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Finding Multiple Possible Critical Paths Using Fuzzy PERT

Shyi-Ming Chen and Tao-Hsing Chang

Abstract—Program evaluation and review techniques (PERT) is an efficient tool for large project management. In actual project control decisions, PERT has successfully been applied to business management, industry production, project scheduling control, logistics support, etc. However, classical PERT requires a crisp duration time representation for each activity. This requirement is often difficult for the decision-makers due to the fact that they usually can not estimate these values precisely. In recent years, some fuzzy PERT methods have been proposed based on fuzzy set theory for project management. However, there is a drawback in the existing fuzzy PERT methods, i.e., sometimes they maybe cannot find a critical path in a fuzzy project network. In this paper, we propose a fuzzy PERT algorithm to find multiple possible critical paths in a fuzzy project network, where the duration time of each activity in a fuzzy project network is represented by a fuzzy number. The proposed algorithm can overcome the drawback of the existing fuzzy PERT methods.

Index Terms—Index of optimism, fuzzy numbers, Fuzzy PERT, fuzzy project networks, possible critical paths, risk levels.

I. INTRODUCTION

It is difficult to manage a large project due to the fact that there are thousands of activities and complex relationships among them. Program evaluation and review techniques (PERT) is an efficient tool for large project management. It can describe the relationships of activities based on a project network structure and calculate the time value of every activity to find a critical path quickly. According to the critical path, the decision-maker can control the time and the cost of the project and improve the efficiency of resource allocation to ensure the project quality. PERT has successfully been used in business management, factory production, logistic support, …, etc [1], [14], [17], [20].

However, the activity duration time often is an uncertain value so that the result of classical PERT computation can not properly match the real-world situation. In [8], Dubois *et al.* extended the fuzzy arithmetic operations model to compute the latest starting time of each activity in a project network. In [9], Gazdik used fuzzy arithmetic operations to compute the earliest starting time of each activity in a project network. In [15], Nasution proposed how to compute total floats and find critical paths in a project network. Many other researchers also presented some calculation methods of fuzzy PERT under different conditions [2]–[5], [14], [18], [20], [21]. In [2], Chanas *et al.* presented a method to use fuzzy variables in PERT. In [3], we have presented a method for finding critical paths using fuzzy PERT. In [4], we have presented a method for finding multiple critical paths based on fuzzy PERT. In [5], Chen *et al.* discussed some issues in fuzzy PERT. In [14], Mon *et al.* proposed fuzzy PERT/Cost with different fuzzy distributions on activity duration under various risk levels and indices of optimism. The

definitions of the risk level and the index of optimism are briefly reviewed from [14] as follows. Mon *et al.* pointed out that the higher the risk level, the more uncertainty in time/cost is involved in the project. Thus, they considered the α-cut as a risk level, where $\alpha \in [0, 1]$. They assumed that $\alpha < 0.3$ is high risk, $0.3 \leq \alpha < 0.7$ is medium risk, and $\alpha \geq 0.7$ is low risk. Let A be a fuzzy number with membership function μ_A, and let the α-cut A_α of A be $[a_L^\alpha, a_R^\alpha]$ (i.e., $A_\alpha = [a_L^\alpha, a_R^\alpha]$), where $a_\lambda^\alpha = \lambda a_L^\alpha + (1 - \lambda) a_R^\alpha$ represents an element in the universe of discourse determined by an index of optimism λ and $\lambda \in [0, 1]$. In [14], Mon *et al.* pointed out that the index of optimism λ indicates the degree of optimism of a decision maker, where a larger value of λ indicates a higher degree of optimism. In [18], Prade applied fuzzy set theory to deal with a scheduling problem. In [20], Wang *et al.* presented fuzzy scheduling models under inflation conditions. In [21], Yao *et al.* presented a fuzzy critical path method based on signed distance ranking of fuzzy numbers.

However, there is a drawback of the existing fuzzy PERT methods, i.e., sometimes they maybe cannot find a critical path in a fuzzy project network [3]. Furthermore, there is an increasing demand that the decision-maker needs more "possible critical paths" to decrease the decision risk for project management [5]. It is obvious that a critical path is a path in which the sum of the total float of all the activities in the path is zero. On the other hand, if the sum of the total float of the activities in a path is smaller than the others, then the delay of any activity in the path will be the highest risk that the project can not be finished on time. In a fuzzy project network, the higher risk path is called a "possible critical path."

In this paper, we propose a fuzzy PERT algorithm for project management, where the duration time of each activity in a fuzzy project network is represented by a fuzzy number. It can overcome the drawback of the existing fuzzy PERT methods. The proposed algorithm can find multiple possible critical paths in a fuzzy project network and can increase the decision quality of project management.

The rest of this paper is organized as follows. In Section II, we briefly review the basic definitions of fuzzy sets and the arithmetic operations of fuzzy numbers from [6], [7], [11], [12], [15], and [22]. In Section III, we present a method for calculating fuzzy time values in a fuzzy project network. In Section IV, we use an example to illustrate the concept of finding multiple possible critical paths in a defuzzfied fuzzy project network. We also present a fuzzy PERT algorithm to find multiple possible critical paths in a fuzzy project network. In Section V, we illustrate some examples to find multiple possible critical paths in fuzzy project networks. The conclusions are discussed in Section VI.

II. ARITHMETIC OPERATIONS OF FUZZY NUMBERS

In 1965, Zadeh proposed the theory of fuzzy sets [22]. Roughly speaking, a fuzzy set is a set with fuzzy boundaries. Let U be the universe of discourse, $U = \{u_1, u_2, \ldots, u_n\}$. A fuzzy set A of U can be represented by

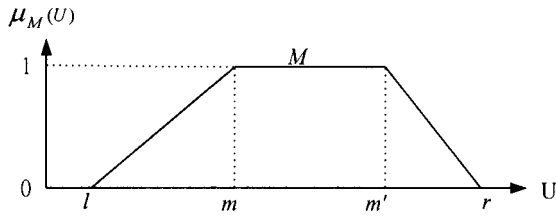$$A = \mu_A(u_1)/u_1 + \mu_A(u_2)/u_2 + \cdots + \mu_A(u_n)/u_n$$

where μ_A is the membership function of the fuzzy set A and $\mu_A(u_i)$ indicates the grade of membership of u_i in the fuzzy set A, where $\mu_A(u_i) \in [0, 1]$.

