1-Rotationally Resolvable 4-Cycle Systems of $2K_v$

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Abstract: **In this article, it is shown that there exists a 1-rotationally resolvable 4-cycle system of $2K_v$ if and only if $v \equiv 0 \pmod 4$. To prove that, some special sequences of integers are utilized.** © 2002 Wiley Periodicals, Inc. J Combin Designs 10: 116–125, 2002; DOI 10.1002/jcd.10006

1. INTRODUCTION

For a graph G, let $V(G)$ be the vertex-set of G and $\mathcal{C}$ be a collection of cycles of length m (*m-cycles*) whose edges partition the edges of G. Then the pair $(V(G), \mathcal{C})$ is called an *m-cycle system* of G. An *m-cycle system* of λK_v is also referred to as a *λ-fold m-cycle system of order v.* Here λK_v is the graph on v vertices in which each pair of vertices is joined by exactly λ edges.

Let a pair $(V, \mathcal{C})$ be an *m-cycle system* of λK_v and Π be an *automorphism group* of the *m-cycle system* $(V, \mathcal{C})$, i.e., a group of permutations on v vertices leaving the collection $\mathcal{C}$ of cycles invariant. If there is an automorphism $\pi \in \Pi$ of order v, then the *m-cycle system* $(V, \mathcal{C})$ is said to be *cyclic.* If π is an automorphism of order $v - 1$ with a single fixed point, then the system $(V, \mathcal{C})$ is said to be 1-*rotational.* For

a 1-rotational m-cycle system of λK_v, the vertex-set V can be identified with $\{\infty\} \cup \mathbb{Z}_{v-1}$, i.e., a fixed point ∞ and the residue group of integers modulo $v - 1$. In this case, the automorphism can be represented by

$$\pi : \infty \mapsto \infty, i \mapsto i + 1 \pmod{(v-1)} \quad \text{or} \quad \pi = (\infty)(0, 1, \ldots, v-2)$$

acting on the vertex-set $V = \{\infty\} \cup \mathbb{Z}_{v-1}$.

Let $C \in \mathcal{C}$ be a cycle of a 1-rotational m-cycle system of λK_v, $(V, \mathcal{C})$. A *cycle orbit* of C is defined by $\{C + y : y \in \mathbb{Z}_{v-1}\}$. The *length* of a cycle orbit is its cardinality. A cycle orbit of length $v - 1$ is said to be *full*, otherwise *short*. A *base cycle* of a cycle orbit $\mathcal{O}$ is a cycle $C \in \mathcal{O}$ which is chosen arbitrarily. Any 1-rotational m-cycle system is generated from base cycles.

For an m-cycle system of λK_v, $(V, \mathcal{C})$, if the collection $\mathcal{C}$ of cycles can be partitioned into $s(= \lambda(v-1)/2)$ 2-factors (in terms of block designs, *resolution classes* or *parallel classes*), $R_1, \ldots, R_s$, then the system $(V, \mathcal{C})$ is said to be *resolvable* or to have *resolvability* and $\mathcal{R} = \{R_1, \ldots, R_s\}$ is called a *resolution* of the system. Obviously, for the existence of a resolvable m-cycle system of λK_v, m must divide v and $\lambda(v - 1) \equiv 0 \pmod 2$.

A 1-rotational m-cycle system is said to be 1-*rotationally resolvable* when it admits $\pi = (\infty)(0, 1, \ldots, v-2)$ as an automorphism leaving a resolution invariant. A *base resolution class* can be defined in a manner similar to a base cycle.

For m-cycle systems of λK_v, v is said to be (m, λ)-*admissible* if m divides $\lambda v(v-1)/2$, $\lambda(v-1) \equiv 0 \pmod 2$, and either $v = 1$ or $v \geq m$. The spectrum problem for m-cycle systems of λK_v has been investigated by many people (for the history of the problem, see [5]). Rodger [7] surveyed the existence results of m-cycle systems of λK_v and those with several properties including resolvability. However, as far as the authors know, necessary and sufficient conditions for an m-cycle system of λK_v with $\lambda \geq 2$ to be 1-rotational or resolvable are not available.

In this article, concerning the case $(m, \lambda) = (4, 2)$, we will show that there exists a 1-rotationally resolvable 4-cycle system of $2K_v$ if and only if $v \equiv 0 \pmod 4$, by use of extended Skolem sequences and some other similar sequences.

