the invariant optimal assignment. However, their proof is incomplete, pointed out recently by Jalali et al. [*The Optimal Consecutive-k-out-of-n: G Line for $n \leq 2k$*, manuscript, 1999]. We present a complete proof in this paper.

Key words. invariant optimal assignment, consecutive-k-out-of-n: G cycle

AMS subject classifications. 60K10, 90B25

PII. S0895480100375041

1. Introduction. A cyclic consecutive-k-out-of-n: G system $con_C(k, n : G)$ is a cycle of $n(\geq k)$ components such that the system works if and only if some k consecutive components all work. Suppose n components with working probabilities $p_{[1]} \leq p_{[2]} \leq \cdots \leq p_{[n]}$ are all exchangeable. How can they be assigned to the n positions on the cycle to maximize the reliability of the system? Kuo, Zhang, and Zuo [10] showed that if $k = 2$, then the optimal assignment is *invariant*; i.e., it depends only on the ordering of working probabilities of the components but not their value. They also claimed that for $k \geq 3$ and $n > 2k + 1$, $Con_C(k, n : G)$ has no invariant optimal assignment. For $n \leq 2k + 1$, Zuo and Kuo [13] claimed that there exists an invariant optimal assignment

$$(p_{[1]}, p_{[3]}, p_{[5]}, \ldots, p_{[6]}, p_{[4]}, p_{[2]}).$$

However, Jalali et al. [9] found that their proof is incomplete. A proof in case $n = 2k + 1$ was given in [6]. In this paper, we give a complete proof for this invariant optimal assignment with $n \leq 2k + 1$.

2. Main result. In this section, we show the following.

THEOREM 2.1. *For $k \leq n \leq 2k + 1$, there exists an invariant optimal assignment*

$$(p_{[1]}, p_{[3]}, p_{[5]}, \ldots, p_{[6]}, p_{[4]}, p_{[2]}).$$

Let $p_1, p_2, \ldots, p_n$ be reliabilities of the n components on the cycle in the counter-clockwise direction. For simplicity of the proof, we first assume that

$$0 < p_{[1]} < p_{[2]} < \cdots < p_{[n]} < 1.$$

Our proof is based on the following representation of the reliability of consecutive-k-out-of-n: G cycle for $n \leq 2k + 1$.

LEMMA 2.2. *The reliability of consecutive-k-out-of-n: G cycle for $n \leq 2k + 1$ under assignment C can be represented as*

$$R(C) = p_1 \cdots p_n + \sum_{i=1}^{n} q_i p_{i+1} \cdots p_{i+k}$$

$$= p_1 \cdots p_n + \sum_{i=1}^{n} p_i \cdots p_{i+k-1} - \sum_{i=1}^{n} p_i \cdots p_{i+k},$$

where $q_i = 1 - p_i$ and $p_{n+i} = p_i$.

Proof. The system works if and only if all components work or for some i, the ith component fails, and the $(i+1)$st component, ..., the $(i+k)$th component all work. Since $n \leq 2k + 1$, there exists at most one such i. Therefore,

$$R(C) = p_1 \cdots p_n + \sum_{i=1}^{n} q_i p_{i+1} \cdots p_{i+k}.$$

Note that

$$q_i p_{i+1} \cdots p_{i+k} = p_{i+1} \cdots p_{i+k} - p_i \cdots p_{i+k}.$$

This implies the second representation.   □

This representation is a key point to show the main theorem. It explains why invariant optimal assignment exists for $n \leq 2k + 1$ but does not exist for $n > 2k + 1$.

For $n = k, k + 1$, by Lemma 2.2, $R(C)$ has the same value for all assignment C, and hence Theorem 2.1 is trivially true. Next, we assume $k + 2 \leq n \leq 2k + 1$.