Fig. 1. Membership function curve of fuzzy number M.

A fuzzy number is a fuzzy set which is both convex and normal. A fuzzy set A of the universe of discourse U is convex if and only if for all u_1, u_2 in U,

$$\mu_A(\lambda u_1 + (1-\lambda)u_2) \geq \text{Min}(\mu_A(u_1), \mu_A(u_2)) \tag{1}$$

where $\lambda \in [0, 1]$. A fuzzy set of the universe of discourse U is called a normal fuzzy set if $\exists\, u_i \in U$, $\mu_A(u_i) = 1$.

In a fuzzy project network, the duration time of each activity is represented by a fuzzy number.

Definition 2.1: Assume that M is a trapezoidal fuzzy number in the universe of discourse U parameterized by

$$M = (l, m, m', r)$$

where

$\forall i$ $\in [m, m']$, we can see that $\mu_M(i) = 1$;
l left spread of M;
r right spread of M.

Fig. 1 shows the membership function curve of the trapezoidal fuzzy number M.

The definitions of fuzzy number arithmetic operations of trapezoidal fuzzy numbers are shown as follows.

Definition 2.2: Let A and B be two trapezoidal fuzzy numbers where $A = (l_1, m_1, m'_1, r_1)$ and $B = (l_2, m_2, m'_2, r_2)$.

1) Fuzzy Numbers Addition $\oplus$:

$$A \oplus B \approx (l_1 + l_2, m_1 + m_2, m'_1 + m'_2, r_1 + r_2) \tag{2}$$

where the symbol "$\approx$" means "approximately equal."

2) Fuzzy Numbers Subtraction $\ominus$:

$$A \ominus B \approx (l_1 - r_2, m_1 - m'_2, m'_1 - m_2, r_1 - l_2) \tag{3}$$

where the symbol "$\approx$" means "approximately equal."

3) Fuzzy Numbers Maximum $f\max$:

$$f\max(A, B) \approx (\max(l_1, l_2), \max(m_1, m_2) \\ \max(m'_1, m'_2), \max(r_1, r_2)) \tag{4}$$

where the symbol "$\approx$" means "approximately equal."
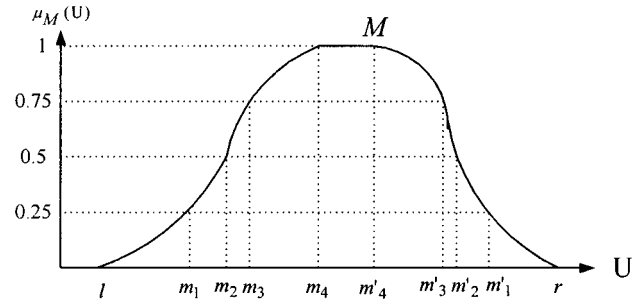
4) Fuzzy Numbers Minimum $f\min$:

$$f\min(A, B) \approx (\min(l_1, l_2)\ \min(m_1, m_2) \\ \min(m'_1, m'_2), \min(r_1, r_2)) \tag{5}$$

where the symbol "$\approx$" means "approximately equal."

Definition 2.3: Let $M = (l, m, m', r)$ be a trapezoidal fuzzy number, then the defuzzified value M_e [6], [7], [12] of the trapezoidal fuzzy number M is

$$M_e = \frac{l + m + m' + r}{4} \tag{6}$$

where M_e coincides with the center of gravity (COA) defuzzification method if M is symmetric.

Fig. 2. p equal dividing α-cuts form M_α^p of M (when $p = 4$).

Definition 2.4: Let M be a fuzzy number in the universe of discourse U characterized by the membership function μ_M, μ_M: $U \rightarrow [0, 1]$. The α-cut M_α [11], [12] of the fuzzy number M is defined by

$$M_\alpha = \{u | \mu_M(u) \geq \alpha \text{ and } u \in U\}$$

where $\alpha \in [0, 1]$.

Definition 2.5: Let M be an arbitrary fuzzy number in the universe of discourse U and p be a positive integer. Then, the fuzzy subset M_α^p of U is called the p equal dividing α-cuts form of M, where $\alpha \in [0, 1]$ and M_α^p can be parameterized by

$$M_\alpha^p = (l, m_1, m_2, \ldots, m_p, m'_p, \ldots, m'_2, m'_1, r) \tag{7}$$

where

$\forall k$ $\leq p$;
m_k left bound of the k/p-cut $M_{k/p}$ of the fuzzy number M;
m'_k right bound value of k/p-cut $M_{k/p}$ of the fuzzy number M;
l left spread of M;
r right spread of M.

For example, Fig. 2 shows the p equal dividing α-cuts form M_α^p of M, where $p = 4$.

Proposition 2.1: Let A and B be two fuzzy numbers and let the p equal dividing α-cuts form of A and B be A_α^p and B_α^p where

$$A_\alpha^p = (l_a, m_{a1}, \ldots, m_{ap}, m'_{ap}, \ldots, m'_{a1}, r_a)$$
$$B_\alpha^p = (l_b, m_{b1}, \ldots, m_{bp}, m'_{bp}, \ldots, m'_{b1}, r_b)$$

and $\alpha \in [0, 1]$, then

$$\begin{aligned}(A \oplus B)_\alpha^p \\ = A_\alpha^p \oplus B_\alpha^p \\ \approx (l_a + l_b, m_{a1} + m_{b1}, \ldots, m'_{a1} + m'_{b1}, r_a + r_b).\end{aligned} \tag{8}$$

$$\begin{aligned}(A \ominus B)_\alpha^p \\ = A_\alpha^p \ominus B_\alpha^p \\ \approx (l_a - r_b, m_{a1} - m'_{b1}, \ldots, m'_{a1} - m_{b1}, r_a - l_b).\end{aligned} \tag{9}$$

$$\begin{aligned}[f\max(A, B)]_\alpha^p \\ = f\max(A_\alpha^p, B_\alpha^p) \\ \approx (\max(l_a, l_b), \max(m_{a1}, m_{b1}), \ldots, \\ \max(m'_{a1}, m'_{b1}), \max(r_a, r_b)).\end{aligned} \tag{10}$$

$$\begin{aligned}[f\min(A, B)]_\alpha^p \\ = f\min(A_\alpha^p, B_\alpha^p) \\ \approx (\min(l_a, l_b), \min(m_{a1}, m_{b1}), \ldots, \\ \min(m'_{a1}, m'_{b1}), \min(r_a, r_b)).\end{aligned} \tag{11}$$

