$a \in A(i)$. Then, we have that $p(i-1|i, 1) = 1$ for all $i \geq 1$, and $p(1|0, 2) = 1$, $p(i|0, 1) = \tilde{p}(i)$ for all $i \in S$; and, $c(i, 1) = 1$ for all $i \geq 1$, $c(0, 2) = 1$, $c(0, 1) = 0$. Obviously, this discrete-time MDPs model is the same as in [4, Prop. 3.3], therefore, (5.16) contradicts with [4, Prop. 3.3]. □

Remark 5.2: This example shows that the conditions to guarantee the existence of a solution to the optimality inequality don't imply the existence of a solution to the optimality equation.

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A General Invariance Principle for Nonlinear Time-Varying Systems and Its Applications

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***Abstract*—A general invariance principle, from the output-to-state point of view, is proposed for the dynamical analysis of nonlinear time-varying systems. This is achieved by the construction of a simple and intuitive criterion using integral inequality of the output function and modified detectability conditions. The proposed scheme can be viewed as an extension of the integral invariance principle (Byrnes and Martin, 1995) for time-invariant systems to time-varying systems. Such extension is nontrivial and can be used in various research areas such as adaptive control, tracking control and the control of driftless systems. An application to global tracking control of four-wheeled mobile robots is given to demonstrate the feasibility and validity of the proposed approach.**

***Index Terms*—Invariance principle, mobile robots, time-varying systems.**

I. Introduction

Since the 1960s, Lyapunov function based approaches have been well developed for the analysis of system stability (see [1]–[5], [7], [8], and [11]–[15]). Among these, a very useful criterion, called the "LaSalle invariance principle," was proposed in [7] and has been applied and extended to the study of many diverse areas in the recent literature. For instance, Byrnes and Martin [4] proposed an integral invariance principle to study the stability of nonlinear time-invariant systems. However, neither the LaSalle invariance principle nor the integral invariance principle can be applied to time-varying systems directly. This is due to the fact that the ω-limit set is not an invariant set in general time-varying systems (see, e.g., [5, p. 193]). Since the invariance principles have been proved to be important and useful in the analysis of system dynamics, the extension of these principles to general time-varying systems has attracted much attention (e.g., [1], [2], [7], [12]). In [12], results for some classes of time-varying systems such as almost periodic systems and asymptotically autonomous systems were obtained using the concept of pseudo-invariant set. However, no simple method was given for the determination of the pseudo-invariance set. Instead of using the concept of the invariance principles, two interesting results employing the concept of "limit equations" [2] and the direct Lyapunov approach [1] were obtained for time-varying systems. Although the stability criteria proposed in previous literature can be used in some time-varying systems, their approaches are, in general, hard to check. The development of simple stability criteria for easy checking remains an important issue.

In this note, a simple stability criterion for time-varying systems is proposed. Instead of using the existence of ω-limit set, the concept of limit systems is defined for time-varying systems. Two detectability conditions will be given in terms of limit systems. Based on these con-

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ditions and an integral inequality for the observer function, bounded solutions of system dynamics are shown to approach a pre-specified equilibrium set. The relationships between the proposed scheme and LaSalle invariance principle as well as the integral invariance principle are also studied. Finally, we revisit the tracking control problem for a 4-wheeled mobile robot studied in [9]. In that paper, it has been shown that the error model of the tracking problem is feedback-equivalent to a passive time-varying system. However, a complete stability analysis was not given. In this study, a novel stability analysis of 4-wheeled mobile robot system will be presented from the concept of limit system. Through such an application, it can be seen that, just like the LaSalle invariance principle being feasible to the stability study of time-invariant systems, the approach presented in this note is applicable to analyze the stability of time-varying systems.

II. Preliminaries

In this section, we give an example to illustrate that the LaSalle invariance principle and the integral invariance principle can not be applied directly to time-varying systems for determining system stability. Then the definition of limit systems and two modified detectability conditions are presented, which will be used in the next section for the derivation of the main result. In this note, $|v| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$, for all $v = (v_1, v_2, \ldots, v_n) \in \Re^n$, the distance function is defined as $|v|_\Omega = \inf\{|w - v||w \in \Omega \subseteq \Re^n\}$ and a function $x(t)$: $[t_0\, \infty) \to X \subseteq \Re^n$ is said to be bounded if $x(t)$ lies within a compact subset of X.