2. TRANSLATION OF THE PROBLEM

It is known that for any $(4, 2)$-admissible v, i.e., $v \equiv 0, 1 \pmod 4$, there exists a 4-cycle system of $2K_v$. For a 4-cycle system of $2K_v$ to be resolvable, it is necessary that 4 divides v. On the other hand, by noting that any 1-rotational 4-cycle system of $2K_v$ consists of $v/4$ full cycle orbits, $v \equiv 0 \pmod 4$ is a necessary condition also for the existence of a 1-rotational 4-cycle system of $2K_v$. Therefore we have the following.

Lemma 2.1. *A necessary condition for the existence of a* 1-*rotationally resolvable 4-cycle system of* $2K_v$ *is that* $v \equiv 0 \pmod 4$.

Here we should remark that a 1-rotationally resolvable 4-cycle system of $2K_v$ is closely related to a $\mathbb{Z}$-*cyclic whist tournament* Wh(v) with $v \equiv 0 \pmod 4$ (see [1] for the definition of a ($\mathbb{Z}$-cyclic) whist tournament). If the condition on partner pairs of a $\mathbb{Z}$-cyclic Wh$(4n)$ is omitted, then the design is regarded as a 1-rotationally resolvable

4-cycle system of $2K_{4n}$. Therefore, the existence of a $\mathbb{Z}$-cyclic $\mathrm{Wh}(4n)$ implies that of a 1-rotationally resolvable 4-cycle system of $2K_{4n}$, although the converse is not true in general. In fact, the known result (Theorem 53.8 in [1]) on a $\mathbb{Z}$-cyclic $\mathrm{Wh}(4n)$ ensures at least the following.

Lemma 2.2. *There exists a* 1-*rotationally resolvable* 4-*cycle system of* $2K_{4n}$ *when* $n \leq 16$.

A *k-extended Skolem sequence* of order t is a sequence $(s_1, \ldots, s_{2t+1})$ of $2t + 1$ integers in which $s_k = 0$ and for each $j \in \{1, \ldots, t\}$, there exists a unique $i \in \{1, \ldots, 2t + 1\} \setminus \{k\}$ such that $s_i = s_{i+j} = j$. A k-extended Skolem sequence of order t is also represented as a collection of ordered pairs $\{(a_j, b_j) : 1 \leq j \leq t, \; b_j - a_j = j\}$ with $\cup_{j=1}^{t} \{a_j, b_j\} = \{1, 2, \ldots, 2t + 1\} \setminus \{k\}$. If $k = t + 1$, the sequence is often referred to as a *Rosa sequence* or a *split Skolem sequence* (see [4] and [8]). Baker [2] settled the spectrum of k-extended Skolem sequences of order t.

It is well-known that base blocks for cyclic Steiner triple systems can be obtained from Skolem sequences and split Skolem sequences. For more details, the reader may see [9]. Skolem sequences and their generalizations are quite useful to get other combinatorial designs as well (see [2], [3], [6], etc.).

In this section, we will show that extended Skolem sequences with certain properties can also provide base cycles for 1-rotationally resolvable 4-cycle systems of $2K_v$, which is, in fact, a primary idea for proving the sufficiency of Lemma 2.1.

In what follows, a k-extended Skolem sequence of order t is denoted by k-ext $\mathcal{S}_t$ for simplicity.

Theorem 2.3 ([2]). *There exists a* k-ext $\mathcal{S}_t, 1 \leq k \leq 2t + 1$, *if and only if either*

(1) *k is odd and $t \equiv 0$ or 1 (mod 4); or*
(2) *k is even and $t \equiv 2$ or 3 (mod 4).*

Now, we shall show how to utilize a k-ext $\mathcal{S}_t$ to obtain a 1-rotationally resolvable 4-cycle system of $2K_{4t+4}$. Since any 1-rotationally resolvable 4-cycle system of $2K_{4t+4}$ consists of $t + 1$ full cycle orbits, it suffices to find the $t + 1$ base cycles which partition the vertex-set of K_{4t+4}.

