Elastic Constants Identification of Shear Deformable Laminated Composite Plates

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Abstract: A constrained minimization method is presented for the identification of elastic constants of shear deformable laminated composite plates. Strains and/or displacements obtained from static testing of laminated composite plates are used in the proposed method to identify the elastic constants of the plates. In the identification process, the trial elastic constants of a laminated composite plate are used in a finite-element analysis to predict the strains and displacements of the plate. An error function is established to measure the differences between the experimental and theoretical predictions of strains and/or displacements. A constrained minimization technique is used to minimize the error function and update the trial elastic constants. The best estimates of the elastic constants of the plate are then determined by subsequently reducing the size of the feasible region of the elastic constants and making the error function a global minimum. The accuracy and applications of the proposed method are demonstrated by means of a number of examples. A sensitivity analysis is also performed to study the effects of variations of experimental data on the accuracy of the identified elastic constants.

INTRODUCTION

The extensive use of composite laminates in the fabrication of high-performance structural systems has made the reliability of composite laminates an important topic of research. To ensure high reliability of the composite structural systems, the actual behaviors of the constituted laminated composite parts in service must be accurately predicted and carefully monitored. The attainment of the actual behavioral predictions of the structures depends on the correctness of the elastic constants of the composite laminates. It is well known that there are many methods for manufacturing laminated composite components (Lubin 1982; Schwartz 1983), and different manufacturing or curing processes may produce different mechanical properties for the components. Furthermore, the material properties determined from standard specimens tested in the laboratory may deviate significantly from those of actual laminated composite components manufactured in the factory. On the other hand, laminated composite structures subjected to dynamic loads or operated in severe environments may experience progressive stiffness reduction or material degradation, which will finally lead to the failure of the structures. It is easy to perceive that accurate determination of current stiffness or material properties of a laminated composite structure can help prevent sudden failure of the structure. In the past two decades, a number of nondestructive evaluation techniques have been proposed for the determination of material properties of laminated composite parts (Bar-Cohen 1986). Nevertheless, these techniques have their own limitations or specific difficulties in the identification of elastic constants, especially for composite materials when in use. On the other hand, a number of researchers have presented methods to identify or improve the analytical system matrices of a structure using vibration test data. For instance, Berman and Nagy (1993) developed a method that used measured normal modes and natural frequencies to improve an analytical mass and stiffness matrix model of a structure. Their method could find minimum changes in the analytical model to make it agree exactly with the set of measured modes and frequencies. Kam and his associates (Kam and Lee 1994; Kam and Liu 1998) developed methods to identify the element bending stiffnesses of beam structures using measured natural frequencies and mode shapes or displacements alone. Recently, a number of researchers used 12–16 experimental eigenfrequencies to identify elastic properties of laminated composites (Wilde and Sol 1987; Mota Soars et al. 1993; Ip et al. 1998; Rikards et al. 1999). Since larger errors may be obtained in the measurement of higher frequencies, the use of over 10 eigenfrequencies in the elastic constants identification has thus made the accuracy as well as the efficiency of the previously proposed identification method questionable.

In this paper, a constrained minimization method is presented for the identification of elastic constants of shear deformable laminated composite plates. A shear deformable finite element is used to predict the deformations of the laminated composite plates. Strains and displacements at several points on the laminated composite plates are measured in the static tests of the plates. An error function is established to measure the sum of the differences between the experimental and theoretical predictions of the deformations of the laminated composite plates. The identification of elastic constants is then formulated as a constrained minimization problem in which the elastic constants are determined by making the error function a global minimum. A bounding technique is used to reduce the size of the feasible region for the trial elastic constants subsequently during the minimization process to improve the convergence rate of the solution. The accuracy and applications of the proposed method are demonstrated by means of several examples. Finally, a sensitivity analysis is proposed to investigate the variations of identified elastic constants induced by those of experimental data.

ANALYSIS OF SHEAR DEFORMABLE LAMINATED COMPOSITE PLATE

Consider a rectangular plate of area $a \times b$ and constant thickness h subject to transverse load $p(x, y)$, as shown in Fig. 1. The plate is composed of a finite number of layer groups in which each layer group contains several orthotropic layers of the same fiber angle and uniform thickness. The x- and y-coordinates of the plate are taken in the midplane of the plate. The displacement field is assumed to be of the form

$$u_1(x, y, z) = u_0(x, y) + z \cdot \psi_x(x, y) \quad (1a)$$

$$u_2(x, y, z) = v_0(x, y) + z \cdot \psi_y(x, y); \quad u_3(x, y, z) = w(x, y) \quad (1b,c)$$

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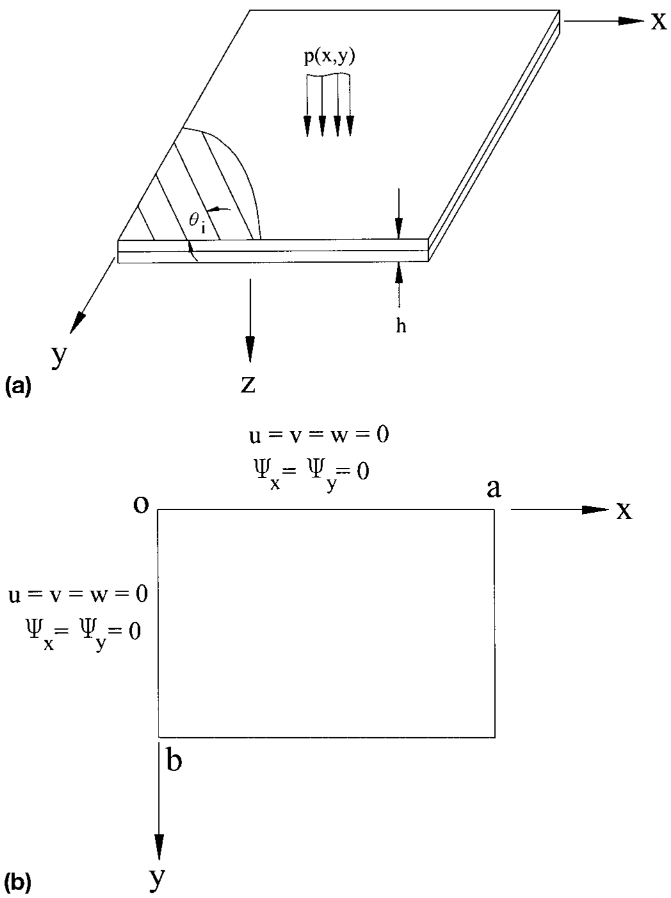

FIG. 1. Laminated Composite Plate: (a) Loading Condition; (b) Boundary Condition

where u_1, u_2, and u_3 = displacements in the x-, y-, and z-directions, respectively; u_0, v_0, and w = associated midplane displacements; and ψ_x and ψ_y = shear rotations.