Example 1: Consider the following system:

$$\begin{aligned}\dot{x}_1 &= e^{-2t} x_2\\ \dot{x}_2 &= -e^{-2t} x_1 - x_2\\ y &= x_2\end{aligned} \tag{1}$$

where $x_1, x_2, y \in \Re$. Choose $V(x_1, x_2) = (1/2)(x_1^2 + x_2^2)$ as a Lyapunov function candidate. Taking the time derivative of V along the state trajectory of system (1), we have $\dot{V}(x_1, x_2) = -x_2^2 = -y^2 \le 0$. It is clear that $\int_0^\infty |y(t)|^2\, dt = -\int_0^\infty \dot{V}(x_1, x_2)\, dt < \infty$. From (1), the set $S = \{(x_1, x_2)|\dot{V}(x_1, x_2) = 0\}$ contains only the trivial equilibrium solution. If LaSalle invariant principle or integral invariant principle is attempted to study the stability of system (1), one will have $\lim_{t\to\infty} x_1(t) = 0$ and $\lim_{t\to\infty} x_2(t) = 0$. However, we will check that $\lim_{t\to\infty} x_1(t) \ne 0$ for any solution $(x_1(t), x_2(t))$ starting from $x_1(0) \ne 0$ and $x_2(0) = 0$. Since $\dot{V} \le 0$ and $x_2(0) = 0$, we then have $V(x_1, x_2) = (1/2)[x_1^2(t) + x_2^2(t)] \le V(x_1(0), x_2(0)) = (1/2)x_1^2(0)$. This implies that $|x_1(t)| \le |x_1(0)|$ for all $t \ge 0$. Moreover, the second differential equation of system (1) gives $|x_2(t)| = |e^{-t}\int_0^t e^{-\tau} x_1(\tau)\, d\tau| \le |x_1(0)|$ for all $t \ge 0$. By solving the first differential equation of system (1), we then have

$$|x_1(t)| = |x_1(0) + \int_0^t e^{-2\tau} x_2(\tau)\, d\tau| \ge \tfrac{1}{2}|x_1(0)|, \qquad \text{for all } t \ge 0.$$

This implies that $\lim_{t\to\infty} x_1(t) \ne 0$. Thus, both the LaSalle invariance principle and the integral invariance principle need a modification for determining the stability of time-varying systems. Now, we present the definition of limit systems, which will be applied in Section III to the construction of invariant principle for time-varying systems. In this note, denote X an open subset of $\Re^n$. Consider a class of systems as given by

$$\dot{x} = f(a(t), x) \tag{2}$$

$$y = h(b(t), x) \tag{3}$$

where $x \in X$, $y \in \Re^m$, $f(a, x) \in \Re^n$ and $h(b, x) \in \Re^m$ with $a(t)$ and $b(t)$ being $\Re^p$-valued function and $\Re^q$-valued function defined on $[0\, \infty)$, respectively. Here, assume both $f(a, x)$ and $h(b, x)$ are continuous with $a(t)$ and $b(t)$ being uniformly continuous and bounded vector functions. Note that, many systems take the form of (2)–(3). For instance, linear time-varying systems and tracking control of autonomous systems all take the extended form of (2)–(3). Since invariance principles guarantee the limit behavior of a bounded solution, it is intuitive to consider the dynamics of the "limit system" for a given system. That is the behavior of system at $t \to \infty$. The definition of limit system will be given below. First, we present the definition of limit function.

Definition 1: Let $c(t)$: $[0\, \infty) \to \Re^p$, with $p \in \aleph$, be any continuous function. A sequence $\gamma = \{t_n\}$ of real number with $\lim_{n\to\infty} t_n = \infty$ is said to be an admissible sequence associated with $c(t)$ if there exists a continuous function $c_\gamma(t)$ defined on $[0\, \infty)$ such that $\{c(t+t_n)\}$ uniformly converges to $c_\gamma(t)$ on every compact subset of $[0\, \infty)$.The function $c_\gamma(t)$ is called a limit function of $c(t)$ and is uniquely defined. ■

Denote $\Lambda(c)$ the set of all admissible sequences associated with $c(t)$. It is not difficult to check that every subsequence of an admissible sequence is also an admissible sequence and all these subsequences provide the same limit function of $c(t)$. Now, we are ready to give the definition of limit system.