It follows from Theorem 2.3 that there exist a $(t + 1)$-ext $\mathcal{S}_t$ if $t \equiv 0, 3$ (mod 4) and a t-ext $\mathcal{S}_t$ if $t \equiv 1, 2$ (mod 4). It should be mentioned that the necessary and sufficient condition for the existence of a $(t + 1)$-ext $\mathcal{S}_t$ was first shown by Rosa [8] in 1966. By using these facts, we will present two constructions for 1-rotationally resolvable 4-cycle systems of $2K_{4t+4}$ depending on the value of t.

Construction I (When $t \equiv 0, 3$ (mod 4)). Let $\{(a_j, b_j) : 1 \leq j \leq t\}$ be a $(t + 1)$-ext $\mathcal{S}_t$ and take $t + 1$ 4-cycles as follows:

$$\{(2a_j - 1, 2b_j - 1, 2a_j, 2b_j) : 1 \leq j \leq t\} \cup \{(\infty, 0, 2t + 1, 2t + 2)\}. \qquad (2.1)$$

Then it is easily verified that (2.1) can be the set of base cycles (and thus the base resolution class) for a 1-rotationally resolvable 4-cycle system of $2K_{4t+4}$.

Example 2.4. When $t = 7$. The sequence $(5, 3, 4, 7, 3, 5, 4, 0, 6, 2, 7, 2, 1, 1, 6)$ is an 8-ext $\mathcal{S}_7$, which is equivalently expressed by the collection $\{(13, 14), (10, 12), (2, 5),$

$(3, 7), (1, 6), (9, 15), (4, 11)\}$. Then the set of base cycles of the form (2.1) for a 1-rotationally resolvable 4-cycle system of $2K_{32}$ will be as follows:

$$\{(25, 27, 26, 28), (19, 23, 20, 24), (3, 9, 4, 10), (5, 13, 6, 14),$$
$$(1, 11, 2, 12), (17, 29, 18, 30), (7, 21, 8, 22), (\infty, 0, 15, 16)\}.$$

Construction II (When $t \equiv 1, 2$ (mod 4)). Assume that $\{(a_j, b_j) : 1 \leq j \leq t\}$ is a t-ext $\mathcal{S}_t$ satisfying the condition $(a_t, b_t) = (t + 1, 2t + 1)$. This time, take $t + 1$ base cycles in the following way:

$$\{(2a_j - 1, 2b_j - 1, 2a_j, 2b_j) : 1 \leq j \leq t - 1\}$$
$$\cup \{(0, 2t, 4t + 1, 2t - 1), (\infty, 2t + 1, 2t + 2, 4t + 2)\}. \tag{2.2}$$

Then it is straightforward to check that (2.2) is the set of base cycles (and thus the base resolution class) for the desired 4-cycle system.

Example 2.5. When $t = 6$. The collection $\{(10, 11), (2, 4), (9, 12), (1, 5), (3, 8), (7, 13)\}$ is a 6-ext $\mathcal{S}_6$ containing the pair $(t + 1, 2t + 1)$. It is remarked that the 6-ext $\mathcal{S}_6$ is also expressed as the sequence $(4, 2, 5, 2, 4, 0, 6, 5, 3, 1, 1, 3, 6)$. According to (2.2), the set of base cycles for a 1-rotationally resolvable 4-cycle system of $2K_{28}$ can be given as follows:

$$\{(19, 21, 20, 22), (3, 7, 4, 8), (17, 23, 18, 24), (1, 9, 2, 10),$$
$$(5, 15, 6, 16), (0, 12, 25, 11), (\infty, 13, 14, 26)\}.$$

We now know that by letting $v = 4t + 4$, the existence problem for 1-rotationally resolvable 4-cycle systems of $2K_v$ can be translated to that for suitable extended Skolem sequences of order t. That is, the existence of a 1-rotationally resolvable 4-cycle system of $2K_v$ when $v \equiv 0, 4$ (mod 16) is equivalent to that of a $(t + 1)$-ext $\mathcal{S}_t$ when $t \equiv 3, 0$ (mod 4), and so is the cases $v \equiv 8, 12$ (mod 16) if there exists a t-ext $\mathcal{S}_t$ satisfying the required condition in Construction II when $t \equiv 1, 2$ (mod 4).