The constitutive equations of a shear deformable laminated composite plate can be written as (Ochoa and Reddy 1992)

$$\begin{bmatrix} N_1 \\ N_2 \\ Q_y \\ Q_x \\ N_6 \\ M_1 \\ M_2 \\ M_6 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & 0 & 0 & A_{16} & B_{11} & B_{12} & B_{16} \\ A_{12} & A_{22} & 0 & 0 & A_{26} & B_{12} & B_{22} & B_{26} \\ 0 & 0 & A_{44} & A_{45} & 0 & 0 & 0 & 0 \\ 0 & 0 & A_{45} & A_{55} & 0 & 0 & 0 & 0 \\ A_{16} & A_{26} & 0 & 0 & A_{66} & B_{16} & B_{26} & B_{66} \\ B_{11} & B_{12} & 0 & 0 & B_{16} & D_{11} & D_{12} & D_{16} \\ B_{12} & B_{22} & 0 & 0 & B_{26} & D_{12} & D_{22} & D_{26} \\ B_{16} & B_{26} & 0 & 0 & B_{66} & D_{16} & D_{26} & D_{66} \end{bmatrix} \cdot \begin{bmatrix} u_{0,x} \\ v_{0,y} \\ w_{,y} + \psi_y \\ w_{,x} + \psi_x \\ u_{0,y} + v_{0,x} \\ \psi_{x,x} \\ \psi_{y,y} \\ \psi_{x,y} + \psi_{y,x} \end{bmatrix} \quad (2)$$

where $N_1, N_2, \ldots, M_6$ = stress resultants; A_{ij}, B_{ij}, and D_{ij} = material components; and the comma preceding a subscript denotes the partial derivative with respect to the subscript. The material components are given by

$$(A_{ij}, B_{ij}, D_{ij}) = \int_{-h/2}^{h/2} Q_{ij}^{(m)}(1, z, z^2)\, dz \quad i, j = 1, 2, 6 \quad (3a)$$

$$A_{ij} = k_\alpha \cdot k_\gamma \cdot \bar{A}_{ij}, \bar{A}_{ij} = \int_{-h/2}^{h/2} Q_{ij}^{(m)}\, dz \quad i, j = 4, 5; \alpha = 6 - i; \gamma = 6 - j \quad (3b)$$

The stiffness coefficients $Q_{ij}^{(m)}$ depend on the material properties and orientation of the mth layer group. For a layer with zero fiber angle, the stiffness coefficients are expressed as

$$Q_{11} = \frac{E_1}{1 - \nu_{12}\nu_{21}}; \quad Q_{12} = \frac{\nu_{12}E_2}{1 - \nu_{12}\nu_{21}} = Q_{21} \quad (4a,b)$$

$$Q_{22} = \frac{E_2}{1 - \nu_{12}\nu_{21}}; \quad Q_{44} = G_{23}; \quad Q_{55} = G_{12} = Q_{66} \quad (4c\text{–}e)$$

with

$$\frac{\nu_{12}}{E_1} = \frac{\nu_{21}}{E_2}$$

where E_1 and E_2 = Young's moduli in the fiber and matrix directions, respectively; ν_{ij} = Poisson's ratio for transverse strain in the jth direction when stressed in the ith direction; and G_{23} and G_{12} = shear moduli in the 2-3 and 1-2 planes, respectively. The parameters k_i are shear correction factors that are determined using the expressions given by Whitney (1973). In the plate analysis, the shear deformable finite element developed by Kam and Chang (1993) is adopted to evaluate the deformation of the plate. The element contains five degrees of freedom (three displacements and two slopes, i.e., shear rotations) per node. In the evaluation of the element stiffness matrix, a quadratic element of a serendipity family and the reduced integration are used. The accuracy of the element in predicting strains and displacements has been verified by the theoretical and experimental results available in the literature (Kam et al. 1996; Kam and Lai 1999). It has also been found that the use of a 6 × 6 mesh over the plate can yield very good results. In the following identification of material constants, normal strains (ε_x, ε_y) and deflection w at some particular points on a plate are defined as deformational parameters.

IDENTIFICATION OF ELASTIC CONSTANTS

The problem of elastic constants identification of a shear deformable laminated composite plate is formulated as a minimization problem. In mathematical form it is stated as

$$\text{Minimize} \quad e(\mathbf{x}) = (\mathbf{D}^*)^t(\mathbf{D}^*)$$

$$\text{Subject to} \quad x_i^L \le x_i \le x_i^U \quad i = 1, \ldots, 5 \quad (5)$$

where $\mathbf{x} = [E_1, E_2, G_{12}, G_{23}, \nu_{12}]$ = material constants; $\mathbf{D}^*$ = $N \times 1$ vector containing the differences between the measured and predicted values of the deformational parameters; $e(\mathbf{x})$ = error function measuring the sum of differences between the predicted and measured data; and x_i^L and x_i^U = lower and upper bounds of the material constants. It is noted that the lower and upper bounds of the material constants are chosen in such a way that the lower bound of E_1 is larger than the upper bounds of E_2, G_{23}, and G_{12}. The elements in $\mathbf{D}^*$ are expressed as

$$D_i^* = \frac{D_{pi} - D_{mi}}{D_{mi}} \quad i = 1, \ldots, N \quad (6)$$

where D_{pi} and D_{mi} = predicted and measured values of the deformational parameters, respectively. It is noted that the error function in (5) may produce a number of local minima in the feasible region. Therefore, the above minimization problem cannot be solved by merely using any of the conventional local minimization algorithms. Another difficulty one may encounter in solving the above minimization problem is the dominance of E_1 over the search direction during the minimization process. Since E_1 is much larger than the other elastic constants, it will force the search to converge to a solution that can only produce a good estimate of E_1. To overcome the above difficulties, a global minimization method together with a normalization technique are used to solve the problem of (5). The above problem of (5) is first converted into an unconstrained minimization problem by creating the following general augmented Lagrangian (Vanderplaats 1984):