To prove Theorem 2.1, it suffices to show that in any optimal assignment,

$$(2.1) \qquad (p_i - p_j)(p_{i-1} - p_{j+1}) > 0 \quad \text{for } 1 < i < j < n.$$

In fact, the optimal assigment described in Theorem 2.1 can be determined uniquely by condition (2.1) (see [9]). Selecting any component to be labeled p_1, we always have from condition (2.1) that

$$(2.2) \qquad (p_i - p_{n-i+1})(p_{i+1} - p_{n-i}) > 0 \quad \text{for } i = 1, \ldots, h,$$

where $h = \lfloor n/2 \rfloor$. For simplicity of representation, we denote $i' = n - i + 1$. When n is odd, $(\frac{n+1}{2})' = \frac{n+1}{2}$. Furthermore, without loss of generality, we assume $p_1 > p_{1'}$ throughout this proof. Then the condition (2.2) can be rewritten as

$$p_i > p_{i'} \quad \text{for } i = 1, \ldots, h.$$

Let $I = \{i \mid 1 < i \leq h, p_i < p_{i'}\}$. Let C_I be the assignment obtained from C by exchanging components i and i' for all $i \in I$. To prove (2.1), it suffices to show that for any assignment C, if $I \neq \emptyset$,

$$R(C) < R(C_I).$$

Denote $I' = \{i' \mid i \in I\}$ and

$$(y_{i_1} \cdots y_{i_d})^I$$
$$= \left(\prod_{1 \le j \le d, i_j \notin I \cup I'} y_{i_j} - \prod_{1 \le j \le d, i_j \notin I \cup I'} y_{i'_j} \right) \left(\prod_{1 \le j \le d, i_j \in I \cup I'} y_{i'_j} - \prod_{1 \le j \le d, i_j \in I \cup I'} y_{i_j} \right),$$

where $y_i = p_i$ or q_i. It is easy to verify that

$$\left(y_{i_1} \cdots y_{i_d} \prod_{1 \le j \le d, i_j \in I \cup I'} \frac{y_{i'_j}}{y_{i_j}} + y_{i'_1} \cdots y_{i'_d} \prod_{1 \le j \le d, i_j \in I \cup I'} \frac{y_{i_j}}{y_{i'_j}} \right)$$
$$- (y_{i_1} \cdots y_{i_d} + y_{i'_1} \cdots y_{i'_d})$$
$$= (y_{i_1} \cdots y_{i_d})^I.$$

Denote $Q_k(C) = \sum_{i=1}^n p_i \cdots p_{i+k-1}$. Then

$$R(C) = p_1 \cdots p_n + Q_k(C) - Q_{k+1}(C).$$

Let $a = \lfloor k/2 \rfloor$ and

$$s = \left\{ \begin{array}{ll} h & \text{if } n \text{ is even and } k \text{ is odd,} \\ h+1 & \text{otherwise.} \end{array} \right.$$

Then we have the following.

LEMMA 2.3. $Q_k(C_I) - Q_k(C) = \sum_{i=1}^s (p_{-a+i} \cdots p_{-a+i+k-1})^I.$

Proof. Consider four cases.

Case 1. n and k both are even. In this case, $s = h + 1 = 1 + n/2$ and $a = k/2$. Note that

$$\sum_{i=h+2}^n p_{-a+i} \cdots p_{-a+i+k-1}$$
$$= \sum_{i=2}^h p_{-a+(2h+2-i)} \cdots p_{-a+(2h+2-i)+k-1}$$
$$= \sum_{i=2}^h p_{n-(-a+i+k-1)+1} \cdots p_{n-(-a+i)+1}$$
$$= \sum_{i=2}^h p_{(-a+i+k-1)'} \cdots p_{(-a+i)'}.$$

Thus,

$$Q_k(C)$$
$$= p_{-a+1} \cdots p_a + \sum_{i=2}^h (p_{-a+i} \cdots p_{-a+i+k-1} + p_{(-a+i)'} \cdots p_{(-a+i+k-1)'})$$
$$+ p_{-a+h+1} \cdots p_{-a+h+k}$$
$$= \prod_{j=1}^a (p_j p_{j'}) + \sum_{i=2}^h (p_{-a+i} \cdots p_{-a+i+k-1} + p_{(-a+i)'} \cdots p_{(-a+i+k-1)'})$$
$$+ \prod_{j=1}^a (p_{h-a+j} p_{(h-a+j)'}).$$

So,

$$Q_k(C_I) - Q_k(C) = \sum_{i=2}^{h} (p_{-a+i} \cdots p_{-a+i+k-1})^I.$$

However,

$$(p_{-a+1} \cdots p_{-a+k})^I = (p_{-a+h+1} \cdots p_{-a+h+k})^I = 0.$$

Therefore,

$$Q_k(C_I) - Q_k(C) = \sum_{i=1}^{h+1} (p_{-a+i} \cdots p_{-a+i+k-1})^I.$$

Case 2. n is even and k is odd. In this case, $s = h = n/2$ and $a = (k-1)/2$. Note that

$$Q_k(C) = \sum_{i=1}^{h} (p_{-a+i} \cdots p_{-a+i+k-1} + p_{(-a+i)'} \cdots p_{(-a+i+k-1)'}).$$