3. WHEN $v \equiv 0, 4$ (mod 16)

Since the existence of a $(t + 1)$-ext $\mathcal{S}_t$ when $t \equiv 0, 3$ (mod 4) implies that of a 1-rotationally resolvable 4-cycle system of $2K_v$ when $v \equiv 4, 0$ (mod 16), Theorem 2.3 and Construction I ensure a half of the sufficiency of Lemma 2.1.

Proposition 3.1. *There exists a* 1-*rotationally resolvable* 4-*cycle system of* $2K_v$ *whenever* $v \equiv 0, 4$ (mod 16).

4. WHEN $v \equiv 8, 12$ (mod 16)

Although Theorem 2.3 assures the existence of a t-ext $\mathcal{S}_t$ when $t \equiv 1, 2$ (mod 4), it does not guarantee that the sequence satisfies the condition assumed at the beginning of Construction II. That means, if only we can show the existence of a t-ext $\mathcal{S}_t$ which includes the pair $(t + 1, 2t + 1)$ when $t \equiv 1, 2$ (mod 4), then the other half of the

sufficiency of Lemma 2.1, i.e., the cases $v \equiv 8, 12 \pmod{16}$, will be proved. Actually it suffices to care for $t \geq 16$, since we have Lemma 2.2. However, the existence problem of a t-ext $\mathcal{S}_t$ satisfying the required condition is of interest in its own right and therefore we will investigate it for all $t \geq 1$ in this section.

Before going into the discussion, we will give a couple of definitions which were brought by Baker [2] as generalizations of an extended Skolem sequence. Those sequences will be useful in constructing the required t-ext $\mathcal{S}_t$.

For odd n, a k-ext $\mathcal{O}_n$ is a sequence $(s_1, \ldots, s_{n+2})$ of $n + 2$ integers, $s_k = 0$ and all the remaining entries odd with the property that for every $j \in \{0, \ldots, (n-1)/2\}$, there exists a unique $i \in \{1, \ldots, n+2\} \setminus \{k\}$ such that $s_i = s_{i+2j+1} = 2j + 1$. Baker [2] proved that a 3-ext $\mathcal{O}_n$ exists for any odd $n \geq 5$.

A $\{p, q\}$-ext $\mathcal{S}_m$ for $p, q \in \{1, \ldots, 2m + 2\}$ is a sequence $(s_1, \ldots, s_{2m+2})$ of $2m + 2$ integers satisfying that $s_p = s_q = 0$ and for every $j \in \{1, \ldots, m\}$, there exists a unique $i \in \{1, \ldots, 2m + 2\} \setminus \{p, q\}$ such that $s_i = s_{i+j} = j$. Of course, we may write these two sequences as collections of ordered pairs as we did in Section 2.

The following technique will also be helpful as it is in [2]: a sequence can be *doubled* and used to fill a sequence of either even or odd positions in a larger sequence. For example, $(1, 1, 2, 3, 2, 0, 3)$ can be doubled to give $(2,-,2,-,4,-,6,-,4,-,0,-,6)$.

Let $(s_1, \ldots, s_{2t+1})$ be the required t-ext $\mathcal{S}_t$. Since s_{t+1} and s_{2t+1} are fixed as assumed in Construction II, i.e., $s_t = s_{2t+1} = t$, we can describe the existence problem of such a t-ext $\mathcal{S}_t$ as that of a $\{t, t+1\}$-ext $\mathcal{S}_{t-1}$ by omitting s_{2t+1} and putting $s_{t+1} = 0$. Now, partition $2t - 2$ entries $\{s_i : i = 1, \ldots, t-1, t+2, \ldots, 2t\}$ of the $\{t, t+1\}$-ext $\mathcal{S}_{t-1}$ into two subsets

$$S_1 = \{s_1, \ldots, s_{t-1}\} \quad \text{and} \quad S_2 = \{s_{t+2}, \ldots, s_{2t}\}$$

of size $t - 1$ each. We will look at the cases $t \equiv 1$ and $2 \pmod 4$ independently and will fix some entries in the $\{t, t+1\}$-ext $\mathcal{S}_{t-1}$ beforehand at which certain integers are allocated. In what follows, conforming to the expression as in [2], we say that j is in $(i, i+j)$, or, just that the sequence contains the pair $(i, i+j)$, if $s_i = s_{i+j} = j$.