$$\bar{\Psi}(\tilde{\mathbf{x}}, \boldsymbol{\mu}, \boldsymbol{\eta}, r_p) = e(\mathbf{x}) + \sum_{j=1}^{5} [\mu_j z_j + r_p z_j^2 + \eta_j \phi_j + r_p \phi_j^2] \quad (7)$$

with

$$z_j = \max\left[g_j(\tilde{x}_j), \frac{-\mu_j}{2r_p}\right]; \quad g_j(\tilde{x}_j) = \tilde{x}_j - \tilde{x}_j^U \le 0 \qquad (8a,b)$$

$$\phi_j = \max\left[H_j(\tilde{x}_j), \frac{-\eta_j}{2r_p}\right] \qquad (8c)$$

$$H_j(\tilde{x}_j) = \tilde{x}_j^L - \tilde{x}_j \le 0 \quad j = 1, \ldots, 5 \qquad (8d)$$

where μ_j, η_j, and r_p = multipliers; max[*, *] takes on the maximum value of the numbers in the brackets $\tilde{x}_i$ = modified design variables; and $\tilde{x}_i^L$ and $\tilde{x}_i^U$ = modified lower and upper bounds of the modified design variables, respectively. The modified design variables $\tilde{\mathbf{x}}$ are defined as

$$\tilde{\mathbf{x}} = \left[\frac{E_1}{\alpha_1}, \frac{E_2}{\alpha_2}, \frac{G_{12}}{\alpha_3}, \frac{G_{23}}{\alpha_4}, \nu_{12}\right] \qquad (9)$$

where α_i = normalization factors. The normalization factors are chosen in such a way that the differences among the modified design variables are <10 and the value of $\tilde{x}_1$ is larger than those of the other modified design variables. Unsatisfactory results may be obtained if the above rule for selecting the values of the normalization factors is violated. It is noted that the modified design variables $\tilde{\mathbf{x}}$ are only used in the minimization algorithm while the original design variables $\mathbf{x}$ are used in the finite-element analysis of the plate. The update formulas for the multipliers μ_j, η_j, and r_p given in the literature (Vanderplaats 1984) are

$$\mu_j^{n+1} = \mu_j^n + 2r_p^n z_j^n; \quad \eta_j^{n+1} = \eta_j^n + 2r_p^n \phi_j^n \quad j = 1, \ldots, 5 \qquad (10a,b)$$

$$r_j^{n+1} = \begin{cases} \gamma_0 r_p^n & \text{if } r_j^{n+1} < r_p^{\max} \\ r_p^{\max} & \text{if } r_j^{n+1} \ge r_p^{\max} \end{cases} \qquad (10c)$$

where the superscript n denotes iteration number; γ_0 = constant; and $r_p^{\max}$ = maximum value of r_p. From experience, the parameters μ_j^0, η_j^0, r_p^0, γ_0, and $r_p^{\max}$ are chosen as

$$\mu_j^0 = 1.0, \ \eta_j^0 = 1.0 \quad j = 1, \ldots, 5 \qquad (11a)$$

$$r_p^0 = 0.4, \ \gamma_0 = 1.25, \ r_p^{\max} = 100 \qquad (11b)$$

It is noted that slight variations in the values of the parameters in the above equation do not have significant effects on the convergence of the solution.

The constrained minimization problem of (7) has thus become the solution of the following unconstrained optimization problem:

$$\text{Minimize} \quad \bar{\boldsymbol{\Psi}}(\tilde{\mathbf{x}}, \boldsymbol{\mu}, \boldsymbol{\eta}, r_p) \qquad (12)$$

The above unconstrained optimization problem is to be solved using a two-stage multistart global optimization algorithm. In the adopted optimization algorithm, the objective function of (12) is treated as the potential energy of a traveling particle, and the search trajectories for locating the global minimum are derived from the equation of motion of the particle in a conservative force field (Snyman and Fatti 1987). The design variables (i.e., elastic constants) that make the potential energy of the particle (i.e., objective function) the global minimum constitute the solution of the problem. At the first stage of the optimization process, the side constraints in (5) are observed and a series of starting points for the design variables of (9) are selected at random from the region of interest. The lowest local minimum along the search trajectory initiated from each starting point is determined and recorded. A Bayesian argument is then used to establish the probability of the current overall minimum value of the objective function being the global minimum, given the number of starts and the number of times this value has been achieved. The multistart optimization procedure at this stage is terminated when two conditions are met, i.e., a target probability, typically 0.67, has been exceeded, and the value of the error function is $<10^{-8}$. The estimates of the global optimal values of the design variables obtained at this stage are defined as $\tilde{x}_i^*$ ($i = 1, \ldots, 5$). When comparing the estimates of the elastic constants obtained at this stage with their true values, it has been observed that the error for the estimate of E_1 is <2%, those for the estimates of E_2 and G_{12} are <5%, and those for the estimates of G_{23} and ν_{12} are <10%. Further improvement of the accuracy