Therefore,

$$Q_k(C_I) - Q_k(C) = \sum_{i=1}^{h} (p_{-a+i} \cdots p_{-a+i+k-1})^I.$$

Case 3. n and k both are odd. In this case, $s = h+1 = (n+1)/2$ and $a = (k-1)/2$. Note that

$$
\begin{aligned}
Q_k(C) \\
= \sum_{i=1}^{h} (p_{-a+i} \cdots p_{-a+i+k-1} + p_{(-a+i)'} \cdots p_{(-a+i+k-1)'}) + p_{-a+h+1} \cdots p_{-a+h+k} \\
= \sum_{i=1}^{h} (p_{-a+i} \cdots p_{-a+i+k-1} + p_{(-a+i)'} \cdots p_{(-a+i+k-1)'}) \\
+ p_{-a+h-1} p_{(-a+h-1)'} \cdots p_h p_{h'} p_{h+1}.
\end{aligned}
$$

Thus,

$$Q_k(C_I) - Q_k(C) = \sum_{i=1}^{h} (p_{-a+i} \cdots p_{-a+i+k-1})^I.$$

However,

$$(p_{-a+h+1} \cdots p_{-a+h+k})^I = 0.$$

Therefore,

$$Q_k(C_I) - Q_k(C) = \sum_{i=1}^{h+1} (p_{-a+i} \cdots p_{-a+i+k-1})^I.$$

Case 4. n is odd and k is even. In this case, $s = h + 1 = (n+1)/2$ and $a = k/2$. Note that

$$Q_k(C) = p_{-a+1} \cdots p_{-a+k} + \sum_{i=2}^{h+1} (p_{-a+i} \cdots p_{-a+i+k-1} + p_{(-a+i)'} \cdots p_{(-a+i+k-1)'})$$

$$= \prod_{j=1}^{a} (p_j p_{j'}) + \sum_{i=2}^{h+1} (p_{-a+i} \cdots p_{-a+i+k-1} + p_{(-a+i)'} \cdots p_{(-a+i+k-1)'}).$$

Thus

$$Q_k(C_I) - Q_k(C) = \sum_{i=2}^{h+1} (p_{-a+i} \cdots p_{-a+i+k-1})^I.$$

However,

$$(p_{-a+1} \cdots p_{-a+k})^I = 0.$$

Therefore,

$$Q_k(C_I) - Q_k(C) = \sum_{i=1}^{h+1} (p_{-a+i} \cdots p_{-a+i+k-1})^I. \qquad \square$$

Define

$$t = \begin{cases} h & \text{if } n \text{ is even and } k+1 \text{ is odd}, \\ h+1 & \text{otherwise} \end{cases}$$

and $b = \lfloor (k+1)/2 \rfloor$. We have a useful representation of $R(C_I) - R(C)$ as follows.

LEMMA 2.4.

$$R(C_I) - R(C) = \sum_{i=2}^{t} (p_{-b+i} \cdots p_{-b+i+k-1})^I - \sum_{i=1}^{t} (p_{-b+i} \cdots p_{-b+i+k})^I.$$

Proof. By Lemma 2.3, we have

$$R(C_I) - R(C) = \sum_{i=1}^{s} (p_{-a+i} \cdots p_{-a+i+k-1})^I - \sum_{i=1}^{t} (p_{-b+i} \cdots p_{-b+i+k})^I.$$

Note that if k is even and n is odd, then $a = b$, $s = t$, and

$$(p_{-a+1} \cdots p_{-a+k})^I = (p_{-k/2+1} \cdots p_{k/2})^I = 0;$$

if k is odd and n is odd, then $a = b - 1$, $s = t$, and

$$(p_{-a+s} \cdots p_{-a+s+k-1})^I = (p_{-(n-k)/2} \cdots p_{-(n+k)/2-1})^I = 0;$$

if k is even and n is even, then $a = b$, $s = t + 1$, and

$$(p_{-a+1} \cdots p_{-a+k})^I = (p_{-a+s} \cdots p_{-a+s+k-1})^I = 0;$$

if k is odd and n is even, then $a = b - 1$ and $s = t - 1$. Therefore, we always have

$$\sum_{i=1}^{s}(p_{-a+i}\cdots p_{-a+i+k-1})^I = \sum_{i=2}^{t}(p_{-a+i}\cdots p_{-a+i+k-1})^I. \qquad \square$$