Case I. When $t \equiv 2 \pmod 4$. Let $t = 4n + 2$ for $n \geq 1$ and fix $2n - 1$ ordered pairs of the $\{t, t+1\}$-ext $\mathcal{S}_{t-1}$ as follows:

$$2n + 3 + r \text{ in } (4n - 1 - 2r, 6n + 2 - r), \quad 0 \leq r \leq 2n - 2. \tag{4.1}$$

Note that $s_{4n-1-2r} \in S_1$ and $s_{6n+2-r} \in S_2$. Then the $4n + 4$ remaining entries in S_1 and S_2, more precisely, the $2n + 2$ entries $s_1, s_2, s_4, \ldots, s_{4n-2}, s_{4n}, s_{4n+1}$ in S_1 and the $2n + 2$ consecutive entries $s_{6n+3}, \ldots, s_{8n+4}$ in S_2, are left for the $2n + 2$ integers, $1, \ldots, 2n + 2$, to be allocated.

Case II. When $t \equiv 1 \pmod 4$. Let $t = 4n + 5$ for $n \geq 1$ and fix $2n + 2$ ordered pairs of the sequence in the following way:

$$
\begin{aligned}
2n + 3 \quad &\text{in} \quad (4n + 4, 6n + 7); \\
2n + 4 \quad &\text{in} \quad (4n + 2, 6n + 6); \\
2n + 5 \quad &\text{in} \quad (4n + 3, 6n + 8); \\
2n + 6 + r \quad &\text{in} \quad (4n - 1 - 2r, 6n + 5 - r), \quad 0 \leq r \leq 2n - 2. \quad (4.2)
\end{aligned}
$$

Again $4n + 4$ entries remain for the $2n + 2$ integers, $1, \ldots, 2n + 2$, to be allocated. In this case, the same $2n + 2$ entries as in Case I are left in S_1 and the $2n + 2$ consecutive entries $s_{6n+9}, \ldots, s_{8n+4}$ are left in S_2.

Now, it is turned out that the two cases I and II eventually lead to the same allocation problem.

Lemma 4.1. *For $n \geq 1$, let J_1 and J_2 be two disjoint sets of $n + 1$ integers each such that $J_1 \cup J_2 = \{1, \ldots, 2n + 2\}$. If there exist*

(a) *a sequence $\mathcal{A} = (\alpha_1, \ldots, \alpha_{4n+1})$ of $4n + 1$ integers in which $\alpha_3 = \alpha_5 = \cdots = \alpha_{4n-3} = \alpha_{4n-1} = 0$ and for each $j \in J_1$, there exists a unique i such that $i, i + j \in \{1, \ldots, 4n + 1\} \setminus \{3, 5, \ldots, 4n - 3, 4n - 1\}$ and $\alpha_i = \alpha_{i+j} = j$; and*

(b) *a sequence $\mathcal{B} = (\beta_1, \ldots, \beta_{2n+2})$ of $2n + 2$ integers in which for each $j \in J_2$, there exists a unique $i \in \{1, \ldots, 2n + 2\}$ such that $\beta_i = \beta_{i+j} = j$,*

then there exists a t-ext $\mathcal{S}_t$ containing the pair $(t + 1, 2t + 1)$ whenever $t \geq 6$ and $t \equiv 1, 2 \pmod{4}$.

Note that the odd positions, except the 1st and the $(4n + 1)$-th entries, of the sequence $\mathcal{A}$ are supposed to be filled with (4.1) or (4.2). Then to prove the other half of the sufficiency of Lemma 2.1, we have only to show the existence of the sequences $\mathcal{A}$ and $\mathcal{B}$ of Lemma 4.1, if we admit the setting of Cases I and II.