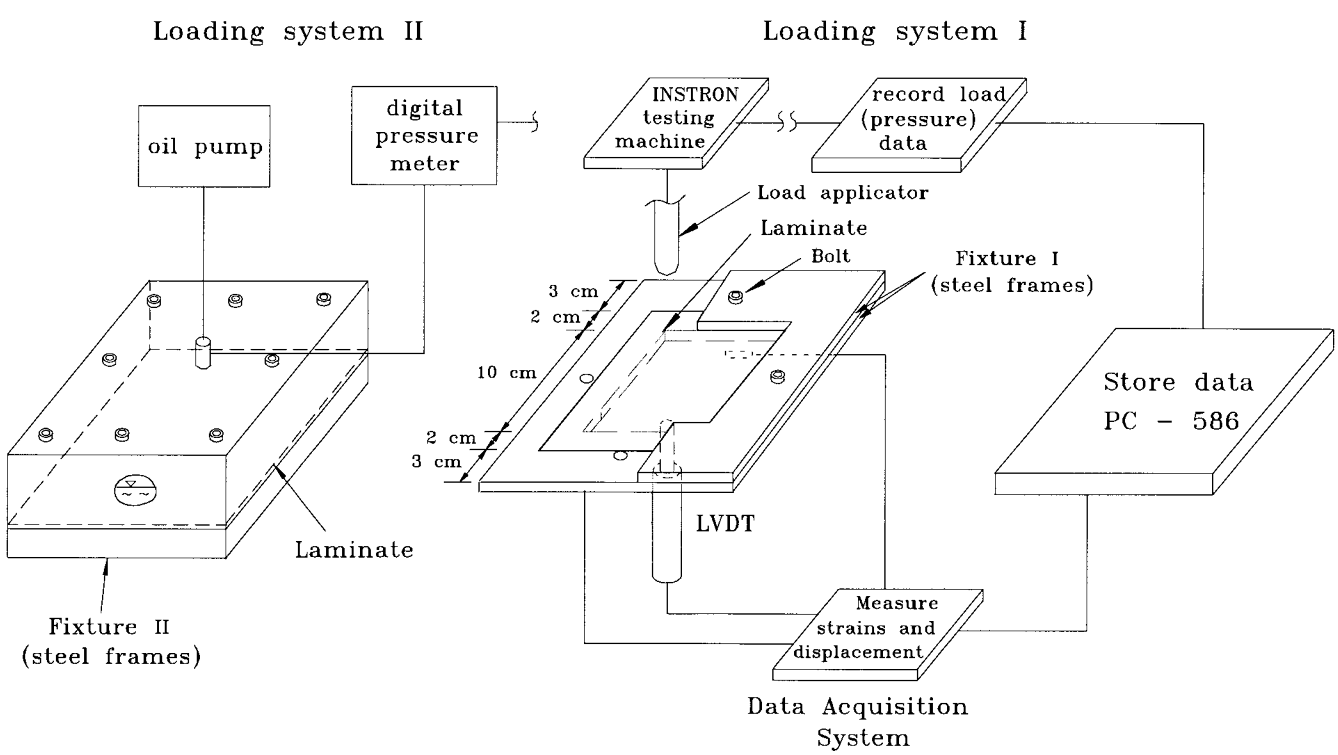

FIG. 2. Schematic Description of Experimental Setup

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of the solution can be achieved by performing the second-stage optimization process. At the second stage, the modified lower and upper bounds of the modified design variables in (8) are selected as

$$\tilde{x}_i^L = \tilde{x}_i^* - \beta_i \tilde{x}_i^*; \quad \tilde{x}_i^U = \tilde{x}_i^* + \beta_i \tilde{x}_i^* \qquad (13a,b)$$

The correction factors β_i ($i = 1, \ldots, 5$) may have different values. In general, β_1 is in the range from 0.05 to 1.0, β_2 and β_3 are in the range from 0.1 to 1.0, and β_4 and β_5 are in the range from 0.2 to 1.0. It is noted that at this stage of optimization the magnitudes of the correction factors only have slight effects on the convergence rate of the final solution. In general, the adoption of smaller values for β_i ($i = 1, \ldots, 5$) can make the solution converge faster. The multistart global minimization algorithm is again used to solve the minimization problem of (12) with the inclusion of the side constraints in (13). The multistart optimization procedure is terminated once a target probability, typically 0.99, has been exceeded. In general, when the conditions of (13) are observed, the global optimum can be determined in a very efficient way.

EXPERIMENTAL INVESTIGATION

The composite materials under consideration are T300/2500 graphite/epoxy produced by Torayca Co., Japan. The properties of the graphite/epoxy material are first determined experimentally in accordance with the relevant ASTM (1990) specifications. Each material constant was determined from tests using five specimens. The mean values and coefficients of variation of the experimentally determined material constants are given as

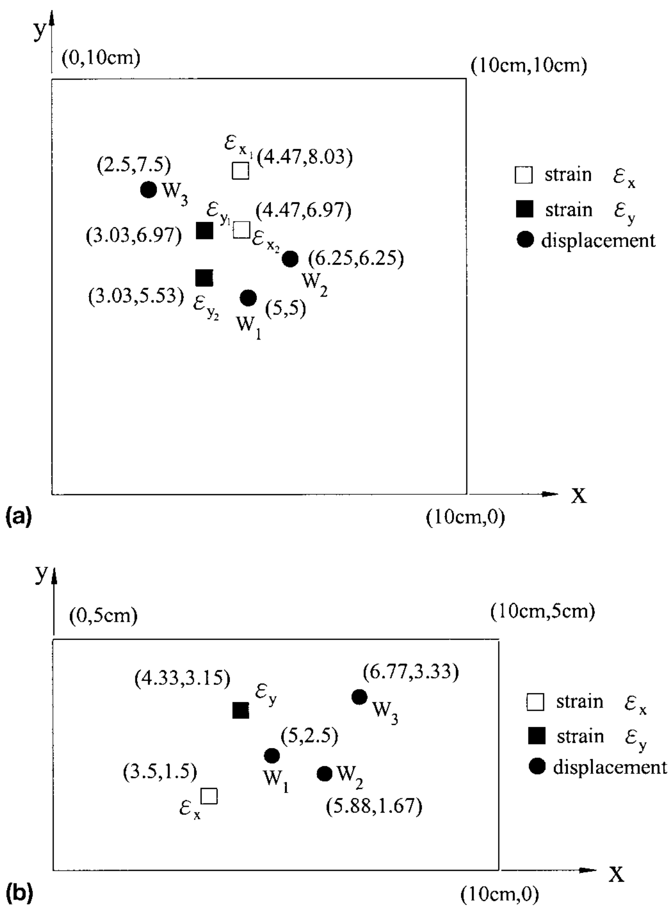

FIG. 3. Locations on Laminated Plates for Measurements of Displacements and Strains: (a) Square Plate (a/b = 1); (b) Rectangular Plate (a/b = 2)

$$E_1 = 124.68 \text{ GPa } (2.75\%), \quad E_2 = 9.6 \text{ GPa } (3.24\%),$$
$$G_{12} = 8.64 \text{ GPa } (2.8\%), \quad G_{23} = 2.32 \text{ GPa } (6.82\%),$$
$$\nu_{12} = 0.33 \ (5.1\%) \qquad (14)$$

where the values in parentheses denote coefficients of variation.