Definition 2: Let γ be an admissible sequence associated with both $a(t)$ and $b(t)$ [i.e., $\gamma \in \Lambda(a) \cap \Lambda(b)$]. Then the following associated system

$$\dot{\overline{x}} = f\,(a_\gamma(t), \overline{x}) \tag{4}$$

$$\overline{y} = h\,(b_\gamma(t), \overline{x}) \tag{5}$$

is called a "limit system" of system (2)–(3) where $a_\gamma(t)$ and $b_\gamma(t)$ denote the limit functions of $a(t)$ and $b(t)$ determined by the sequence γ, respectively. ■

As an example, by virtue of $\lim_{t\to\infty} e^{-2t} = 0$, a limit system of that in Example 1 can be described by the following:

$$\begin{aligned}\dot{\overline{x}}_1 &= 0\\ \dot{\overline{x}}_2 &= -\overline{x}_2\end{aligned}$$

and

$$\overline{y} = \overline{x}_2. \tag{6}$$

A condition to guarantee the existence of limit functions is given as follows.

Lemma 1: Let $c(t)$: $[0\, \infty) \to \Re^p$, with $p \in \aleph$, be a uniformly continuous and bounded function and $\{t_n\}$ be a sequence approaching infinity. Then, there exists a subsequence $\{t_{n_k}\}$ of $\{t_n\}$ such that $\{c(t + t_{n_k})\}$ converges uniformly to a limit function $\overline{c}(t)$ on every compact subset of $[0\, \infty)$.

Proof: Denote $c_n(t) = c(t + t_n)$. Then, by the assumption, the sequence $\{c_n(t)\}$ is totally bounded and equi-continuous. Thus, according to Arzela–Ascoli lemma (see [6]), there exists a subsequence $\{n_k\}$ of $\{n\}$ such that $\{c_{n_k}(t)\}$ converges uniformly to a continuous function $\overline{c}(t)$ on every compact subset of $[0\, \infty)$. This completes the proof. ■

As motivated by Lemma 1, we can show that the set $\Lambda(a) \cap \Lambda(b)$ is nonempty. Let $\{t_n\}$ be any sequence approaching infinity. Then, from Lemma 1 and the assumptions of system (2)–(3) there exists a subsequence $\gamma = \{t_{n_k}\}$ of $\{t_n\}$ such that $\gamma \in \Lambda(a)$. Similarly, we have a subsequence $\overline{\gamma}$ of γ such that $\overline{\gamma} \in \Lambda(b)$. It is clear that $\overline{\gamma} \in \Lambda(a) \cap \Lambda(b)$. Thus, by Definition 2, Lemma 1 provides the existence of limit systems. Throughout this note, for simplicity, any sequence $\gamma \in \Lambda(a) \cap \Lambda(b)$ is said to be an admissible sequence of system (2)–(3).

It is known that (e.g., [3]) the zero-state detectability is used in time-invariant systems to determine system stability. In the following, two zero-state detectability conditions for limit systems will be given. In the remainder of this note, denote Ω a subset of X and $\phi(t_0, t, x_0)$ a bounded solution of (2) starting from $\phi(t_0, t_0, x_0) = x_0$ at $t = t_0$ for all $t \geq t_0 \geq 0$. We then have the following two detectability conditions with respect to the trajectory ϕ:

(C1): System (2)–(3) is weakly detectable w. r. t. ϕ. That is, there exists an admissible sequence γ of system (2)–(3) such that every solution $\overline{x}(t)$ of limit system (4), starting at $t = 0$, approaches the given se Ω, i.e., $\lim_{t\to\infty} |\overline{x}(t)|_\Omega = 0$, when $\overline{x}(t)$ lies within the ω-limit set of ϕ and satisfies $h(b_r(t), \overline{x}(t)) \equiv 0$.