Note that $(p_{-b+i}\cdots p_{-b+i+k-1})^I \geq 0$ for $2 \leq i \leq t$ and $(p_{-b+i}\cdots p_{-b+i+k})^I \geq 0$ for $1 \leq i \leq t$. Therefore, to prove $R(C_I) < R(C)$, we need to compare $(p_{-b+i}\cdots p_{-b+i+k-1})^I$ with $(p_{-b+i}\cdots p_{-b+i+k})^I$.

LEMMA 2.5. *Suppose* $I = \{i \mid 1 < i \leq h, p_i < p_{i'}\}$. *Then, for* $i = 1, \ldots, b$,

$$(q_{-b+i}p_{-b+i+1}\cdots p_{-b+i+k})^I = (p_{-b+i+1}\cdots p_{-b+i+k})^I - (p_{-b+i}p_{-b+i+1}\cdots p_{-b+i+k})^I \geq 0,$$

and the strict inequality sign holds if and only if

$$\{j \mid b - i + 1 \leq j \leq \min(-b+i+k, n+b-i-k), j \in I\} \neq \emptyset,$$
$$\{j \mid b - i + 1 \leq j \leq \min(-b+i+k, n+b-i-k), j \notin I\} \neq \emptyset.$$

Proof. First, assume $-b + i \in I \cup I'$. Then we have

$$(q_{-b+i}p_{-b+i+1}\cdots p_{-b+i+k})^I$$

$$= \left(q_{-b+i} \prod_{-b+i+1\leq j\leq -b+i+k, j\notin I\cup I'} p_j - q_{(-b+i)'} \prod_{-b+i+1\leq j\leq -b+i+k, j\notin I\cup I'} p_{j'} \right)$$

$$\cdot \left(\prod_{-b+i+1\leq j\leq -b+i+k, j\in I\cup I'} p_{j'} - \prod_{-b+i+1\leq j\leq -b+i+k, j\in I\cup I'} p_j \right)$$

$$= \left(\prod_{-b+i+1\leq j\leq -b+i+k, j\notin I\cup I'} p_j - \prod_{-b+i+1\leq j\leq -b+i+k, j\notin I\cup I'} p_{j'} \right)$$

$$\cdot \left(\prod_{-b+i+1\leq j\leq -b+i+k, j\in I\cup I'} p_{j'} - \prod_{-b+i+1\leq j\leq -b+i+k, j\in I\cup I'} p_j \right)$$

$$- \left(\prod_{-b+i\leq j\leq -b+i+k, j\notin I\cup I'} p_j - \prod_{-b+i\leq j\leq -b+i+k, j\notin I\cup I'} p_{j'} \right)$$

$$\cdot \left(\prod_{-b+i+1\leq j\leq -b+i+k, j\in I\cup I'} p_{j'} - \prod_{-b+i+1\leq j\leq -b+i+k, j\in I\cup I'} p_j \right)$$

$$= (p_{-b+i+1}\cdots p_{-b+i+k})^I - (p_i \cdots p_{i+k})^I.$$

If $-b + i + k < n + b - i - k + 1$, then $-b + i + k \leq h$, and hence we have

$$(q_{-b+i}p_{-b+i+1}\cdots p_{-b+i+k})^I$$

$$= \left(\prod_{j=1}^{b-i} p_j p_{j'} \right) \left(\prod_{b-i+2\leq j\leq -b+i+k, j\notin I} p_j - \prod_{b-i+1\leq j\leq -b+i+k, j\notin I} p_{j'} \right)$$

$$\left(p_{b-i+1}q_{(b-i+1)'} \prod_{b-i+1\leq j\leq -b+i+k, j\in I} p_{j'} - p_{(b-i+1)'}q_{b-i+1} \prod_{b-i+1\leq j\leq -b+i+k, j\in I} p_j \right)$$

$$\geq 0$$

since $-b + i \in I \cup I'$ implies that

$$p_{b-i+1}q_{(b-i+1)'} - p_{(b-i+1)'}q_{b-i+1} = p_{b-i+1} - p_{(b-i+1)'} > 0.$$