Here, suppose that for the sequence $\mathcal{A}$,

$$
\begin{aligned}
2n - 3 &\quad \text{in} \quad (1, 2n - 2); \\
2n + 1 &\quad \text{in} \quad (2n, 4n + 1),
\end{aligned}
\tag{4.3}
$$

and all the remaining entries at even positions are even. This means that finding a proper allocation for those even positions of $\mathcal{A}$ is equivalent to finding a sequence $\mathcal{A}' = (\alpha'_1, \ldots, \alpha'_{2n})$ of $2n$ integers in which $\alpha'_{n-1} = \alpha'_n = 0$ and for each j of suitable $n - 1$ integers in $\{1, \ldots, n + 1\}$, there exists a unique $i \in \{1, \ldots, 2n\} \setminus \{n - 1, n\}$ such that $\alpha'_i = \alpha'_{i+j} = j$. That is, if only there exists such a sequence $\mathcal{A}'$, then it will be doubled and combined with (4.3) to give the desired sequence $\mathcal{A}$. It may be mentioned that the required property for the sequence $\mathcal{A}'$ is quite similar to that for a $\{m, m + 1\}$-ext $\mathcal{S}_m$. It differs only on the range of integers j to be allocated.

Theorem 4.2. *Whenever $m \geq 2$, there exists a sequence $(s_1, \ldots, s_{2m+2})$ of $2m + 2$ integers such that $s_m = s_{m+1} = 0$ and there exists a unique $i \in \{1, \ldots, 2m + 2\} \setminus \{m, m + 1\}$ satisfying $s_i = s_{i+j} = j$ for each*

(1) *$j \in \{1, \ldots, m - 1, m + 2\}$ if $m \equiv 0, 1 \pmod{4}$; or*
(2) *$j \in \{1, \ldots, m - 2, m, m + 2\}$ if $m \equiv 2, 3 \pmod{4}$.*

Proof.   **Case 1.** $m \equiv 0$ (mod 4). Let $m = 4M$. For $M \geq 2$, put

$$
\begin{aligned}
4M + 2 \quad &\text{in} \quad (2M, 6M + 2); \\
2M - 1 \quad &\text{in} \quad (4M + 2, 6M + 1); \\
2r + 1 \quad &\text{in} \quad (6M + 1 - r, 6M + 2 + r), \quad 1 \leq r \leq M - 2; \\
&\phantom{\text{in} \quad} (6M + 2 - r, 6M + 3 + r), \quad M \leq r \leq 2M - 1; \\
2r \quad &\text{in} \quad (2M - r, 2M + r), \quad 1 \leq r \leq 2M - 1; \\
1 \quad &\text{in} \quad (7M + 1, 7M + 2); \text{ and} \\
0 \quad &\text{at} \quad 4M \text{ and } 4M + 1.
\end{aligned}
$$

For $M = 1$, the desired sequence is given by $(6, 1, 1, 0, 0, 3, 6, 2, 3, 2)$.

   Case 2. $m \equiv 1$ (mod 4). Let $m = 4M + 1$. For $M \geq 1$, put

$$
\begin{aligned}
4M + 3 \quad &\text{in} \quad (2M + 1, 6M + 4); \\
2M + 1 \quad &\text{in} \quad (2M + 2, 4M + 3); \\
2r + 1 \quad &\text{in} \quad (2M + 1 - r, 2M + 2 + r), \quad 1 \leq r \leq M - 1; \\
&\phantom{\text{in} \quad} (2M - r, 2M + 1 + r), \quad M + 1 \leq r \leq 2M - 1; \\
2r \quad &\text{in} \quad (6M + 4 - r, 6M + 4 + r), \quad 1 \leq r \leq 2M; \\
1 \quad &\text{in} \quad (M, M + 1); \text{ and} \\
0 \quad &\text{at} \quad 4M + 1 \text{ and } 4M + 2.
\end{aligned}
$$

   Case 3. $m \equiv 2$ (mod 4). Let $m = 4M + 2$. For $M \geq 2$, put

$$
\begin{aligned}
4M + 4 \quad &\text{in} \quad (2M + 1, 6M + 5); \\
2M - 1 \quad &\text{in} \quad (2M + 2, 4M + 1); \\
2r + 1 \quad &\text{in} \quad (2M + 1 - r, 2M + 2 + r), \quad 1 \leq r \leq M - 2; \\
&\phantom{\text{in} \quad} (2M - r, 2M + 1 + r), \quad M \leq r \leq 2M - 1; \\
2r \quad &\text{in} \quad (6M + 5 - r, 6M + 5 + r), \quad 1 \leq r \leq 2M + 1; \\
1 \quad &\text{in} \quad (M + 1, M + 2); \text{ and} \\
0 \quad &\text{at} \quad 4M + 2 \text{ and } 4M + 3.
\end{aligned}
$$

For $M = 0$ and 1, the desired sequences are given by $(4, 0, 0, 2, 4, 2)$ and $(8, 1, 1, 4, 6, 0, 0, 4, 8, 3, 6, 2, 3, 2)$, respectively.

