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A GLOBAL PINCHING THEOREM FOR SURFACES WITH CONSTANT MEAN CURVATURE IN S^3

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ABSTRACT. Let M be a compact immersed surface in the unit sphere S^3 with constant mean curvature H. Denote by ϕ the linear map from $T_p(M)$ into $T_p(M)$, $\phi = A - \frac{H}{2}I$, where A is the linear map associated to the second fundamental form and I is the identity map. Let Φ denote the square of the length of ϕ. We prove that if $||\Phi||_{L^2} \leq C$, then M is either totally umbilical or an $H(r)$-torus, where C is a constant depending only on the mean curvature H.

1. Introduction

Let M be a compact immersed hypersurface in the unit sphere S^{n+1} with constant mean curvature H. Denote by $h = [h_{ij}]$ the second fundamental form of M and by ϕ the tensor $\phi_{ij} = h_{ij} - \frac{H}{n}\delta_{ij}$. Let Φ denote the square of the length of ϕ. It is well known that if $H = 0$ and $0 \leq \Phi \leq n$, then M is either the equatorial sphere or a Clifford torus [3]. Recently, H. Alencar and M. do Carmo extended the above result to a hypersurface M with constant mean curvature H [1]. They proved that M is either totally umbilical or an $H(r)$-torus if Φ satisfies a certain pointwise pinching condition. In 1989, C. L. Shen proved that a minimal hypersurface M is totally geodesic if M is of nonnegative sectional curvature, and Φ satisfies a certain global pinching condition [8]. Later, the first author improved a result of Shen in the case of $n = 2$ and found a sharp bound concerning the global pinching condition [6]. The purpose of this paper is to extend our global theorem to a surface M with constant mean curvature H and obtain the best constant.

Before stating our main result, let B be the constant $B = 2 + \frac{H^2}{2}$ and $m(B)$ be the maximum value of the function $q(x) = 2\sqrt{2}\frac{\sqrt{B}x^2+2(B-2)x+B\sqrt{B}}{(\sqrt{B}+x)^2((x^2-B)^2+8x^2)}$ on $[0, \infty)$. The following is our main result.

Theorem 1.1. *Let M be a compact immersed surface in the unit sphere S^3 with constant mean curvature H. Then*

$$||\Phi||_2 \geq 2\pi\sqrt{\frac{2g}{M(B)}},$$

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where g is the genus of M and $||\cdot||_2$ is the L^2-norm. The equality holds if and only if M is either totally umbilical or an $H(r)$-torus. In particular, if $||\Phi||_2 \leq 2\pi\sqrt{\frac{2}{M(B)}}$, then M is either totally umbilical or an $H(r)$-torus.

It should be noted that for M to be either totally umbilical or an $H(r)$-torus, $||\Phi||_2 = 2\pi\sqrt{2g}(\frac{B}{2})^{\frac{3}{4}}$. It turns out that there exists a constant $H_0 \geq 2$ such that $m(B) = (\frac{2}{B})^{\frac{3}{2}}$ for all $|H| \leq H_0$. As a immediate consequence of the above result, we state

Corollary 1.2. *Let M be a compact immersed surface in the unit sphere S^3 with constant mean curvature H, $|H| \leq H_0$. Then*

$$||\Phi||_2 \geq 2\pi\sqrt{2g}(\frac{B}{2})^{\frac{3}{4}},$$

where g is the genus of M and $||\cdot||_2$ is the L^2-norm. The equality holds if and only if M is either totally umbilical or an $H(r)$-torus. In particular, if $||\Phi||_2 \leq 2\pi\sqrt{2}(\frac{B}{2})^{\frac{3}{4}}$, then M is either totally umbilic or an $H(r)$-torus.

For the proof of the main theorem, we shall need the following Bernstein-Hopf theorem (see [2], [5]).

Theorem 1.3. *Let M be a compact immersed surface in the unit sphere S^3 with constant mean curvature H. If M is a topological sphere, then M is totally umbilical.*

2. Notations and auxiliary results

Let M be a compact connected immersed surface in the unit sphere S^3. Following the notations of [1] and [3],

Lemma 2.1. $\frac{1}{2}\Delta\Phi = \Phi(B-\Phi)+\sum\phi_{ijk}^2$ *where ϕ_{ijk} denote the covariant derivative of ϕ_{ij}.*

Lemma 2.2. $|\nabla\Phi|^2 = 2\Phi\sum\phi_{ijk}^2$.