(C2): System (2)–(3) is uniformly detectable w. r. t. ϕ. That is, for every positive constant ε, there exists a positive constant T such that for every admissible sequence γ of system (2)–(3), every solution $\overline{x}(t)$ of limit system (4), starting at $t = 0$, will satisfy the inequality $|\overline{x}(t)|_\Omega < \varepsilon$ for all $t \geq$ T when $\overline{x}(t)$ lies within the ω-limit set of ϕ with $h(b_r(t), \overline{x}(t)) \equiv 0$.

Remark 1: For time-invariant systems, the zero-state detectability only concerns the set $\Omega = \{0\}$. Moreover, every limit system of a time-invariant system is the same as the original system. Thus, it is clear that conditions (C1) and (C2) for such case are, respectively, implied by the zero-state detectability condition and zero-state observability condition introduced in [3].

III. Main Results

In this section, a general invariant principle will be proposed and used to guarantee the attractivity of an equilibrium set using the modified detectability conditions (C1)–(C2) given in Section II. An application to the tracking control problem for mobile robots is also presented to demonstrate the use of the main results. Details are given as follows.

A. A Modified Invariant Principle

Before deriving the modified invariant principle, for simplicity, we have the following hypothesis for a bounded solution $\phi(t_0, t, x_0)$ of (2).

Hypothesis 1: Suppose $\phi(t_0, t, x_0)$ satisfies the following inequality

$$\int_{t_0}^{\infty} w(h(b(t), \phi(t_0, t, x_0)))\, dt < \infty \tag{7}$$

for the output map (3), where w is a positive definite continuous function with $\lim_{|y|\to\infty} w(y) = \infty$.

Since $\dot{\phi} = f(a(t), \phi(t_0, t, x_0))$ is bounded, $\phi(t_0, t, x_0)$ is uniformly continuous. Let $\tilde{h}(t) = h(b(t), \phi(t_0, t, x_0))$ for all $t \geq t_0$. Then, $w(\tilde{h}(t))$ is also uniformly continuous. From Hypothesis 1 and Barbalat's Lemma [5], we have $\lim_{t\to\infty} w(\tilde{h}(t)) = 0$. This implies that $\lim_{t\to\infty} h(t) = 0$. We then have the next result.

Theorem 1: Suppose Hypothesis 1 holds. Then the following two results hold for system (2)–(3):

i) The set Ω contains a ω-limit point of $\phi(t_0, t, x_0)$ if condition (C1) holds.

ii) Condition (C2) implies that the equality $\lim_{t\to\infty} |\phi(t_0, t, x_0)|_\Omega = 0$ holds.

Proof: First, we prove i) by contradiction. Suppose statement i) is false. Then, by the definition of ω-limit point (see [5]), there exist a $T > 0$ and a $\varepsilon > 0$ such that $|\phi(t_0, t+t_0, x_0)|_\Omega \geq \varepsilon$ for all $t \geq$ T. Let γ be an admissible sequence of system (2)–(3) such that the conclusion of (C1) holds. Denote $a_r(t)$ and $b_r(t)$ the corresponding limit functions of $a(t)$ and $b(t)$, respectively. Using a similar proof of Lemma 1 and the boundedness and uniform continuity of ϕ, there exists a subsequence $\{t_{n_k}\}$ of γ such that $\{\phi(t_0, t + t_{n_k}, x_0)\}$ converges uniformly to a continuous function $\overline{x}(t)$ on every compact subset of $[0\,\infty)$. Note that, $\{a(t + t_{n_k})\}$ and $\{b(t + t_{n_k})\}$ also converge uniformly to the limit functions $a_r(t)$ and $b_r(t)$ on every compact subset of $[0\,\infty)$ since every subsequence of an admissible sequence is also an admissible sequence and yields the same limit function. Observe that $\dot{\phi}(t_0, t+t_{n_k}, x_0) = f(a(t+t_{n_k}), \phi(t_0, t+t_{n_k}, x_0))$ and the sequences of functions relating to ϕ and f appearing in the differential equations are uniformly convergent on every compact subset of $[0\,\infty)$. We can then take the limit of differential equations, see [6]. By taking the limit of differential equations, we hence have $\dot{\overline{x}}(t) = f(a_\gamma(t), \overline{x}(t))$. Moreover, by the fact of $t + t_{n_k} \to \infty$ and $\lim_{t\to\infty} h(t) = 0$, $h(b_\gamma(t), \overline{x}(t)) = \lim_{k\to\infty} h(b(t + t_{n_k}), \phi(t_0, t + t_{n_k}, x_0)) = 0$ for each $t \geq 0$. Note that $\overline{x}(t)$ lies within the ω-limit set of ϕ since $\overline{x}(t) = \lim_{k\to\infty} \phi(t_0, t + t_{n_k}, x_0)$. Thus, $\overline{x}(t)$ is a solution of the limit system (4)–(5) starting at $t = 0$ and lies within the ω-limit set of ϕ with $h(b_\gamma(t), \overline{x}(t)) \equiv 0$. From condition (C1), we have $\lim_{t\to\infty} |\overline{x}(t)|_\Omega = 0$, which contradicts the presumption that $|\overline{x}(t)|_\Omega = \lim_{k\to\infty} |\phi(t_0, t + t_{n_k}, x_0)|_\Omega \geq \varepsilon$ since $t + t_k \geq T + t_0$ for each t and large enough k. The result of i) is hence proved.