For experimental investigation, a number of symmetric rectangular laminates, namely $[45_2^\circ/0_3^\circ/-45_2^\circ/0_3^\circ/45_2^\circ]_s$ and $[0_2^\circ/90_2^\circ/0_2^\circ/90_2^\circ]_s$ of dimensions 14 × 14 cm or 14 × 9 cm were manufactured and subjected to static tests in accordance with the test procedure described in the literature (Kam and Lai 1999). The lamina thickness for the laminated plates is 0.125 mm. A schematic description of the experimental setup is shown in Fig. 2, in which the laminate is clamped at all edges and the actual dimensions of the laminate are either 10 × 10 cm or 10 × 5 cm. The laminated plates were subjected to two types of loadings, namely a center point load or a uniformly distributed load. A number of displacement transducers (LVDT) and strain gauges were placed beneath the bottom surface of the laminate for measuring the deformational parameters of the laminate. The load-displacement and load-strain curves of the laminates were constructed using the data measured from the displacement transducers (LVDT) and strain gauges, respectively. Each laminate was repeatedly tested six times and six readings were recorded for each deformational parameter. The mean values of the deformational parameters are then used in the present method to determine the elastic constants of the laminates. It is noted that the coefficients of variation of the measured deformational parameters are <3%. The points on the bottom surfaces of the laminated plates at which the displacements and strains were measured are shown in Fig. 3.

SENSITIVITY ANALYSIS

The existence of noise in experimental data is inevitable. It is therefore, likely that the measured deformation parameters may deviate from the true values for the laminated composite plates of which the elastic constants are to be identified. Herein, an approximate analysis in the field of probability (Benjamin and Cornell 1970) is used to investigate the effects of the variations in measured deformational parameters on the accuracy of the identified elastic constants. The deformational parameters are assumed to be measured independently and they can thus be treated as independent random variables. Let $(\bar{Y}_i, \sigma_{Y_i})$ be the expected value and standard deviation pair of measured deformational parameter Y_i. The elastic constants can then be expressed as

$$x_i = G_i(\mathbf{Y}) \quad i = 1, \ldots, 5 \qquad (15)$$

where $\mathbf{Y}$ = row vector containing the measured deformational parameters. The expansion of x_i at the mean values of the measured deformational parameters in a truncated Taylor series gives

$$x_i \cong G_i(\bar{\mathbf{Y}}) + \sum_{k=1}^{n} (Y_k - \bar{Y}_k) \left. \frac{\partial G_i}{\partial Y_k} \right|_{\bar{Y}} \qquad (16)$$

where n = number of deformational parameters; and $\bar{Y}$ stands for the collection of the expected values of all measured deformation parameters. It is noted that the gradients $(\partial G_i/\partial Y_k)|_{\bar{Y}}$ are evaluated at the mean values of the measured deformational parameters. The above minimization method can provide the gradients $\partial Y_k/\partial G_i$, which can be used to determine the gradients $\partial G_i/\partial Y_k$ via the following relations:

$$\frac{\partial G_i}{\partial Y_k} = \frac{1}{\dfrac{\partial Y_k}{\partial G_i}} \quad i = 1, \ldots, 5; \quad k = 1, \ldots, n \tag{17}$$

The first-order approximation to the variance of each elastic constant is

$$\mathrm{var}[x_i] \cong \sum_{k=1}^{n} \left(\frac{\partial G_i}{\partial Y_k} \bigg|_{\bar{Y}} \right)^2 \mathrm{var}[Y_k] \quad i = 1, \ldots, 5 \tag{18}$$

where var[ ] = variance of the random variable in the brackets.

RESULTS AND DISCUSSION

The aforementioned minimization method will be applied to the material characterization of the laminated composite plates that have been tested. The upper and lower bounds of the material constants are chosen based on experience

$$40 \le E_1 \le 400 \text{ GPa}, \quad 0 \le E_2 \le 40 \text{ GPa},$$
$$0 \le G_{12} \le 40 \text{ GPa}, \quad 0 \le G_{23} \le 40 \text{ GPa} \quad 0 \le \nu_{12} \le 0.5 \tag{19}$$

The modified design variables of (9) are obtained via the use of the following normalization factors:

$$\alpha_1 = 100; \quad \alpha_i = 10 \quad i = 2, 3, 4 \tag{20a,b}$$

Since very good estimates of E_1, E_2, and G_{12} can be obtained at the first stage of the optimization process, the correction factors used to modify the lower and upper bounds of the design variable at the second stage of minimization are chosen as β_i (i = 1, 2, 3) = 1/10 and β_j (j = 4, 5) = 3/10.

A detailed numerical investigation of the proposed method for elastic constants identification has revealed that the accuracies of the identified elastic constants converge when the number of deformational parameters adopted in the present method is three or more. Three deformational parameters are, therefore, used in the present identification method to determine the material elastic constants of the laminated composite plates. About four and three starting points have been randomly selected at the first and second stages, respectively, of the minimization process to obtain estimates of the global minimum. The estimates of the actual material elastic constants as well as the percentage differences between the theoretically predicted and experimentally determined material elastic constants at different stages for various cases are listed in Tables 1–4. It is noted that the present method can produce excellent results for all the cases under consideration. In particular, even the mere use of target probability 0.67 at the first stage of the optimization process can still produce results of acceptable ac-

TABLE 1. Material Constants Identification of Laminated Composite Plates Subjected to Uniform Load (a/b = 1)