Proposition 2.2: Let M be a fuzzy number, where the p equal dividing α-cuts form of M is $M_\alpha^p = (l, m_1, m_2, \ldots, m_p, m'_p, \ldots,$

m'_2, m'_1, r). Then, the defuzzified value M^p_e of the p equal dividing α-cuts form M^p_α of M is

$$M^p_e = \frac{l + m_1 + m_2 + \cdots + m_p + m'_p + \cdots + m'_2 + m'_1 + r}{2(p+1)} \quad (12)$$

where M^p_e also coincides with the center of gravity of the fuzzy subset M^p_α if it is symmetric.

For example, from Fig. 2, we can see that $p = 4$, and based on (12), we can see that the defuzzified value M^4_e of M is as follows:

$$M^4_e = \frac{l + m_1 + m_2 + m_3 + m_4 + m'_4 + m'_3 + m'_2 + m'_1 + r}{2(4+1)}$$
$$= \frac{l + m_1 + m_2 + m_3 + m_4 + m'_4 + m'_3 + m'_2 + m'_1 + r}{10}.$$

Furthermore, let's consider the trapezoidal fuzzy number shown in Fig. 1. In this case, we can see that $p = 1$. Based on (12), we can see that the defuzzified value M^1_e of M is as follows:

$$M^1_e = \frac{l + m + m' + r}{2(1+1)}$$
$$= \frac{l + m + m' + r}{4}.$$

It is obvious that this result coincides with the defuzzified value of the trapezoidal fuzzy number shown in (6).

III. CALCULATING FUZZY TIME VALUES IN A FUZZY PROJECT NETWORK

In fuzzy PERT, we must calculate the earliest starting time, latest starting time, earliest finishing time, latest finishing time, and total float to find a critical path in a fuzzy project network. The critical path can provide the decision-maker to control the progress of a project. In this paper, we use the p equal dividing α-cuts form of fuzzy numbers and their fuzzy arithmetic operations to find the critical paths of a fuzzy project network.

First, we must choose a suitable value of p. Each activity time in a fuzzy project network is represented by the p equal dividing α-cuts form M^p_α of the fuzzy number M as shown in (7). Let D_j be the set of immediately preceding nodes of node j in a fuzzy project network. The method for calculating fuzzy time values in a fuzzy project network is reviewed from [3] as follows. For any node j in the fuzzy project network, its earliest starting time TE_j is as follows:

$$TE_j = \begin{cases} f \max\{TE_i \oplus P_{ij} \mid i \in D_j\}, & \text{if } D_j \neq \varnothing \\ 0, & \text{if } D_j = \varnothing \end{cases} \quad (13)$$

where P_{ij} is the activity duration time of activity $i - j$.

For every node i and activity $i - j$ in the fuzzy project network, when the earliest starting time of node i was obtained, we can calculate the earliest starting time ES_{ij} and earliest finishing time EF_{ij} of activity $i - j$ as follows:

$$ES_{ij} = TE_i \quad (14)$$
$$EF_{ij} = ES_{ij} \oplus P_{ij}. \quad (15)$$

Let Z be the set of activities in a fuzzy project network. When the earliest starting time and the earliest finishing time of every activity in a fuzzy project network have been calculated, we can obtain the project finishing time T_W as follows:

$$T_W = f \max\{EF_{ij} \mid \text{Activity } i - j \in Z\}. \quad (16)$$

The earliest starting time of the ending node is equal to its earliest finishing time and also equal to its latest starting time. Clearly, the ear-

liest starting time of the ending node is equal to the project finishing time, too. Now, the latest starting time can be calculated in the following way. Let S_j be a set of immediately succeeding nodes of node j. The latest starting time TL_j of node j can be calculated as follows:

$$TL_j = \begin{cases} f \min\{TL_k \ominus P_{jk} \mid k \in S_j\}, & \text{if } S_j \neq \varnothing \\ T_w, & \text{if } S_j = \varnothing \end{cases} \quad (17)$$

where P_{jk} is the activity duration time of activity $j - k$.

For every node j and every activity $i - j$ in a fuzzy project network, when the latest finishing time of node j has been calculated, we can obtain the latest starting time LS_{ij} and the latest finishing time LF_{ij} of activity $i - j$ as follows:

$$LF_{ij} = TL_j \quad (18)$$
$$LS_{ij} = LF_{ij} \ominus P_{ij}. \quad (19)$$

When the earliest starting time and the latest starting time of activity $i - j$ have been obtained, we can calculate the total float of each activity. For activity $i - j$ in a fuzzy project network, the total float TF_{ij} of the activity $i - j$ can be computed as follows:

$$TF_{ij} = LS_{ij} \ominus ES_{ij}. \quad (20)$$

Hence we can obtain the earliest starting time, the latest starting time, the earliest finishing time, the latest finishing time, and the total float of every activity by using (13)–(20). Calculating these fuzzy time values can let the decision-maker understand what time the activity can begin, how much time it can be delayed, and what the deadline for the activity is, etc. These fuzzy time values also let the decision-maker get the possible critical paths based on the total float of every activity, control the project progress, and know which activity will cause delay if the project cannot be finished on time. In the next section, we will propose a method to find multiple possible critical paths in a fuzzy project network.

IV. FINDING POSSIBLE CRITICAL PATHS IN A FUZZY PROJECT NETWORK

In a traditional PERT network, there exists at least one critical path between the starting node and the ending node where the total float of each activity in the path is zero. Therefore, the project's finishing time will be delayed if any activity was delayed in the critical path. Thus, how to find the critical path is a very important task of PERT.