Similarly, we next prove ii) by contradiction . Suppose statement ii) is false. Then, there exist an $\varepsilon > 0$ and a sequence $\{t_n\}$ approaching infinity such that $|\phi(t_0, t_n, x_0)|_\Omega \geq \varepsilon$. Let T be the positive constant given in condition (C2), which depends only on ε. Using the similar argument in the proof of Lemma 1, it is concluded that there exists a subsequence $\gamma = \{t_{n_k} - T\}$ of $\{t_n - T\}$ such that all three sequences $\{a(t + t_{n_k} - T)\}$, $\{b(t + t_{n_k} - T)\}$ and $\{\phi(t_0, t + t_{n_k} - T, x_0)\}$, respectively, converge uniformly to their limit functions $a_\gamma(t)$, $b_\gamma(t)$ and $\overline{x}(t)$. We then have $\dot{\overline{x}}(t) = f(a_\gamma(t), \overline{x}(t))$ and $h(b_\gamma(t), \overline{x}(t)) \equiv 0$ using the fact of $\lim_{t\to\infty} h(t) = 0$, along with a similar proof of i). Thus, $\overline{x}(t)$ is a solution of the limit system, starting at $t = 0$, and lies within the ω-limit set of ϕ with $h(b_\gamma(t), \overline{x}(t)) \equiv 0$. By condition (C2), we have $|\overline{x}(T)|_\Omega < \varepsilon$. This contradicts the assumption of $|\overline{x}(T)|_\Omega = \lim_{k\to\infty} |\phi(t_0, t_{n_k}, x_0)|_\Omega \geq \varepsilon$. The proof of ii) is then completed. ■

Remark 2: The function w given in Hypothesis 1 is usually taken as $w(y) = |y|^p$ for $0 < p < \infty$ (see [4]). For such case, w is positive definite and $\lim_{|y|\to\infty} w(y) = \infty$.

Now, we re-examine the analysis of the system given in Example 1 to demonstrate the possible application of Theorem 1. For such system, every solution is bounded since the Lyapunov function V is proper and satisfying $\dot{V} \leq 0$. Moreover, Hypothesis 1 holds for any solution by choosing $w(y) = |y|^2$. The corresponding limit system is given in (6). If we take $\Omega = \{(x_1, 0) | x_1 \in \Re\}$, condition (C2) also holds. Then by Theorem 1, $x_2(t) \to 0$. However, if we take $\Omega = \{(0, 0)\}$, condition (C1) does not hold for any solution starting from the initial conditions: $x_1(0) \neq 0$ and $x_2(0) = 0$. The reason is that it was shown in Section II that $|x_1(t)| \geq (1/2)|x_1(0)|$ for all $t \geq 0$. Thus, every solution $(\overline{x}_1(t), \overline{x}_2(t))$ of (6), lying within the ω-limit set of the original solution and satisfying $\overline{x}_2(t) \equiv 0$, will have $|\overline{x}_1(t)| \geq (1/2)|x_1(0)|$ for all $t \geq 0$. It is observed from this example that conditions (C1) and (C2) can be used to predict the dynamical behavior of a time-varying system better than that obtained from time-invariant systems.