Lemma 2.3. *If $\Phi \geq B$, then Φ is a constant function, $\Phi \equiv B$, and M is an $H(r)$-torus.*

Proof. By Lemmas 2.1 and 2.2, we have $\frac{1}{2}\Delta\log\Phi = B - \Phi$ at the points where Φ is positive. It follows that $\Phi = B$ on M. □

According to Lemma 2.1 and Lemma 2.3, we see that if Φ is a constant function, then either $\Phi = 0$ or $\Phi = B$.

In the minimal case, H. B. Lawson proved that the set of all zeros of Φ is either the whole space M or at most a finite set of points [7]. We need the following analogous result for the case that M is with constant mean curvature.

Lemma 2.4. *The set of all zeros of Φ is either the whole M or at most a finite set of points.*

Proof. The proof of the lemma is similar to that of Lawson. Let us sketch the proof for completeness. We use an isothermal coordinate (u, v) on a neighborhood D in M. Denote the position vector of this immersion by X and the unit normal of M in S^3 by N. Then the mean curvature H and the Gaussian curvature K are given

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by $\frac{1}{F^2}(X_{uu}+X_{vv})\cdot N$ and $\frac{1}{F^4}[(X_{uu}\cdot N)(X_{vv}\cdot N)-(X_{uv}\cdot N)^2]$ respectively, where $F^2 = X_u\cdot X_u = X_v\cdot X_v$.

Since M is of constant mean curvature, the Weingarten equations imply that $X_{uv}\cdot N$ is harmonic in the (u,v) coordinate, $X_{uu}\cdot N$ and $X_{vv}\cdot N$ differ by a constant if $X_{uv}\cdot N$ is constant on D, and the zero set of $X_{uv}\cdot N$ and that of $(X_{uu}-X_{vv})\cdot N$ intersect transversely at the points where the gradient of $X_{uv}\cdot N$ does not vanish. Let G be the set of all points where the gradient of $X_{uv}\cdot N$ vanishes. Since $X_{uv}\cdot N$ is harmonic, G is either isolated or the whole D. For G being isolated, the set of all zeros of Φ is isolated. In the other case, the set of all zeros of Φ is either empty or the whole D. □

Lemma 2.5. *If M is not totally umbilical, then*

$$\lim_{\epsilon\to 0}\sum_{i=1}^{k}\int_{\partial B_\epsilon(p_i)}\frac{\Phi_r}{\Phi} = 16\pi(g-1)$$

where $p_1, p_2, \cdots, p_k$ constitute all the zeros of Φ and Φ_r denotes the derivative of Φ on $\partial B_\epsilon(p_i)$ in the radial direction from p_i. In particular, if Φ is positive on M, then M is a topological torus.

Proof. At the points where Φ is positive, by Lemma 2.2, we get

$$\Delta\log\Phi = B-\Phi. \tag{2.1}$$

Integrating (2.1) over $M_\epsilon = M\setminus\bigcup_{i=1}^{k}B_\epsilon(p_i)$, we get, from the Gauss equation

$$2K = B-\Phi, \tag{2.2}$$

where K is the Gaussian curvature of M, the assertion by Stokes's theorem and the theorem of Gauss-Bonnet. □

Lemma 2.6.

$$\int_M\Big(\frac{\pi}{2}+\tan^{-1}\frac{H-\sqrt{2\Phi}}{2}-\tan^{-1}\frac{H+\sqrt{2\Phi}}{2}\Big)(B-\Phi)+2\sqrt{2\Phi}\geq 4\pi^2(1+g).$$

Proof. Regard M as an immersed surface of $\mathbb{R}^4$. Then the total absolute curvature of M in the sense of [4] is given by

$$\begin{aligned} T(M) &= \int_M\int_0^{2\pi}\Big|\Big(\sin\theta+\frac{H+\sqrt{2S-H^2}}{2}\cos\theta\Big)\Big(\sin\theta+\frac{H-\sqrt{2S-H^2}}{2}\cos\theta\Big)\Big|d\theta dV \\ &= \int_M\Big(\frac{\pi}{2}+\tan^{-1}\frac{H-\sqrt{2\Phi}}{2}-\tan^{-1}\frac{H+\sqrt{2\Phi}}{2}\Big)(B-\Phi)+2\sqrt{2\Phi}. \end{aligned}$$