However, according to the fuzzy number arithmetic operations presented in Section III, sometimes we cannot obtain a critical path where the sum of the total float of all the activities in the path is zero. From a different point of view, if the sum of the total float of the activities in a path is smaller than the others, then the delay of any activities in the path will be the highest risk that the project cannot be finished on time. In a fuzzy project network, the higher risk path is called a "possible critical path." Sometimes we may need to provide several possible critical paths to the decision-maker for project management. According to this viewpoint, we defuzzify the total float of each activity by (6) or (12) and find the possible critical path such that the sum of the total floats of the activities in the path is smaller than the others.

The depth-first search method can find several possible critical paths in a fuzzy project network. We can obtain all paths by using the depth-first search method, compute the sum of the total floats of the activities, and sort them to find several paths which have smaller values of the sum of total floats than the other paths. Thus, we can find multiple possible critical paths in a fuzzy project network. Assume that we want to find n possible critical paths in a fuzzy project network, and assume that for any activity k in a fuzzy project network, there are at most
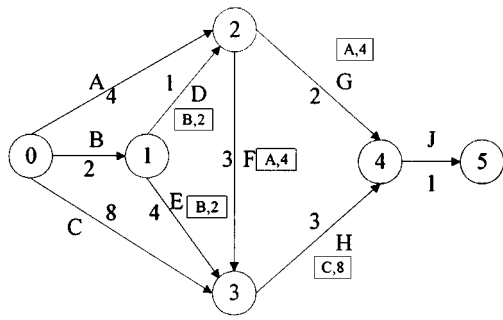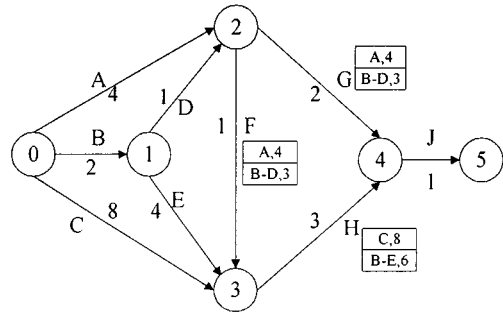
Fig. 3.   Finding possible critical paths.

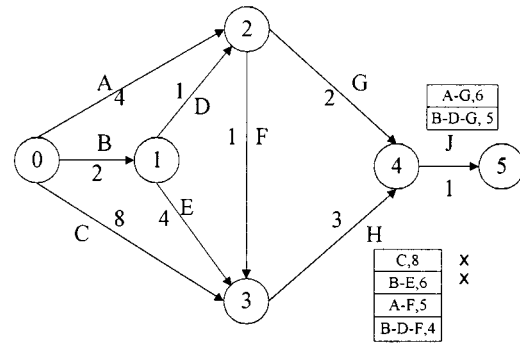Fig. 5.   Finding possible critical paths (continued from Fig. 4).

Fig. 4.   Finding possible critical paths (continued from Fig. 3).

Fig. 6.   Finding possible critical paths (continued from Fig. 5).

n possible critical paths in a fuzzy project network. Furthermore, we assume that there are m paths from the starting node to activity k. Then at most n paths are possible critical paths in these m paths. The paths after activity k can be any combinations, but only n paths become the possible critical paths at most. Thus, we rank these paths based on the sum of the total floats of activities in these paths, where the path with the smallest value of the sum of the total floats has the best ranking, and the paths whose ranking above n are impossible to be possible critical paths.

In the following, we use Fig. 3 to illustrate the process of finding multiple possible critical paths in a fuzzy project network. For simplicity, we assume that the total float of each activity in Fig. 3 has been defuzzified into a crisp value by using (6) or (12), where the number in each edge of Fig. 3 shows the defuzzified total float of each activity. For example, from Fig. 3, we can see that the defuzzified total float of activity A is four. Assume that we need to find two possible critical paths in Fig. 3. First, we can find that the activities A, B, and C do not have any immediately preceding activities. They use a path–message format including the path field and the sum field of the total float to write their activity ID to the path field and write their total float to the sum field of total float and transfer the path message to their immediately succeeding activities. For example, from Fig. 3, we can see that activity A transfers the path message "Path A, sum of total float = 4," denoted by $\boxed{A, 4}$, to activities F and G; activity B transfers the path message "Path B, sum of total float = 2," denoted by $\boxed{B, 2}$, to activities D and E; activity C transfers the path message "Path C, sum of total float = 8," denoted by $\boxed{C, 8}$, to activity H.

Then, the activities D and E add their activity ID and total float values to the path field and the sum field of the total float in the path messages, respectively, and transfer the path messages to their immediately succeeding activities, respectively. For example, from Fig. 4, we can see that activity D transfers the path message "Path B–D, the sum of total float is 3," denoted by $\boxed{B-D, 3}$, to its immediately succeeding activities F and G, respectively; activity E transfers the path message "Path B–E, the sum of total float is 6," denoted by $\boxed{B-E, 6}$, to its immediately succeeding activity H.
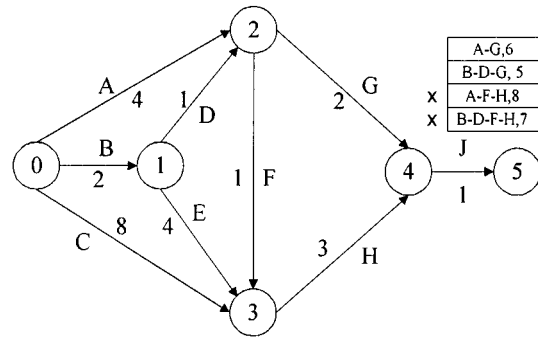
Then, the activities F and G add their activity ID and total float values to the path field and the sum field of total float in the path messages, respectively, and transfer the path messages to their immediately succeeding activities. For example, the activity F transfers the path message "Path B–D–F, the sum of total float is 4," denoted by $\boxed{B-D-F, 4}$, to its immediately succeeding activity H as shown in Fig. 5. From Fig. 5, we also can see that the activity F also transfers the path message "Path A–F, the sum of total float is 5," denoted by $\boxed{A-F, 5}$, to its immediately succeeding activity H; the activity G transfers the path messages "Path A–G, the sum of total float is 6," denoted by $\boxed{A-G, 6}$, and "Path B–D–G, the sum of total float is 5," denoted by $\boxed{B-D-G, 5}$, to its immediately succeeding activity J, respectively.