Moreover, it is easy to verify that

$$(q_{-b+i}p_{-b+i+1}\cdots p_{-b+i+k})^I > 0$$

if and only if

$$\{j \mid b - i \leq j \leq -b + i + k, j \in I\} \neq \emptyset,$$
$$\{j \mid b - i \leq j \leq -b + i + k, j \notin I\} \neq \emptyset.$$

If $-b + i + k \geq n + b - i - k + 1$, then $n + b - i - k \leq h$, and hence

$$(q_{-b+i}p_{-b+i+1}\cdots p_{-b+i+k})^I$$
$$= \left(\prod_{j=1}^{b-i} p_j p_{j'}\right)\left(\prod_{\substack{b-i+2\leq j\leq n+b-i-k, j\notin I}} p_j - \prod_{\substack{b-i+1\leq j\leq n+b-i-k, j\notin I}} p_{j'}\right)$$
$$\left(p_{b-i+1}q_{(b-i+1)'}\prod_{\substack{b-i+1\leq j\leq n+b-i-k, j\in I}} p_{j'} - p_{(b-i+1)'}q_{b-i+1}\prod_{\substack{b-i+1\leq j\leq n+b-i-k, j\in I}} p_j\right)$$
$$\cdot\left(\prod_{\substack{n+b-i-k+1\leq j\leq h}} p_j p_{j'}\right)\gamma$$
$$\geq 0,$$

where

$$\gamma = \begin{cases} 1 & \text{if } n \text{ is even}, \\ p_{h+1} & \text{if } n \text{ is odd}. \end{cases}$$

Moreover,

$$(q_{-b+i}p_{-b+i+1}\cdots p_{-b+i+k})^I > 0$$

if and only if

$$\{j \mid b - i + 1 \leq j \leq n + b - i - k, j \in I\} \neq \emptyset,$$
$$\{j \mid b - i + 1 \leq j \leq n + b - i - k, j \notin I\} \neq \emptyset.$$

Finally, we note that a similar argument works in the case that $-b + i \notin I \cup I'$. □

Similarly, we can show the following.

LEMMA 2.6. *Suppose* $I = \{i \mid 1 \leq i \leq h, p_i < p_{i'}\}$. *Then, for* $i = b, \ldots, t$,

$$(p_{-b+i}\cdots p_{-b+i+k-1}q_{-b+i+k})^I = (p_{-b+i}\cdots p_{-b+i+k-1})^I - (p_{-b+i}p_{-b+i+1}\cdots p_{-b+i+k})^I$$
$$\geq 0.$$

LEMMA 2.7. *Suppose $I = \{i \mid 1 \leq i \leq h, p_i < p_{i'}\}$. Then*

$$(q_0 p_1 \cdots p_k q_{k+1})^I$$
$$= (p_1 \cdots p_k)^I - (p_0 \cdots p_k)^I - (p_1 \cdots p_{k+1})^I + (p_0 \cdots p_{k+1})^I$$
$$\geq 0,$$

and the strict inequality sign holds if and only if

$$\{j \mid 1 \leq j \leq n - k, j \in I\} \neq \emptyset,$$
$$\{j \mid 1 \leq j \leq n - k, j \notin I\} \neq \emptyset.$$

By Lemmas 2.4–2.7, we have

$$R(C_I) - R(C)$$
$$= \sum_{i=1}^{b-1} (q_{-b+i} p_{-b+i+1} \cdots p_{-b+i+k})^I + \sum_{i=b+2}^{t} (p_{-b+i} \cdots p_{-b+i+k-1} q_{-b+i+k})^I$$
$$+ (q_0 p_1 \cdots p_k q_{k+1})^I - \left(\prod_{j=0}^{k+1} p_j \right)^I.$$

Let $d = \lceil (n-k)/2 \rceil - 1$. Note that

$$\left(\prod_{j=0}^{k+1} p_j \right)^I$$
$$= \left(q_{-1} \prod_{j=0}^{k+1} p_j \right)^I + \left(\prod_{j=-1}^{k+1} p_j \right)^I$$
$$= \cdots$$
$$= \sum_{i=1}^{d} \left(q_{-i} \prod_{j=-i+1}^{k+i} p_j \right)^I + \sum_{i=1}^{n-k-2-d} \left(q_{i+k+1} \prod_{j=-i}^{i+k} p_j \right)^I + \left(\prod_{j=-d}^{n-d-1} p_j \right)^I$$