   Case 4. $m \equiv 3$ (mod 4). Let $m = 4M + 3$. For $M \geq 2$, put

$$
\begin{aligned}
4M + 5 \quad &\text{in} \quad (2M + 1, 6M + 6); \\
2M + 3 \quad &\text{in} \quad (4M + 2, 6M + 5); \\
2r + 1 \quad &\text{in} \quad (6M + 5 - r, 6M + 6 + r), \quad 1 \leq r \leq M; \\
&\phantom{\text{in} \quad} (6M + 6 - r, 6M + 7 + r), \quad M + 2 \leq r \leq 2M + 1; \\
2r \quad &\text{in} \quad (2M + 1 - r, 2M + 1 + r), \quad 1 \leq r \leq 2M; \\
1 \quad &\text{in} \quad (7M + 7, 7M + 8); \text{ and} \\
0 \quad &\text{at} \quad 4M + 3 \text{ and } 4M + 4.
\end{aligned}
$$

For $M = 0$ and 1, the desired sequences are given by $(5, 3, 0, 0, 3, 5, 1, 1)$ and $(4, 2, 9, 2, 4, 5, 0, 0, 7, 3, 5, 9, 3, 1, 1, 7)$, respectively. $\square$

It should be remarked that to obtain the sequence $\mathcal{A}'$, Theorem 4.2 can be applied with $m = n - 1$.

We will need a companion result to Theorem 4.2 for the sequence $\mathcal{B}$ in Lemma 4.1.

Theorem 4.3. *Whenever $n \geq 3$, there exists a sequence $(s_1, \ldots, s_{2n+2})$ of $2n + 2$ integers such that there exists a unique $i \in \{1, \ldots, 2n + 2\}$ satisfying $s_i = s_{i+j} = j$ for each*

(1) $j \in \{1, 3, \ldots, 2n - 7, 2n - 5, 2n - 2, 2n - 1, 2n\}$ *if $n \equiv 1, 2 \pmod 4$; or*

(2) $j \in \{1, 3, \ldots, 2n - 7, 2n - 5, 2n - 4, 2n - 1, 2n\}$ *if $n \equiv 0, 3 \pmod 4$.*

Proof. **Case 1.** $n \equiv 1, 2 \pmod 4$. For $n \geq 3$ (thus $n \geq 5$), put

$$
\begin{aligned}
2n & \quad \text{in} \quad (2, 2n + 2); \\
2n - 1 & \quad \text{in} \quad (1, 2n); \\
2n - 2 & \quad \text{in} \quad (3, 2n + 1); \text{ and} \\
2r + 1 & \quad \text{in} \quad (n + 1 - r, n + 2 + r), \quad 0 \leq r \leq n - 3.
\end{aligned}
$$

Case 2. $n \equiv 0, 3 \pmod 4$. When $n = 3$ and 4, the required sequences are given by $(5, 6, 1, 1, 2, 5, 2, 6)$ and $(7, 8, 4, 1, 1, 3, 4, 7, 3, 8)$, respectively. For $n \geq 5$ (thus $n \geq 7$), put $2n$ in $(2, 2n + 2)$ and $2n - 1$ in $(1, 2n)$. Since there exists a 3-ext $\mathcal{O}_p$ for any odd $p \geq 5$ (see Remark of Lemma 2 in [2]), there exists a 3-ext $\mathcal{O}_{2n-5}$ whenever $n \geq 5$. Fill the 3-ext $\mathcal{O}_{2n-5}$ in $(s_3, \ldots, s_{2n-1})$. Then $(s_3, \ldots, s_{2n-1})$ consists of odd integers $2r + 1$, $0 \leq r \leq n - 3$, and $s_5 = 0$. Thus by putting $2n - 4$ in $(5, 2n + 1)$, the desired sequence can be obtained. $\square$

In consequence of Lemma 4.1, Theorems 4.2 and 4.3, the existence of a t-ext $\mathcal{S}_t$ satisfying the condition required in Construction II is guaranteed whenever $t \equiv 1, 2 \pmod 4$ and $t \geq 14$. This result, together with Lemma 2.2, proves the other half of the sufficiency of Lemma 2.1. But for our interest, we will further investigate the existence of a t-ext $\mathcal{S}_t$ containing the pair $(t + 1, 2t + 1)$ for the rest cases, i.e., $t = 1, 2, 5, 6, 9, 10,$ and 13.