From Fig. 5, we can see that there are four path messages in the activity H, i.e., "Path C, the sum of total float is 8," "Path B–E, the sum of total float is 6," "Path A–F, the sum of total float is 5," and "Path B–D–F, the sum of total float is 4." In this example, we only need two possible critical paths, so we rank the path messages based on the sum of total float in the path messages, where the path with the smallest value of the sum value of total float has the best ranking. Then, from the path messages in the activity H of Fig. 5, we can obtain two path messages whose rank is smaller than the others (i.e., $\boxed{A-F, 5}$ and $\boxed{B-D-F, 4}$), and add their activity ID and total float value to the path field and the sum field of total float in the path messages, and transfer the path messages to its immediately succeeding activity J as shown in Fig. 6. For example, in Fig. 6, the activity H transfers the path messages "Path A–F–H, the sum of total float is 8," denoted by $\boxed{A-F-H, 8}$, and "Path B–D–F–H, the sum of total float is 7," denoted by $\boxed{B-D-F-H, 7}$, to its immediately succeeding activity J, respectively.

From Fig. 6, we can see that there are four path messages in the activity J, i.e., "Path A–G, the sum of total float is 6," "Path B–D–G, the sum of total float is 5," "Path A–F–H, the sum of total float is 8," and "Path B–D–F–G, the sum of total float is 7." We can obtain two path messages where the sum field of total float is smaller than the others (i.e., $\boxed{A-G, 6}$ and $\boxed{B-D-G, 5}$). Finally, because activity J does not
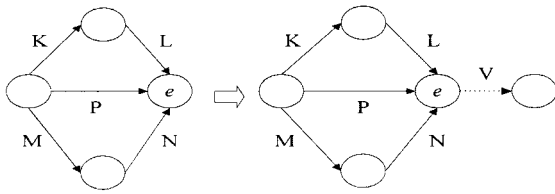
Fig. 7.   Adding a virtual ending activity V into a fuzzy project network.

Fig. 8.   Crisp project network [13].

have any immediately succeeding activities, we can obtain two possible critical paths: "path B–D–G–J" and "path A–G–J."

Assume that the decision-maker needs to find n possible critical paths in a fuzzy project network. For any activity that got m path messages from its immediately preceding activities, where $n \leq m$, it only needs to transfer n path messages to its immediately succeeding activities. According to the proposed method, we can discard the impossible paths in every activity computation and transfer less path messages to its succeeding activities. Thus, the computation time for searching possible critical paths can be improved efficiently and we can find n possible critical paths in a fuzzy project network more quickly.

When we search possible critical paths using the proposed method, there may exist some activities which do not have immediately succeeding activities. In order to get possible critical paths, we must rank the path messages of these activities again. To avoid the situation, we can add an extra "virtual ending activity" V at the tail of the fuzzy project network, and let the activity duration time of the virtual ending activity V be zero. For example, in the fuzzy project network shown in Fig. 7, activities L, P, and N, transfer their path messages to the virtual ending activity V, and the virtual ending activity V can get the possible critical paths.

In the following, we present a fuzzy PERT algorithm to compute the earliest starting time, the latest starting time, the earliest finishing time, the latest finishing time, the total float of each activity, and the possible critical paths in a fuzzy project network. Assume that the activity duration time of each activity in a fuzzy project network is represented by fuzzy numbers using p equal dividing α-cuts form, where $p \geq 2$ and $\alpha \in [0, 1]$. Furthermore, assume that the decision-maker wants to find n possible critical paths in a fuzzy project network, where $n \geq 1$. The fuzzy PERT algorithm is now presented as follows.

```
Fuzzy PERT Algorithm:
INPUT: A fuzzy project network.
OUTPUT: Multiple possible critical paths in the
  fuzzy project network.
Step 1: /* Calculate the earliest starting time and
  the earliest finishing time of each activity. */
  Add a virtual ending activity V into a fuzzy
  project network;
  repeat
    choose an activity which does not have immedi-
  ately preceding activities or where all of its
  immediately preceding activities have been marked;
    calculate the earliest starting time and the ear-
  liest finishing time of the activity by formulas
  (13)-(15), and mark the activity
  until all activities have been marked;
  Let all activities be unmarked.
Step 2: /* Calculate the latest starting time, the
  latest finishing time, and the total float of each
  activity. */
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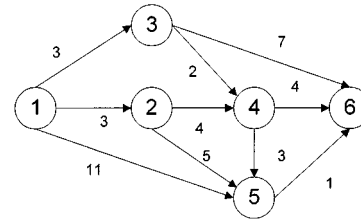
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  Let the earliest starting time of the virtual
  ending activity V be the latest starting time, and
  let virtual ending activity V be marked;
  repeat
    choose an activity in which all of its immedi-
  ately succeeding activities have been marked;
    calculate the latest starting time and the
  latest finishing time of the activity by formulas
  (16)-(19);
    calculate the total float of the activity by for-
  mula (20) and mark the activity
  until all activities have been marked;
  Let all activities be unmarked.
Step 3: /* Find n possible critical paths. */
  repeat
    choose an activity that does not have any imme-
  diately preceding activities or where all of its
  immediately preceding activities have been marked;
    rank the path messages based on the defuzzified
  sum values of the total floats in the path mes-
  sages, where the path with the smallest value of
  the sum value of the total floats has the best
  ranking;
    discard the path messages whose ranking is larger
  than n;
    add the activity to the path field of path mes-
  sages;
    add the activity time to the defuzzified sum
  value of the total floats of the path messages;
    send all path messages to all of its immediately
  succeeding activities and mark the activity
  until all activities have been marked;
  The path fields of n path messages of the virtual
  ending activity V form the n possible critical
  paths.
```