and

$$\left(\prod_{j=-d}^{n-d-1} p_j \right)^I = 0.$$

Thus, we have

(2.3)　　　$R(C_I) - R(C)$

$$= \sum_{i=1}^{b-d-1} \left(q_{-b+i} \prod_{j=-b+i+1}^{-b+i+k} p_j \right)^I$$
$$+ \sum_{i=1}^{d} \left[\left(q_{-i} \prod_{j=-i+1}^{-i+k} p_j \right)^I - \left(q_{-i} \prod_{j=-i+1}^{i+k} p_j \right)^I \right]$$

$$+ \sum_{i=b+n-k-d}^{t} \left(q_{-b+i+k} \prod_{j=-b+i}^{-b+i+k-1} p_j \right)^I$$

$$+ \sum_{i=1}^{n-k-2-d} \left[\left(q_{i+k+1} \prod_{j=i+1}^{i+k} p_j \right)^I - \left(q_{i+k+1} \prod_{j=-i}^{i+k} p_j \right)^I \right]$$

$$+ (q_0 p_1 \cdots p_k q_{k+1})^I.$$

This representation suggests that we show Lemmas 2.8 and 2.9.

LEMMA 2.8. *For* $i = 1, \ldots, d,$

$$\left(q_{-i} \prod_{j=-i+1}^{-i+k} p_j \right)^I \geq \left(q_{-i} \prod_{j=-i+1}^{i+k} p_j \right)^I,$$

and the inequality holds strictly if and only if

$$\left(q_{-i} \prod_{j=-i+1}^{-i+k} p_j \right)^I > 0.$$

Proof. First, consider the case that $-i \in I'$ and n is even. Denote

$$A = \prod_{-i+1 \leq j \leq -i+k, j \notin I \cup I'} p_j - \prod_{-i+1 \leq j \leq -i+k, j \notin I \cup I'} p_{j'},$$

$$B = \prod_{-i+1 \leq j \leq i+k, j \notin I \cup I'} p_j - \prod_{-i+1 \leq j \leq i+k, j \notin I \cup I'} p_{j'},$$

$$A' = q_{(-i)'} \prod_{-i+1 \leq j \leq -i+k, j \in I \cup I'} p_{j'} - q_{-i} \prod_{-i+1 \leq j \leq -i+k, j \notin I \cup I'} p_j,$$

$$B' = q_{(-i)'} \prod_{-i+1 \leq j \leq i+k, j \in I \cup I'} p_{j'} - q_{-i} \prod_{-i+1 \leq j \leq -i+k, j \notin I \cup I'} p_j.$$

Then

$$\left(q_{-i} \prod_{j=-i+1}^{-i+k} p_j \right)^I - \left(q_{-i} \prod_{j=-i+1}^{i+k} p_j \right)^I = AA' - BB'.$$

If $-i+k < n+i-k+1$, then $-i+k \leq h$. Hence

$$A - B$$

$$= \left(\prod_{1 \leq j \leq i, j \notin I} p_j p_{j'} \right) \left[\left(\prod_{i+1 \leq j \leq n-i-k, j \notin I} p_j \right) \alpha(1 - \beta\delta) \right.$$

$$\left. - \left(\prod_{i+1 \leq j \leq n-i-k, j \notin I} p_{j'} \right) \beta(1 - \alpha\delta) \right],$$

where

$$\alpha = \prod_{n-i-k+1 \le j \le -i+k, j \notin I} p_j,$$

$$\beta = \prod_{n-i-k+1 \le j \le -i+k, j \notin I} p_{j'},$$

$$\delta = \prod_{-i+k+1 \le j \le h, j \notin I} p_j p_{j'},$$

and

$$\alpha(1 - \beta\delta) - \beta(1 - \alpha\delta) = \alpha - \beta \ge 0.$$

Thus, $A \ge B$.

If $-i + k \ge n + i - k + 1$, then $n + i - k \le h$. Hence

$$A - B$$

$$= \left(\prod_{1 \le j \le i, j \notin I} p_j p_{j'} \right) \left(\prod_{n+i-k+1 \le j \le h, j \notin I} p_j p_{j'} \right) \left[\left(\prod_{i+1 \le j \le n-i-k, j \notin I} p_j \right) \alpha(1 - \beta) \right.$$

$$\left. - \left(\prod_{i+1 \le j \le n-i-k, j \notin I} p_{j'} \right) \beta(1 - \alpha) \right],$$

where

$$\alpha = \prod_{n-i-k+1 \le j \le n+i-k, j \notin I} p_j,$$

$$\beta = \prod_{n-i-k+1 \le j \le n+i-k, j \notin I} p_{j'},$$

and

$$\alpha(1 - \beta) - \beta(1 - \alpha) = \alpha - \beta \ge 0.$$

Thus, $A \ge B$.