By the well-known inequality of Chern-Lashof [4], we have

$$T(M)\geq\frac{\pi^2}{2}(b_0+b_1+b_2),$$

where b_i is the ith Betti number relative to the real field, for $i=0,1,2$. Since M is two-dimensional, $b_0=1$, $b_1=2g$ and $b_2=1$. □

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3. Proof of main results

We are now in position to prove the main result of Theorem 1.1. We may assume that Φ is positive except possibly at a finite set of points (see Lemma 2.4). By Lemmas 2.2 and 2.6, we get

$$
\begin{aligned}
&\int_M 2\sqrt{2B} + m(B)\sum \phi_{ijk}^2 - (\frac{\pi}{2} + \tan^{-1}\frac{H-\sqrt{2\Phi}}{2} - \tan^{-1}\frac{H+\sqrt{2\Phi}}{2})(B-\Phi) - 2\sqrt{2\Phi}\\
&= \int_M m(B)\sum \phi_{ijk}^2 + [\frac{2\sqrt{2}}{\sqrt{B}+\sqrt{\Phi}} - (\frac{\pi}{2} + \tan^{-1}\frac{H-\sqrt{2\Phi}}{2} - \tan^{-1}\frac{H+\sqrt{2\Phi}}{2})](B-\Phi)\\
&= \lim_{\epsilon\to 0}\int_{M_\epsilon} m(B)\frac{|\nabla\Phi|^2}{2\Phi} + \frac{1}{2}[\frac{2\sqrt{2}}{\sqrt{B}+\sqrt{\Phi}} - (\frac{\pi}{2} + \tan^{-1}\frac{H-\sqrt{2\Phi}}{2} - \tan^{-1}\frac{H+\sqrt{2\Phi}}{2})]\Delta\log\Phi\\
&= \lim_{\epsilon\to 0}\int_{M_\epsilon} m(B)\frac{|\nabla\Phi|^2}{2\Phi} + \frac{1}{2}\nabla[\frac{2\sqrt{2}}{\sqrt{B}+\sqrt{\Phi}} - (\frac{\pi}{2} + \tan^{-1}\frac{H-\sqrt{2\Phi}}{2} - \tan^{-1}\frac{H+\sqrt{2\Phi}}{2})]\nabla\log\Phi\\
&- \lim_{\epsilon\to 0}\int_{\partial M_\epsilon}[\frac{2\sqrt{2}}{\sqrt{B}+\sqrt{\Phi}} - (\frac{\pi}{2} + \tan^{-1}\frac{H-\sqrt{2\Phi}}{2} - \tan^{-1}\frac{H+\sqrt{2\Phi}}{2})]\frac{\Phi_r}{2\Phi}\\
&= 8\pi(1-g)(\frac{2\sqrt{2}}{\sqrt{B}} - \frac{\pi}{2}) + \int_M [m(B) - 2\sqrt{2}\frac{\sqrt{B}\Phi^2 + 2(B-2)\Phi + B\sqrt{B}}{(\sqrt{B}+\Phi)^2((\Phi^2-B)^2+8\Phi^2)}]\sum\phi_{ijk}^2\\
&\geq 8\pi(1-g)(\frac{2\sqrt{2}}{\sqrt{B}} - \frac{\pi}{2}),
\end{aligned}
$$

where the equality holds if and only if Φ is constant. On the other hand, according to Lemma 2.6, we get

$$2\sqrt{2B}Area(M) + m(B)\int_M \sum \phi_{ijk}^2 \geq 8\pi^2 g - 2\sqrt{\frac{2}{B}}8\pi(g-1). \tag{3.1}$$

By combining (2.2) with the inequality (3.1), it follows from Lemma 2.1 that

$$m(B)\int_M \Phi^2 \geq 8\pi^2 g + (Bm(B) - 2\sqrt{\frac{2}{B}})\int_M \Phi \geq 8\pi^2 g.$$

It remains to show that the second assertion holds. Suppose now that $||\Phi||_2 \leq 2\pi\sqrt{\frac{2}{m(B)}}$. The first assertion implies that $g = 0$ or 1. If $g = 1$, then $||\Phi||_2 = 2\pi\sqrt{\frac{2}{m(B)}}$ and Φ is a constant function. If $g = 0$, then by Theorem 1.3, M is totally umbilical. This completes the proof of Theorem 1.1.

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