Plate layup	Measured deformational parameter	Identified material constant (first stage)	Identified material constant (second stage)
$[45^\circ_2/0^\circ_3/-45^\circ_2/0^\circ_3/45^\circ_2]_s$	w_1 = 0.12 mm	E_1 = 126.75 GPa, E_2 = 10.02 GPa	E_1 = 125.1 GPa (0.34%), E_2 = 9.51 GPa (0.94%)
$[45^\circ_2/0^\circ_3/-45^\circ_2/0^\circ_3/45^\circ_2]_s$	$\varepsilon_{x1} = 0.2 \times 10^{-3}$	G_{12} = 8.12 GPa, G_{23} = 2.48 GPa	G_{12} = 8.53 GPa (1.27%), G_{23} = 2.49 GPa (7.32%)
$[45^\circ_2/0^\circ_3/-45^\circ_2/0^\circ_3/45^\circ_2]_s$	$\varepsilon_{y1} = 0.16 \times 10^{-3}$	ν_{12} = 0.29	ν_{12} = 0.3325 (0.75%)
$[0^\circ_2/90^\circ_2/0^\circ_2/90^\circ_2]_s$	w_1 = 0.348 mm	E_1 = 126.03 GPa, E_2 = 9.56 GPa	E_1 = 124.65 GPa (0.024%), E_2 = 9.55 GPa (0.52%)
$[0^\circ_2/90^\circ_2/0^\circ_2/90^\circ_2]_s$	$\varepsilon_{x1} = 0.2 \times 10^{-3}$	G_{12} = 8.73 GPa, G_{23} = 2.42 GPa	G_{12} = 8.65 GPa (0.12%), G_{23} = 2.24 GPa (3.4%)
$[0^\circ_2/90^\circ_2/0^\circ_2/90^\circ_2]_s$	$\varepsilon_{y1} = 0.238 \times 10^{-3}$	ν_{12} = 0.3325	ν_{12} = 0.332 (0.6%)
$[0^\circ_2/90^\circ_2/0^\circ_2/90^\circ_2]_s$	w_1 = 0.348 mm	E_1 = 125.78 GPa, E_2 = 9.13 GPa	E_1 = 124.7 GPa (0.016%), E_2 = 9.57 GPa (0.33%)
$[0^\circ_2/90^\circ_2/0^\circ_2/90^\circ_2]_s$	w_2 = 0.275 mm	G_{12} = 8.37 GPa, G_{23} = 2.12 GPa	G_{12} = 8.65 GPa (0.12%), G_{23} = 2.5 GPa (7.8%)
$[0^\circ_2/90^\circ_2/0^\circ_2/90^\circ_2]_s$	w_3 = 0.125 mm	ν_{12} = 0.3325	ν_{12} = 0.332 (0.6%)

Note: Values in parentheses denote percentage difference between predicted and measured data.

TABLE 2. Material Constants Identification of Laminated Composite Plates Subjected to Point Load (a/b = 1)

Plate layup	Measured deformational parameter	Identified material constant (first stage)	Identified material constant (second stage)
$[45^\circ_2/0^\circ_3/-45^\circ_2/0^\circ_3/45^\circ_2]_s$	w_1 = 0.595 mm	E_1 = 126.2 GPa, E_2 = 9.13 GPa	E_1 = 124.67 GPa (0.0%), E_2 = 9.28 GPa (3.33%)
$[45^\circ_2/0^\circ_3/-45^\circ_2/0^\circ_3/45^\circ_2]_s$	$\varepsilon_{x1} = 0.7 \times 10^{-3}$	G_{12} = 8.18 GPa, G_{23} = 2.48 GPa	G_{12} = 8.48 GPa (1.85%), G_{23} = 2.41 GPa (3.9%)
$[45^\circ_2/0^\circ_3/-45^\circ_2/0^\circ_3/45^\circ_2]_s$	$\varepsilon_{y1} = 1.1 \times 10^{-3}$	ν_{12} = 0.29	ν_{12} = 0.312 (5.45%)
$[0^\circ_2/90^\circ_2/0^\circ_2/90^\circ_2]_s$	w_1 = 1.63 mm	E_1 = 125.64 GPa, E_2 = 9.13 GPa	E_1 = 125.2 GPa (0.42%), E_2 = 9.51 GPa (0.94%)
$[0^\circ_2/90^\circ_2/0^\circ_2/90^\circ_2]_s$	$\varepsilon_{x2} = 0.65 \times 10^{-3}$	G_{12} = 8.46 GPa, G_{23} = 2.08 GPa	G_{12} = 8.63 GPa (0.118%), G_{23} = 2.51 GPa (8.2%)
$[0^\circ_2/90^\circ_2/0^\circ_2/90^\circ_2]_s$	$\varepsilon_{y2} = 0.53 \times 10^{-3}$	ν_{12} = 0.29	ν_{12} = 0.332 (0.6%)
$[0^\circ_2/90^\circ_2/0^\circ_2/90^\circ_2]_s$	w_1 = 1.63 mm	E_1 = 126.41 GPa, E_2 = 9.8 GPa	E_1 = 125.1 GPa (0.34%), E_2 = 9.27 GPa (3.4%)
$[0^\circ_2/90^\circ_2/0^\circ_2/90^\circ_2]_s$	w_2 = 1.01 mm	G_{12} = 8.71 GPa, G_{23} = 2.43 GPa	G_{12} = 8.54 GPa (1.16%), G_{23} = 2.43 GPa (4.74%)
$[0^\circ_2/90^\circ_2/0^\circ_2/90^\circ_2]_s$	w_3 = 0.352 mm	ν_{12} = 0.34	ν_{12} = 0.331 (0.3%)

Note: Values in parentheses denote percentage difference between predicted and measured data.

TABLE 3. Material Constants Identification of Laminated Composite Plates Subjected to Uniform Load (a/b = 2)