V. EXAMPLES

Example 5.1: Consider the crisp project network shown in Fig. 8 [13], where the crisp number labeled in each edge of Fig. 8 is the activity duration time. In order to simplify calculations, we can represent the crisp values of the activity duration time shown in Fig. 8 using trapezoidal fuzzy numbers. Table I shows a comparison of the results of applying the traditional PERT method [13] and the proposed fuzzy PERT algorithm. From Table I, we can see that the traditional PERT method finds the critical path 1–5–6. We also can see that the proposed fuzzy PERT algorithm finds that the most possible critical path is path 1–5–6, and finds that the other possible critical paths are the path 1–2–4–6 and the path 1–2–4–5–6. According to Table I, we can see that the proposed fuzzy PERT algorithm operating in the crisp project network shown in Fig. 8 can obtain the same result (i.e., the critical path 1–5–6) as that

TABLE I
CALCULATION RESULTS OF TRADITIONAL PERT METHOD AND THE PROPOSED FUZZY PERT ALGORITHM

| Activity	Traditional PERT Method [13]			The Proposed Fuzzy PERT Algorithm			
	Earliest Starting Time	Latest Starting Time	Total Float	Earliest Starting Time	Latest Starting Time	Total Float	Defuzzified Value of Total Float
1-2	0	1	1	(0,0,0,0)	(1,1,1,1)	(1,1,1,1)	1
1-3	0	2	2	(0,0,0,0)	(2,2,2,2)	(2,2,2,2)	2
1-5	0	0	0	(0,0,0,0)	(0,0,0,0)	(0,0,0,0)	0
2-4	3	4	1	(3,3,3,3)	(4,4,4,4)	(1,1,1,1)	1
2-5	3	6	3	(3,3,3,3)	(6,6,6,6)	(3,3,3,3)	3
3-4	3	6	3	(3,3,3,3)	(6,6,6,6)	(3,3,3,3)	3
3-6	3	5	2	(3,3,3,3)	(5,5,5,5)	(2,2,2,2)	2
4-5	7	8	1	(7,7,7,7)	(8,8,8,8)	(1,1,1,1)	1
4-6	7	8	1	(7,7,7,7)	(8,8,8,8)	(1,1,1,1)	1
5-6	11	11	0	(11,11,11,11)	(11,11,11,11)	(0,0,0,0)	0
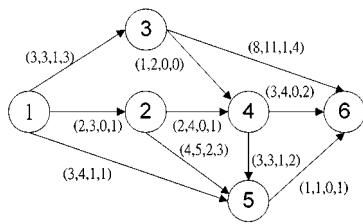
Fig. 9.　Fuzzy project network of Example 5.2 [15].

TABLE II
COMPARISON OF THE ACTIVITY DURATION TIME OF DIFFERENT METHODS

i-j	Nasution's Method [15]	Hapke-and-Slowinski's Method [10]
1-2	(2,3,0,1)	(2,2,3,4)
1-3	(3,3,1,3)	(2,3,3,6)
1-5	(3,4,1,1)	(2,3,4,5)
2-4	(2,4,0,1)	(2,2,4,5)
2-5	(4,5,2,3)	(2,4,5,8)
3-4	(1,2,0,0)	(1,1,2,2)
3-6	(8,11,1,4)	(7,8,11,15)
4-5	(3,3,1,2)	(2,3,3,5)
4-6	(3,4,0,2)	(3,3,4,6)
5-6	(1,1,0,1)	(1,1,1,2)

of the traditional PERT method [13]. Furthermore, the proposed algorithm is better than the traditional PERT method due to the fact that it can provide multiple possible critical paths to the decision-maker for project management.

Example 5.2: Consider the fuzzy project network shown in Fig. 9 [15], where the activity duration time of each activity is represented by a trapezoidal fuzzy number. Table II shows a comparison of the activity duration time of each activity shown in Fig. 9 using different methods [10], [15]. It should be pointed out that the trapezoidal fuzzy number M shown in Fig. 1 represented by Nasution's method [15] is $(m, m', m-l, r-m')$, but the fuzzy number M of Fig. 1 represented by the Hapke-and-Slowinski's method [10] is (l, m, m', r). The calculating results of Hapke-and-Slowinski's method [10] are shown in Table III and the calculating results of Nasution's method [15] are shown in Table IV, where the value h_i shown in Table IV is the indicator of criticality indicating the degree of involvement of an Event i in the critical path. If $h_i = 1$, then $\eta_i = \mu_{\sigma_i}(0)$, where σ_i is a fuzzy number calculated as shown in [15, p. 54]. The calculating results of the proposed fuzzy PERT algorithm are shown in Table V,

TABLE III
COMPUTATION RESULTS OF HAPKE-AND-SLOWINSKI'S METHOD [10]

j	Earliest Starting Time	Latest Starting Time	Total Float
1	0	0	0
2	(2,2,3,4)	(4,5,6,9)	(2,3,3,5)
3	(2,3,3,6)	(2,3,3,6)	0
4	(4,4,7,9)	(6,7,10,14)	(2,3,3,5)
5	(6,7,10,14)	(8,10,13,19)	(2,3,3,5)
6	(9,11,14,21)	(9,11,14,21)	0

where the trapezoidal fuzzy number shown in Fig. 1 is represented by (l, m, m', r). Because different methods use different modes, we can see that Table III and Table IV represent the node's time value using the activity-on-node (AON) mode, and Table V represents the edge's time value using the activity-on-edge (AOE) mode.

From Table III, we can see that by using Hapke-and-Slowinski's method [10], we get the nodes 1, 3, and 6 whose total float is zero, so the critical path is "Path 1–3–6." The method cannot provide more possible critical paths to the decision-maker.

From Table IV, we can see that Nasution's method [15] sometimes needs several α-cuts operations to find critical paths. For example, in Table IV, we can see that the method only decides that node 1 and node 6 lie in a critical path and it cannot find a complete critical path, so it must perform several α-cuts operations to find the complete critical path.

From Table V, we can see that the sum of the total float of activity 1–3 and activity 3–6 is the smallest. Therefore, the proposed fuzzy PERT algorithm obtains the most possible critical path "Path 1–3–6," where this result is the same as that in [10] and [15]. Moreover, the proposed fuzzy PERT algorithm can also obtain the second possible critical path "Path 1–3–4–6."