Similarly, we have $A' \ge B'$. Therefore, $AA' \ge BB'$. By similar arguments, we can prove the inequalities in other cases.

Now it is easy to verify that $AA' > BB'$ if and only if

$$\{j \mid i + 1 \le j \le \min(-i + k, n + i - k), h \in I\} \ne \emptyset,$$
$$\{j \mid i + 1 \le j \le \min(-i + k, n + i - k), h \notin I\} \ne \emptyset$$

if and only if, or, by Lemma 2.5,

$$\left(q_{-i} \prod_{j=-i+1}^{-i+k} p_j \right)^I > 0. \qquad \square$$

Similarly, we can prove the following.

LEMMA 2.9. *For $i = 1, \ldots, n - k - 2 - d$,*

$$\left(q_{i+k+1} \prod_{j=i+1}^{i+k} p_j \right)^I \geq \left(q_{i+k+1} \prod_{j=-i}^{i+k} p_j \right)^I.$$

By Lemmas 2.5–2.9, all terms in the right-hand side of (2.3) are nonnegative. Next, we show that if $I \neq \emptyset$, then at least one term in (2.3) is positive.

Note that $1 \notin I$. Since $I \neq \emptyset$, there exists a positive integer r such that $1 \leq r < h$, $r \notin I$, and $r + 1 \in I$. If $r + 1 \leq n - k$, then

$$r \in \{j \mid 1 \leq j \leq n - k, j \notin I\},$$
$$r + 1 \in \{j \mid 1 \leq j \leq n - k, j \in I\},$$

and hence

$$q_0 p_1 \cdots p_k q_{k+1} > 0.$$

If $r + 1 > n - k$, then choose $i = b + n - k - (r + 1) < b$, and we have

$$b - i + 1 \leq r, r + 1 \leq \min(-b + i + k, n + b - i - k).$$

Hence, by Lemma 2.5,

$$q_i p_{i+1} \cdots p_{i+k} > 0.$$

Finally, we deal with the case that some equality signs hold in $0 \leq p_{[1]} \leq p_{[2]} \leq \cdots \leq p_{[n]} \leq 1$. If $p_{[1]} = p_{[2]} = \cdots = p_{[n]}$, then Theorem 2.1 is trivially true. If $p_{[i]} < p_{[i+1]}$, then, for sufficiently small $\varepsilon > 0$, we have

$$0 < p_{[1]} + \varepsilon < \cdots < p_{[i]} + i\varepsilon < p_{[i+1]} - (n - i)\varepsilon < \cdots < p_{[n]} - \varepsilon < 1.$$

For them, we already proved the optimality of assignment C^* in Theorem 2.1; that is, for any assignment C, $R(C^*) \geq R(C)$. Now we can complete our proof of Theorem 2.1 by setting $\varepsilon \to 0$.

3. Discussion. An invariant optimal assignment is a nice thing to have in practice and also an interesting mathematical problem to solve. The existence of an invariant optimal assignment has been widely studied for the consecutive-k-out-of-n: F systems and G systems, where a F system works if and only if there do not exist k consecutive components that all fail. Usually, the nonexistence of invariant optimal assignments was demonstrated [12, 7, 13]. There are only four nontrivial cases that invariant optimal assignments may exist. The first is an invariant optimal assignment for the consecutive-2-out-of-n: F line conjectured by Derman, Lieberman, and Ross [2] and independently proved by Du and Hwang [3] and Malon [11]. In fact, the former proved the harder cycle version which is the second case of existence. Note that the cycle version implies the line version since by setting $p_{[n]} = 1$ ($p_{[1]} = 0$ in the G system), the line problem is reduced to the cycle problem. The third case is an invariant optimal assignment for the consecutive-k-out-of-n: G line for $n \leq 2k$ conjectured by Kuo, Zhang, and Zuo [10], and proved by Jalali et al. [9]. The fourth case is its cycle version, the current case. Note that again the cycle version implies the line version but is much harder. In the line version, one needs only to prove the case