Plate layup	Measured deformational parameter	Identified material constant (first stage)	Identified material constant (second stage)
$[45^\circ_2/0^\circ_3/-45^\circ_2/0^\circ_3/45^\circ_2]_s$	w_1 = 0.03 mm	E_1 = 122.4 GPa, E_2 = 9.23 GPa	E_1 = 124.5 GPa (0.14%), E_2 = 9.3 GPa (3.13%)
$[45^\circ_2/0^\circ_3/-45^\circ_2/0^\circ_3/45^\circ_2]_s$	$\varepsilon_{x1} = 0.2 \times 10^{-3}$	G_{12} = 8.6 GPa, G_{23} = 2.49 GPa	G_{12} = 8.45 GPa (2.2%), G_{23} = 2.43 GPa (4.74%)
$[45^\circ_2/0^\circ_3/-45^\circ_2/0^\circ_3/45^\circ_2]_s$	$\varepsilon_{y1} = 0.373 \times 10^{-4}$	ν_{12} = 0.3	ν_{12} = 0.331 (0.3%)
$[0^\circ_2/90^\circ_2/0^\circ_2/90^\circ_2]_s$	w_1 = 0.057 mm	E_1 = 124.41 GPa, E_2 = 9.8 GPa	E_1 = 124.8 GPa (0.1%), E_2 = 9.52 GPa (0.83%)
$[0^\circ_2/90^\circ_2/0^\circ_2/90^\circ_2]_s$	$\varepsilon_{x1} = 0.25 \times 10^{-3}$	G_{12} = 8.12 GPa, G_{23} = 2.42 GPa	G_{12} = 8.64 GPa (0.0%), G_{23} = 2.43 GPa (4.74%)
$[0^\circ_2/90^\circ_2/0^\circ_2/90^\circ_2]_s$	$\varepsilon_{y1} = 0.4 \times 10^{-4}$	ν_{12} = 0.31	ν_{12} = 0.32 (3.0%)
$[0^\circ_2/90^\circ_2/0^\circ_2/90^\circ_2]_s$	w_1 = 0.057 mm	E_1 = 123.43 GPa, E_2 = 9.23 GPa	E_1 = 125.1 GPa (0.34%), E_2 = 9.72 GPa (1.25%)
$[0^\circ_2/90^\circ_2/0^\circ_2/90^\circ_2]_s$	w_2 = 0.053 mm	G_{12} = 8.7 GPa, G_{23} = 2.42 GPa	G_{12} = 8.65 GPa (0.12%), G_{23} = 2.44 GPa (5.2%)
$[0^\circ_2/90^\circ_2/0^\circ_2/90^\circ_2]_s$	w_3 = 0.047 mm	ν_{12} = 0.4	ν_{12} = 0.32 (3.0%)

Note: Values in parentheses denote percentage difference between predicted and measured data.

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TABLE 4. Material Constants Identification of Laminated Composite Plates Subjected to Point Load ($a/b = 2$)

Plate layup	Measured deformational parameter	Identified material constant (first stage)	Identified material constant (second stage)
$[45^\circ_2/0^\circ_3/-45^\circ_2/0^\circ_3/45^\circ_2]_s$	$w_1 = 0.33$ mm	$E_1 = 127.12$ GPa, $E_2 = 9.87$ GPa	$E_1 = 125.78$ GPa (0.9%), $E_2 = 9.28$ GPa (3.3%)
$[45^\circ_2/0^\circ_3/-45^\circ_2/0^\circ_3/45^\circ_2]_s$	$\varepsilon_{x1} = 0.121 \times 10^{-3}$	$G_{12} = 8.15$ GPa, $G_{23} = 2.49$ GPa	$G_{12} = 8.46$ GPa (2.1%), $G_{23} = 2.43$ GPa (4.74%)
$[45^\circ_2/0^\circ_3/-45^\circ_2/0^\circ_3/45^\circ_2]_s$	$\varepsilon_{y1} = 0.185 \times 10^{-3}$	$\nu_{12} = 0.3$	$\nu_{12} = 0.331$ (0.3%)
$[0^\circ_2/90^\circ_2/0^\circ_2/90^\circ_2]_s$	$w_1 = 0.7$ mm	$E_1 = 125.41$ GPa, $E_2 = 9.8$ GPa	$E_1 = 125.41$ GPa (0.59%), $E_2 = 9.28$ GPa (3.3%)
$[0^\circ_2/90^\circ_2/0^\circ_2/90^\circ_2]_s$	$\varepsilon_{x1} = 0.25 \times 10^{-2}$	$G_{12} = 8.21$ GPa, $G_{23} = 2.42$ GPa	$G_{12} = 8.46$ GPa (2.08%), $G_{23} = 2.18$ GPa (6.0%)
$[0^\circ_2/90^\circ_2/0^\circ_2/90^\circ_2]_s$	$\varepsilon_{y1} = 0.4 \times 10^{-4}$	$\nu_{12} = 0.32$	$\nu_{12} = 0.332$ (0.6%)
$[0^\circ_2/90^\circ_2/0^\circ_2/90^\circ_2]_s$	$w_1 = 0.7$ mm	$E_1 = 123.43$ GPa, $E_2 = 9.23$ GPa	$E_1 = 125.4$ GPa (0.6%), $E_2 = 9.24$ GPa (3.75%)
$[0^\circ_2/90^\circ_2/0^\circ_2/90^\circ_2]_s$	$w_2 = 0.4$ mm	$G_{12} = 8.7$ GPa, $G_{23} = 2.42$ GPa	$G_{12} = 8.62$ GPa (0.02%), $G_{23} = 2.43$ GPa (5.2%)
$[0^\circ_2/90^\circ_2/0^\circ_2/90^\circ_2]_s$	$w_3 = 0.29$ mm	$\nu_{12} = 0.4$	$\nu_{12} = 0.31$ (6.0%)

Note: Values in parentheses denote percentage difference between predicted and measured data.

TABLE 5. Variation of Identified Elastic Constants for Laminated Composite Plates ($a/b = 1$)