Remark 3: The concept of limit equations similar to that in (4) was first introduced by Artstein [2]. The goal of [2] is to give a sufficient and necessary condition in terms of limit equations to guarantee the uniformly asymptotic stability of the origin. The result is very interesting, however, the stability checking of limit equations yields the same difficulty as that of the original systems in many time-varying systems. On the contrast, in a spirit like LaSalle invariance principle, the order of systems constrained on the zero-locus of the limit functions for output map can be effectively reduced by introducing the concept of limit systems and limit functions of output map as presented above. An interesting example for robot systems will be given in the next subsection to illustrate such point of view.

For general applications, we have $\Omega = \{0\}$ and the uniform Lyapunov stability is usually attainable *a priori*. Under this condition, it is easy to check that the attractivity of the origin is implied by the fact of the origin being a ω-limit point. Next corollary follows readily from Theorem 1.

Corollary 1: Let $\Omega = \{0\}$ and suppose Hypothesis 1 holds. Then, $\phi(t_0, t, x_0) \to 0$ as $t \to \infty$ if the origin is uniformly Lyapunov stable and condition (C1) holds. ■

Note that, several well-known invariance principles for time-invariant systems can be deduced from Theorem 1. For instance, let $w(y) = |y|^p$ and Ω be the largest invariant subset of the zero-locus of the output function for time-invariant systems. It is not difficult to check that both Hypothesis 1 and Condition (C2) hold. Next corollary follows readily from Theorem 1.

Corollary 2 (Integral Invariance Principle [4]): Consider a time-invariant system in the form of (2)–(3), i.e., $a(t)$ and $b(t)$ are both constant functions. Suppose $\int_{t_0}^{\infty} |h(b(t), \phi(t_0, t, x_0))|^p \, dt < \infty$ for $0 < p < \infty$. Then $\phi(t_0, t, x_0)$ approaches the largest invariant subset of the zero-locus of the output function. ■

It was shown in [4] that the integral invariance principle is reduced to the LaSalle invariance principle by choosing the time derivative of Lyapunov function as a virtual output. The LaSalle invariance principle can hence be implied by Theorem 1.

Although in the previous discussions above, we have restricted our attention to systems having the form (2)–(3), similar results can be obtained for more general time-varying systems. For instance, consider a system consisting of asymptotically almost periodic or periodic functions, see [12] for the definitions. Limit systems and conditions (C1)–(C2) for these systems can be defined similarly and Theorem 1 is also true under new conditions.

B. Application to Globally Tracking Control of 4-Wheeled Mobile Robots

In our previous paper [9], a globally tracking control problem of 4-wheeled mobile robots was studied by constructing a simple tracking controller. However, a complete stability analysis was not given. In the following, Corollary 1 will be applied to the stability study of the mobile robots. Before the further discussion, let

$$\alpha(s) = \frac{1 - \cos s}{s}$$

and

$$\beta(s) = \frac{\sin s}{s}$$

for $s \neq 0$. Also, let $\alpha(0) = 0$ and $\beta(0) = 1$. It is obvious that both $\alpha(s)$ and $\beta(s)$ are smooth functions. An error model of the tracking system can then be recalled from [9] as given by

$$\dot{x}_e = v_r(t) f(\psi_r(t), x_e) + G(\psi_r(t), x_e) u_e \tag{8}$$

where $x_e = (x_1, x_2, x_3, x_4)^T \in \Re^4$, $u_e \in \Re^2$; $v_r\colon [0\,\infty) \to \Re$ and $\psi_r\colon [0\,\infty) \to \Re$ are two uniformly continuous and bounded functions. Let $\lambda = x_4 - \alpha(x_3)x_1 - \beta(x_3)x_2 + \psi_r(t)$. Then functions f and G in (8) can be described as follows:

$$f = \begin{bmatrix} x_2\lambda + x_3\alpha(x_3) \\ -x_1\lambda + x_3\beta(x_3) \\ -x_1\alpha(x_3) - x_2\beta(x_3) + x_4 \\ -x_3 \end{bmatrix} \quad G = \begin{bmatrix} 1 + x_2\lambda & 0 \\ -x_1\lambda & 0 \\ \lambda & 0 \\ 0 & 1 \end{bmatrix}. \tag{9}$$