$n = 2k$, and the $n < 2k$ case can be reduced to the $n = 2k$ case. No similar reduction is possible for the cycle case. One may wonder whether a simpler proof exists by considering other pairings. In the current paper, we break the term $q_i p_{i+1} \cdots p_{i+k}$ into two parts, $p_{i+1} \cdots p_{i+k}$ and $-p_i \cdots p_{i+k}$. Use the pairing of $p_{-a+i} \cdots p_{-a+i+k-1}$ with $p_{(-a+i)'} \cdots p_{(-a+i+k-1)'}$ for the first part and a similar one for the second part; then compare the C assignment with the C_I assignment. However, since the comparison of one part is positive and the other is negative, we have to further compare their sizes, thus complicating the proof. Can we not break the term $q_i p_{i+1} \cdots p_{i+k}$ and find a pairing to work? One such possibility is also to consider the clockwise representation of $R(C)$, namely, $R(C) = p_1 \cdots p_n + \sum_{i=1}^{n} q_i p_{i-1} \cdots p_{i-k}$. We then pair each term $q_i p_{i+1} \cdots p_{i+k}$ from the counterclockwise representation with the term $q_{i'} p_{(i+1)'} \cdots p_{(i+k)'}$ from the clockwise representation. The C_I assignment is better than the C assignment in all cases except when $q_i < q_i'$ and i' does not belong to $i, \ldots, i+k$. The determination of invariant optimal assignments on lines and cycles is an application of the broader problem of finding an optimal permutation, linear or cyclic, under a certain objective function. This type of problem has been considered before [1, 8] when the arguments of the objective function are $|x_i - x_{i+1}|$ for all i. In the optimal assignment problem, the arguments are products like $q_i p_{i+1} \cdots p_{i+k}$, which seems to raise a new type of optimal permutation problem. In this paper we give a solution to one such problem and hope the approach may work for other similar problems [1, 8, 4, 5].

REFERENCES

[1] C.-C. Chao and W.-Q. Liang, *Arranging N distinct numbers on a line or a circle to reach extreme total variations*, European J. Combin., 13 (1992) pp. 325–334.

[2] C. Derman, G.J. Lieberman, and S.M. Ross, *On the consecutive-2-out-of-n: F system*, IEEE Trans. Rel., 31 (1982), pp. 57–63.

[3] D.-Z. Du and F.K. Hwang, *Optimal consecutive-2-out-of-n systems*, Math. Oper. Res., 11 (1986), pp. 187–191.

[4] D.-Z. Du and K.-I. Ko, *Some completeness results on decision trees and group testing*, SIAM J. Alg. Discr. Meth., 8 (1987), pp. 762–777.

[5] D.Z. Du and F.K. Hwang, *Optimal assembly of an s-stage k-out-of-n system*, SIAM J. Discrete Math., 3 (1990), pp. 349–354.

[6] D.-Z. Du, F.K. Hwang, Y. Jung, and H.Q. Ngo, *Optimal consecutive-k-out-of-$(2k+1)$: G cycle*, J. Global Optim., 19 (2001), pp. 51–60.

[7] F.K. Hwang, *Invariant permutations for consecutive-k-out-of-n cycles*, IEEE Trans. Rel., 38 (1989), pp. 65–67.

[8] F.K. Hwang, *Extreme permutations with respect to weak majorizations*, European J. Combin., 17 (1996), pp. 637–645.

[9] A. Jalali, A.G. Hawkes, L. Cui, and F.K. Hwang, *The Optimal Consecutive-k-out-of-n: G Line for $n \leq 2k$*, manuscript, 1999.

[10] W. Kuo, W. Zhang, and M. Zuo, *A consecutive-k-out-of-n: G system: The mirror image of a consecutive-k-out-of-n: F system*, IEEE Trans. Rel., 39 (1990), pp. 244–253.

[11] D.M. Malon, *Optimal consecutive-2-out-of-n: F component sequencing*, IEEE Trans. Rel., 33 (1984), pp. 414–418.

[12] D.M. Malon, *Optimal consecutive-k-out-of-n: F component sequencing*, IEEE Trans. Rel., 34 (1985), pp. 46–49.

[13] M. Zuo and W. Kuo, *Design and performance analysis of consecutive-k-out-of-n structure*, Naval Res. Logist., 37 (1990), pp. 203–230.