Example 5.3: Consider the fuzzy project network shown in Fig. 10 [14], where the activity duration time of each activity shown in Fig. 10 is represented by a fuzzy number. Table VI shows the membership function of the activity duration time of each activity shown in Fig. 10. Table VII shows the computation results of Mon–Cheng–Lu's method [14], where in order to properly represent the membership functions of the activity time shown in Table VI, we use M_α^p fuzzy numbers to represent these membership functions, where $p = 4$ and $\alpha \in [0, 1]$. Tables VIII–XI show the computation results of the proposed fuzzy PERT algorithm. Table VII shows the most possible critical path of Fig. 10 is path A–B–D–F–G under the risk level $\alpha = 0.5$ and the index of optimism $\lambda = 0$ due to the fact that it has the largest value of project time. The second possible critical path is A–C–F–G and the

TABLE IV
COMPUTATION RESULTS OF NASUTION'S METHOD [15]

i	Earliest Starting Time	Latest Starting Time	Total Float	h_i	η_i
1	0	0	0	1	0
2	(2,3,0,1)	max{(2,3,0,1), (3,8,6,8)}	max{0,(0,6,7,8)}	1	1
3	(3,3,1,3)	max{(3,3,1,3),min{(-3,3,5,6),(2,6,0,2)}	max{0,min{(-6,0,8,7), (-1,3,3,3)}	1	1
4	(4,7,0,2)	max{(4,7,0,2), (7,10,5,8)}	max{0,(0,6,7,5)}	1	1
5	(7,10,1,4)	(10,13,3,7)	max{0,(0,6,7,5)}	1	1
6	(11,14,2,7)	(11,14,2,7)	0	1	0

TABLE V
COMPUTATION RESULTS OF THE PROPOSED FUZZY PERT ALGORITHM

Activity	Earliest Starting Time	Latest Starting Time	Total Float	Defuzzified Value of Total Float
1-2	(0,0,0,0)	(-7,0,6,14)	(-7,0,6,14)	3.25
1-3	(0,0,0,0)	(-12,-3,3,12)	(-12,-3,3,12)	0
1-5	(0,0,0,0)	(2,6,10,18)	(2,6,10,18)	9
2-4	(2,2,3,4)	(-3,3,8,16)	(-7,0,6,14)	3.25
2-5	(2,2,3,4)	(-1,5,9,18)	(-5,2,7,16)	5
3-4	(2,3,3,6)	(0,5,9,17)	(-6,2,6,15)	4.25
3-6	(2,3,3,6)	(-6,0,6,14)	(-12,-3,3,12)	0
4-5	(4,4,7,9)	(2,7,10,18)	(-7,0,6,14)	3.25
4-6	(4,4,7,9)	(3,7,11,18)	(-6,0,7,14)	3.75
5-6	(6,7,10,14)	(7,10,13,20)	(-7,0,6,14)	3.25
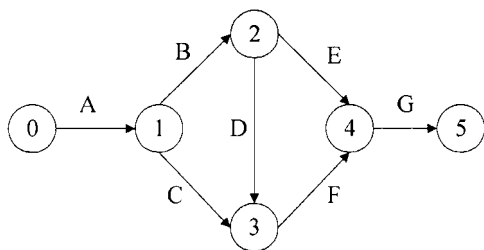
Fig. 10. Fuzzy project network [14].

TABLE VI
FUZZY MEMBERSHIP FUNCTION OF ACTIVITY TIME [14]

Activity	Fuzzy Activity Time Distribution
0-1(A)	1 day
1-2(B)	$\mu_{\tilde{B}}(x) = e^{-(x-3)^2}, \ x > 0$
1-3(C)	$\mu_{\tilde{C}}(x) = \begin{cases} e^{(x-7)}, & x \leq 7 \\ e^{-(x-7)}, & x > 7 \end{cases}$
2-3(D)	$\mu_{\tilde{D}}(x) = \begin{cases} \frac{1}{2}(x-2), & 2 \leq x < 4 \\ 1, & 4 \leq x < 6 \\ \frac{1}{2}(8-x), & 6 \leq x \leq 8 \end{cases}$
2-4(E)	$\mu_{\tilde{E}}(x) = e^{-(1/3)(x-8)^2}, \ x > 0$
3-4(F)	$\mu_{\tilde{F}}(x) = \begin{cases} \frac{1}{2}(x-2), & 2 \leq x < 4 \\ \frac{1}{2}(6-x), & 4 \leq x \leq 6 \end{cases}$
4-5(G)	$\mu_{\tilde{G}}(x) = \begin{cases} e^{(4/3)(x-1)}, & x \leq 1 \\ e^{-(4/3)(x-1)}, & x > 1 \end{cases}$

TABLE VII
THE COMPUTATION RESULT FOR DIFFERENT RISK LEVELS α AND
INDICES OF OPTIMISM λ [14]

α	λ	Critical Path	Project Time
0.05	1	A,B,E,G	9.52
0.05	0.5	A,B,D,F,G	14
0.05	0	A,B,D,F,G	21.28
0.5	1	A,C,F,G	10.79
0.5	0.5	A,B,D,F,G	14
0.5	0	A,B,D,F,G	18.35
0.8	1	A,C,F,G	12.21
0.8	0.5	A,B,D,F,G	14
0.8	0	A,B,D,F,G	16.44

third possible critical path is A–B–E–G. As shown in Table VII, Mon *et al.* [14] used different risk levels α and indices of optimism λ to provide the decision-maker with possible critical paths. However, determining the risk level α and the index of optimism λ is difficult when the decision-maker faces an unfamiliar project, where $\alpha \in [0, 1]$ and $\lambda \in [0, 1]$. Using the defuzzified values of the total floats shown in Table XI, the proposed fuzzy PERT algorithm finds that the possible critical paths of Fig. 10 are the path A–B–D–F–G (i.e., the most possible critical path), the path A–C–F–G (i.e., the second possible critical path), and the path A–B–E–G (i.e., the third possible critical path), where the most possible critical path A–B–D–F–G is also the same as the one presented in [14] under the risk level $\alpha = 0.5$ and the index of optimism $\lambda = 0.5$; the second possible critical path A–C–F–G and the third possible critical path A–B–E–G obtained by the proposed algorithm are the same as those critical paths obtained in [14]. Furthermore, the project finishing time obtained by the proposed fuzzy PERT algorithm is 14.5259. This result is close to the result shown in Table VII [14]. According to these results, for a decision-maker who can't determine the risk level α and the index of optimism λ, the proposed fuzzy PERT algorithm can provide him (her) a useful way to find multiple possible critical paths in a fuzzy project network. Thus, the decision risk of the project can be reduced.