Plate layup	Loading condition	Deformation parameter	Coefficient of Variation (%)				
			E_1	E_2	G_{12}	G_{23}	ν_{12}
$[45^\circ_2/0^\circ_3/-45^\circ_2/0^\circ_3/45^\circ_2]_s$	Point load	$w_1 = 0.59$ mm, $\varepsilon_{x1} = 0.7 \times 10^{-3}$, $\varepsilon_{y1} = 1.1 \times 10^{-3}$	10.7	14.9	15.4	25.1	10.3
$[45^\circ_2/0^\circ_3/-45^\circ_2/0^\circ_3/45^\circ_2]_s$	Uniform load	$w_1 = 0.12$ mm, $\varepsilon_{x1} = 0.2 \times 10^{-3}$, $\varepsilon_{y1} = 0.16 \times 10^{-3}$	6.1	9.14	11.7	15.1	5.3
$[0^\circ_2/90^\circ_2/0^\circ_2/90^\circ_2]_s$	Point load	$w_1 = 1.63$ mm, $\varepsilon_{x2} = 0.65 \times 10^{-3}$, $\varepsilon_{y2} = 0.53 \times 10^{-3}$	9.8	13.7	14.2	21.2	10.1
$[0^\circ_2/90^\circ_2/0^\circ_2/90^\circ_2]_s$	Point load	$w_1 = 1.63$ mm, $w_2 = 1.01$ mm, $w_3 = 0.352$ mm	6.5	10.3	11.7	14.4	5.8
$[0^\circ_2/90^\circ_2/0^\circ_2/90^\circ_2]_s$	Uniform load	$w_1 = 0.348$ mm, $w_2 = 0.275$ mm, $w_3 = 0.125$ mm	10	14.3	16.7	24.4	9.8
$[0^\circ_2/90^\circ_2/0^\circ_2/90^\circ_2]_s$	Uniform load	$w_1 = 0.348$ mm, $\varepsilon_{x1} = 0.2 \times 10^{-3}$, $\varepsilon_{y1} = 0.238 \times 10^{-3}$	5.4	9.6	11.1	14.7	5.8

TABLE 6. Variation of Identified Elastic Constants for Laminated Composite Plates ($a/b = 2$)

Plate layup	Loading condition	Deformation parameter	Coefficient of Variation (%)				
			E_1	E_2	G_{12}	G_{23}	ν_{12}
$[45^\circ_2/0^\circ_3/-45^\circ_2/0^\circ_3/45^\circ_2]_s$	Point load	$w_1 = 0.33$ mm, $\varepsilon_x = 0.121 \times 10^{-3}$, $\varepsilon_y = 0.185 \times 10^{-3}$	10.2	13.5	15.9	27.8	9.7
$[45^\circ_2/0^\circ_3/-45^\circ_2/0^\circ_3/45^\circ_2]_s$	Uniform load	$w_1 = 0.03$ mm, $\varepsilon_x = 0.2 \times 10^{-3}$, $\varepsilon_y = 0.373 \times 10^{-4}$	8.7	11.6	13.9	19.4	7.2
$[0^\circ_2/90^\circ_2/0^\circ_2/90^\circ_2]_s$	Point load	$w_1 = 0.7$ mm, $\varepsilon_x = 0.25 \times 10^{-3}$, $\varepsilon_y = 0.4 \times 10^{-4}$	12.2	15.3	17.6	28.4	10.3
$[0^\circ_2/90^\circ_2/0^\circ_2/90^\circ_2]_s$	Point load	$w_1 = 0.7$ mm, $w_2 = 0.4$ mm, $w_3 = 0.29$ mm	7.9	11.2	11.9	15.3	7.2
$[0^\circ_2/90^\circ_2/0^\circ_2/90^\circ_2]_s$	Uniform load	$w_1 = 0.057$ mm, $\varepsilon_x = 0.25 \times 10^{-3}$, $\varepsilon_y = 0.4 \times 10^{-4}$	7.5	10.2	11.7	14.1	6.9
$[0^\circ_2/90^\circ_2/0^\circ_2/90^\circ_2]_s$	Uniform load	$w_1 = 0.057$ mm, $w_2 = 0.053$ mm, $w_3 = 0.047$ mm	7.8	9.7	11.1	14.2	6.7

curacy. Among the elastic constants, excellent results can be obtained for the predictions of E_1, E_2, and G_{12}, for which the errors are <4%. On the other hand, the error for the prediction of G_{23} can be as large as 8.2%. The cause of the relatively large errors in the prediction of G_{23} is due to the relatively small contribution of transverse shear deformation to the magnitudes of the deformational parameters. It is worth noting that if the present normalization and bounding techniques were not adopted in the material constants identification process, erroneous results would be obtained, or worse, there would be difficulties in making the solution converge. In addition to the laminates that have been used as examples in the above illustration, elastic constants of other laminates have also been identified using the present method and similar accuracies of the identified elastic constants have been observed.

Finally, the effects of uncertainties encountered in deformation measurements on the variations of the identified elastic constants are studied using the aforementioned approximate analysis [see (18)]. The coefficients of variation for the measured deformational parameters are assumed to be 5% in the sensitivity analysis of the elastic constants of the laminated composite plates that have been tested. Tables 5 and 6 list the coefficients of variation of the identified elastic constants for the laminated composite plates. It is noted that the coefficients of variation of the elastic constants are in the range from 5.3% to 28.4%. Among the elastic constants, G_{23} is most sensitive to the variations of the measured deformational parameters. Furthermore, G_{23} becomes much more sensitive to the variations of the measured deformational parameters if the plates are subjected to a center point load. For instance, for the $[45^\circ_2/0^\circ_3/-45^\circ_2/0^\circ_3/45^\circ_2]_s$ plate of $a/b = 1$, the coefficients of variation of G_{23} for the cases of point and uniform loads are 25.1 and 15.1 respectively. Again, the cause of the relatively large variations of G_{23} is due to the relatively small contribution of transverse shear deformation to the magnitudes of the deformational parameters and the small value of G_{23} itself.

CONCLUSIONS

A two-stage constrained minimization method for the identification of material elastic constants of shear deformable laminated composite plates has been presented. The proposed method has been established on the basis of a global optimization method coupled with a two-stage bounding technique. A technique for normalizing the design variables has also been adopted in the present method to increase the convergence rate of the solution. Only three deformational parameters measured from the static test of a laminated composite plate are needed in the present method for the identification of elastic constants of the plate. Static tests of several laminated composite plates with different layups and aspect ratios subjected to various loading conditions have been performed, along with the experimental data used to study the feasibility and accuracy of the present method. The study has shown that the present method can produce good estimates of the elastic constants for the laminated composite plates in an effective and efficient way. The error in the identification of elastic constants E_1, E_2, and G_{12} are <4%, while those for ν_{12} and G_{23} are <8.3%. A sensitivity analysis of the variations of the identified elastic constants has been performed. It has been shown that the elastic constants may have 5.3–28.4% variations when there are 5% variations in the measured deformational parameters. Among the elastic constants, G_{23} is most sensitive to the variations of the measured deformational parameters, and the rea-

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son for this is due to the relatively small contribution of G_{23} to the deformation of the plates.

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