Choose $V = (1/2)|x_e|^2$ as a Lyapunov function candidate for system (8). It is not difficult to check that

$$\frac{\partial V}{\partial x_e} f(\psi_r, x_e) \equiv 0.$$

Let

$$y_e = \left(\frac{\partial V}{\partial x_e} G\right)^T$$

be a virtual output map. Then we have $\dot{V} = y_e^T u_e$. This implies that system (8) is passive. A simple (output feedback) controller can be chosen as

$$u_e = -k y_e \tag{10}$$

for any $k > 0$. We hence have $\dot{V} = -k|y_e|^2 \le 0$, which implies that Hypothesis 1 holds by choosing $w = k|y_e|^2$. It is clear that V is a positive definite and proper function. Thus, solutions of system (8) are concluded to be globally uniformly bounded and the origin is uniformly Lyapunov stable. Under Lyapunov stability condition, we need to verify that the origin is a common ω-limit point of every solution for providing the attractivity of the origin. Before checking the attractivity of the origin, we impose the following hypothesis.

Hypothesis 2: Suppose $v_r(t)$ in system (8) satisfies the inequality:

$$\limsup_{t \to \infty} |v_r(t)| > 0. \tag{11}$$

Note that, the Hypothesis 2 can be referred as "persistent excitation" condition. From Hypothesis 2, there exists a sequence $\{t_n\}$ with $t_n \to \infty$ such that $\lim_{n\to\infty} |v_r(t_n)| \neq 0$. By Lemma 1, there exists a subsequence $\{t_{n_k}\}$ of $\{t_n\}$ such that the two sequence $\{v_r(t + t_{n_k})\}$ and $\{\psi_r(t + t_{n_k})\}$, respectively, converge uniformly to the limit functions $\overline{v}_r(t)$ and $\overline{\psi}_r(t)$ on every compact subset of $[0\,\infty)$. Then, $\{t_{n_k}\}$ is an admissible sequence of the closed-loop system (8) with control

$$u_e = -k\left(\frac{\partial V}{\partial x_e} G\right)^T.$$

The associated limit system for system (8) can then be obtained as

$$\dot{\overline{x}}_e = \overline{v}_r(t) f\left(\overline{\psi}_r(t), \overline{x}_e\right) - kG\left(\overline{\psi}_r(t), \overline{x}_e\right)\overline{y}_e \tag{12}$$

$$\overline{y}_e = \left(\overline{x}_1 + \overline{x}_3\lambda\left(\overline{x}_e, \overline{\psi}_r(t)\right), \overline{x}_4\right). \tag{13}$$

Let $\overline{x}_e(t) = (\overline{x}_1(t), \overline{x}_2(t), \overline{x}_3(t), \overline{x}_4(t))$ be any solution of (12), starting at $t = 0$ with $\overline{y}_e \equiv 0$. Then we have $\overline{x}_4(t) \equiv 0$ and system (12) can be rewritten as

$$\dot{\overline{x}}_e(t) = \overline{v}_r(t) f\left(\overline{\psi}_r(t), \overline{x}_e(t)\right). \tag{14}$$