VI. CONCLUSION

Although there are many fuzzy PERT methods that have been proposed for project management, there is a drawback in the existing fuzzy PERT methods, i.e., sometimes they maybe cannot find a critical path in a fuzzy project network. In this paper, we have presented a fuzzy

TABLE VIII
THE p EQUAL DIVIDING α-CUTS FORMS OF REPRESENTATION OF ACTIVITY DURATION TIME (WHEN $p = 4$)

Activity	Activity Duration Time
0-1(A)	(1, 1, 1, 1, 1, 1, 1, 1, 1, 1)
1-2(B)	(1, 1.8226, 2.17, 2.4636, 3, 3, 3.5364, 3.83, 4.1774, 5)
1-3(C)	(4, 5.6137, 6.31, 6.7123, 7, 7, 7.2877, 7.69, 8.3863, 10)
2-3(D)	(2, 2.5, 3, 3.5, 4, 6, 6.5, 7, 7.5, 8)
2-4(E)	(5, 5.9606, 6.56, 7.071, 8, 8, 8.929, 9.44, 10.0393, 11)
3-4(F)	(2, 2.5, 3, 3.5, 4, 4, 4.5, 5, 5.5, 6)
4-5(G)	(0, 0.0397, 0.48, 0.7842, 1, 1, 1.2158, 1.52, 2.0397, 3)

TABLE IX
CALCULATION RESULTS OF THE ACTIVITY'S EARLIEST STARTING TIME USING THE PROPOSED FUZZY PERT ALGORITHM

Activity	Earliest Starting Time
0-1(A)	(0, 0, 0, 0, 0, 0, 0, 0, 0, 0)
1-2(B)	(1, 1, 1, 1, 1, 1, 1, 1, 1, 1)
1-3(C)	(1, 1, 1, 1, 1, 1, 1, 1, 1, 1)
2-3(D)	(2, 2.8226, 3.17, 3.4636, 4, 4, 4.5364, 4.83, 5.1774, 6)
2-4(E)	(2, 2.8226, 3.17, 3.4636, 4, 4, 4.5364, 4.83, 5.1774, 6)
3-4(F)	(5, 6.6137, 7.31, 7.7123, 8, 10, 11.0364, 11.83, 12.6774, 14)
4-5(G)	(7, 9.1137, 10.31, 11.2123, 12, 14, 15.5364, 16.83, 18.1774, 20)

TABLE X
CALCULATION RESULTS OF THE ACTIVITY'S LATEST STARTING TIME USING THE PROPOSED FUZZY PERT ALGORITHM

Activity	Latest Starting Time
0-1(A)	(-13, -9.0637, -6.52, -4.3241, -2, 2, 4.3241, 6.52, 9.0637, 13)
1-2(B)	(-12, -8.0637, -5.52, -3.3241, -1, 3, 6.0018, 8.1, 10.3942, 14)
1-3(C)	(-9, -4.7726, -2.18, -0.5754, 1, 3, 5.3241, 7.52, 10.0637, 14)
2-3(D)	(-7, -3.8863, -1.69, 0.2123, 2, 6, 8.5364, 10.83, 13.1774, 16)
2-4(E)	(-4, -0.9256, 0.87, 2.2833, 4, 6, 8.4654, 10.27, 12.2168, 15)
3-4(F)	(1, 3.6137, 5.31, 6.7123, 8, 10, 12.0364, 13.83, 15.6774, 18)
4-5(G)	(7, 9.1137, 10.31, 11.2123, 12,14, 15.5364, 16.83, 18.1774, 20)

TABLE XI
THE TOTAL FLOATS AND DEFUZZIFIED VALUES OF THE ACTIVITY'S TOTAL FLOATS USING THE PROPOSED FUZZY PERT ALGORITHM

Activity	Total Float	Defuzzified Value of Total Float
0-1(A)	(-13, -9.0637, -6.52, -4.3241, -2, 2, 4.3241, 6.52, 9.0637, 13)	0
1-2(B)	(-13, -9.0637, -6.52, -4.3241, -2, 2, 5.0018, 7.1, 9.3942, 13)	0.1588
1-3(C)	(-10, -5.7726, -3.18, -1.5754, 0, 2, 4.3241, 6.52, 9.0637, 13)	1.4380
2-3(D)	(-13, -9.0637, -6.52, -4.3241, -2, 2, 5.0728, 7.66, 10.3548,14)	0.4180
2-4(E)	(-10, -6.103, -3.96, -2.2531, 0, 2, 5.0018, 7.1, 9.3942, 13)	1.4180
3-4(F)	(-13, -9.0637, -6.52, -4.3241, -2, 2, 4,3241, 6,52, 9.0637, 13)	0
4-5(G)	(-13, -9.0637, -6.52, -4.3241, -2, 2, 4,3241, 6,52, 9.0637, 13)	0

PERT algorithm to find multiple critical paths in a fuzzy project network to overcome the drawback of the existing fuzzy PERT methods, where the duration time of each activity in a fuzzy project network is represented by a fuzzy number. Because the proposed fuzzy PERT algorithm is based on the depth-first search method according to [10, p. 269] we can see that the time complexity of the proposed algorithm is $O(n + e)$ if adjacency lists are used to represent fuzzy project networks, where n is the number of nodes and e is the number of edges in a fuzzy project network. The proposed fuzzy PERT algorithm can find multiple possible critical paths in a fuzzy project network in a very efficient manner. It can provide more information to the decision-maker for project management and can reduce the decision risk of the project.

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