Lemma 4.4. *When $t = 6, 9, 10,$ and 13, there exists a $\{t, t + 1\}$-ext $\mathcal{S}_{t-1}$.*

Proof. When $t = 6$ and 9. For the case $t = 6$, put

$$
\begin{aligned}
5 & \quad \text{in} \quad (3, 8); \quad\quad 4 \quad \text{in} \quad (1, 5)^*; \quad\quad 3 \quad \text{in} \quad (9, 12)^{**}; \\
2 & \quad \text{in} \quad (2, 4)^*; \quad\quad 1 \quad \text{in} \quad (10, 11)^{**}; \\
0 & \quad \text{at} \quad 6 \text{ and } 7.
\end{aligned}
$$

For the case $t = 9$, the pairs with $*$ are used as they are, the pairs $(x, y)^{**}$ are replaced by $(x, y) + 6$, and the rest are given as follows:

$$8 \quad \text{in} \quad (3, 11); \qquad 7 \quad \text{in} \quad (7, 14); \quad 6 \quad \text{in} \quad (6, 12);$$
$$5 \quad \text{in} \quad (8, 13);$$
$$0 \quad \text{at} \quad 9 \text{ and } 10.$$

When $t = 10$ and 13. For the case $t = 10$, put

$$9 \quad \text{in} \quad (3, 12); \qquad 8 \quad \text{in} \quad (5, 13); \quad 7 \quad \text{in} \quad (9, 16);$$
$$6 \quad \text{in} \quad (14, 20)^{**}; \quad 5 \quad \text{in} \quad (2, 7)^{*}; \quad 4 \quad \text{in} \quad (15, 19)^{**};$$
$$3 \quad \text{in} \quad (1, 4)^{*}; \qquad 2 \quad \text{in} \quad (6, 8)^{*}; \quad 1 \quad \text{in} \quad (17, 18)^{**};$$
$$0 \quad \text{at} \quad 10 \text{ and } 11.$$

For the case $t = 13$, the pairs with $*$ are used as they are, the pairs $(x, y)^{**}$ are replaced by $(x, y) + 6$ and the others are given as follows:

$$12 \quad \text{in} \quad (3, 15); \qquad 11 \quad \text{in} \quad (5, 16); \quad 10 \quad \text{in} \quad (12, 22);$$
$$9 \quad \text{in} \quad (10, 19); \qquad 8 \quad \text{in} \quad (9, 17); \quad 7 \quad \text{in} \quad (11, 18);$$
$$0 \quad \text{at} \quad 13 \text{ and } 14. \hspace{3cm} \square$$

Lemma 4.1, Theorems 4.2 and 4.3, and Lemma 4.4 enable us to state the following.

Theorem 4.5. *Whenever* $t \equiv 1, 2 \pmod 4$ *and* $t \geq 6$, *there exists a* t-*ext* $\mathcal{S}_t$ *containing the pair* $(t + 1, 2t + 1)$.

Unfortunately it can be checked even by hand that when $t = 1, 2$, and 5, there does not exist a t-ext $\mathcal{S}_t$ satisfying the required condition. This means that Construction II cannot be applied for the cases $v = 8, 12$, and 24, which should be covered by Lemma 2.2 instead.

By Lemma 2.2, Proposition 3.1, and Theorem 4.5, the main theorem is established after all.

Theorem 4.6. *There exists a* 1-*rotationally resolvable* 4-*cycle system of* $2K_v$ *if and only if* $v \equiv 0 \pmod 4$.

Remark. Theorem 4.6 eventually shows the necessary and sufficient condition for the existence both of a 1-rotational 4-cycle system of $2K_v$ and a resolvable 4-cycle system of $2K_v$.

systems of 2K_v and $\mathbb{Z}$-cyclic whist tournaments Wh(v). The authors got the idea of this work while they were visiting the University of Queensland and the University of Newcastle, Australia. They are thankful for the generous support from those universities.

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