Note that, $|\overline{v}_r(0)| = \lim_{k\to\infty} |v_r(t_{n_k})| \neq 0$. Thus, by the continuity of $\overline{v}_r(t)$, there exists a positive constant δ such that $\overline{v}_r(t) \neq 0$ for all $t \in [0\,\delta)$. From the fourth state equation of (14), we have $\dot{\overline{x}}_4 = -\overline{v}_r(t)\overline{x}_3(t)$. Since $\overline{x}_4(t) = 0$, this leads to $\overline{x}_3(t) = 0$ for all $t \in [0\,\delta)$. It is not difficult to check from (13) that $\overline{x}_1(t) = 0$ for all $t \in [0\,\delta)$ when $\overline{y}_e \equiv 0$. Similarly, by virtual of $\dot{\overline{x}}_3 = \overline{v}_r(-\overline{x}_1\alpha(\overline{x}_3) - \overline{x}_2\beta(\overline{x}_3) + \overline{x}_4)$ from the third equation of (14) and $\beta(0) = 1$, $\overline{x}_2(t) = 0$ for all $t \in [0\,\delta)$. To conclude the discussions above, we then have $x_e(t) = 0$ for all $t \in [0\,\delta)$. Note that,

$$\frac{\partial V}{\partial x_e} f(\psi_r, x_e) \equiv 0.$$

From (14), this implies $\dot{V}(\overline{x}_e(t)) \equiv 0$. Thus, $V(\overline{x}_e(t)) = V(\overline{x}_e(0)) = 0$ for all $t \geq 0$. By the positive definiteness of V, we have $\overline{x}_e(t) \equiv 0$. Thus, condition (C1) holds. According to Corollary 1, we then have the next theorem.

Theorem 2: Under Hypothesis 2, the origin of system (8) is globally asymptotically stabilizable by the control

$$u_e = -k \left(\frac{\partial V}{\partial x_e} G \right)^T .$$

IV. CONCLUSION

A general invariance principle was proposed in this note for the stability analysis of nonlinear time-varying systems, which cannot be derived from conventional invariance principles. This is achieved by point-set topology approach rather than Lyapunov functions scheme. Thus, it is possible to extend the results in this note to the study of more general dynamical systems. The existing results such as the LaSalle invariance principle [7] and the integral invariance principle [4] was shown to be deduced from the proposed results. Application to the tracking control of 4-wheeled mobile robots was also given to demonstrate the feasibility of the proposed approach.

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Robust Stabilization of Large Space Structures Via Displacement Feedback

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***Abstract*—It has been known that static velocity and displacement feedback with collocated sensors and actuators can stabilize large space structures robustly against "any" uncertainty in mass, damping, and stiffness independently of the number of flexible modes. This note presents dynamic displacement feedback which can achieve such robust stabilization. The proposed control law can be implemented in a decentralized scheme straightforwardly.**

***Index Terms*—Collocated sensors and actuators, displacement feedback, large space structure, robust stabilization.**

I. INTRODUCTION

Large space structures with collocated sensors and actuators can be stabilized robustly against any uncertainty in mass, damping, and stiffness independently of the number of flexible modes using static feedback of the measured velocity and displacement [1], [2]. Such a robust control law has been obtained by utilizing the fact that the space structures possess certain qualitative properties in their parameters independently of the numerical values, and stability can be ensured by a qualitative condition. This result is very important as low authority control [3] which ensures robust stability of the closed-loop systems because the identification errors in large space structures might be quite large.

While velocity sensors are commonly used as well as displacement sensors, if the structure can be controlled without velocity measurements, it is desirable against the failure of velocity sensors and for the cost reduction of the sensing system. Even in the case of the displacement measurements only, it would be expected that using a pseudo differentiator with a sufficiently wide band, the static feedback of velocity and displacement can be realized approximately by dynamic feedback of displacement. However, since the wide-band pseudo differentiator is sensitive to noise and its gain is very large at high frequencies, it may cause unacceptable behaviors of the structure. Therefore, it is not recommended to use such a wide-band pseudo differentiator for approximation of the velocity feedback.

In this note, we present a dynamic displacement feedback control law which stabilizes large space structures under the sensor/actuator collocation. The underlying idea comes from the fact that the unstable modes of structures are the rigid modes only. Then, we can stabilize the whole system by stabilizing the rigid modes using a narrow-band pseudo differentiator around zero frequency without violating stability of the vibration modes.

The proposed control law has the following advantages. It can stabilize structures robustly against any uncertainty in mass, damping, and stiffness independently of the number of flexible modes as the static feedback of velocity and displacement does. The control law can be implemented in a decentralized scheme which generates the control inputs from the measured outputs at each collocated pair of the sensors

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