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Technical note

A fast algorithm for assortment optimization problems

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Abstract

Assortment optimization problems intend to seek the best way of placing a given set of rectangles within a minimum-area rectangle. Such problems are often formulated as a quadratic mixed 0–1 program. Many current methods for assortment problems are either unable to find an optimal solution or being computationally inefficient for reaching an optimal solution. This paper proposes a new method which finds the optimum of assortment problem by solving few linear mixed 0–1 programs. Numerical examples show that the proposed method is more computationally efficient than current methods.

Scope and purpose

Assortment optimization problems aim at cutting given rectangular pieces from a larger rectangle where the wasteful area is minimized. Current assortment optimization methods (Chen et al., European Journal of Operational Research 1993; 63: 362–67; Li and Chang, European Journal of Operational Research 1998; 105: 604–12) are either unable to find optimal solution or being computationally inefficient for reaching the optimal solution. This paper proposes a fast algorithm which only requires to solve three linear programs. Numerical examples demonstrate that the proposed algorithm is much faster than current methods. By utilizing this algorithm, many practical cutting programs in industries could be solved efficiently.

Keywords: Assortment; Cutting; Optimization

1. Introduction

Assortment problems occur when a number of small rectangular pieces need to be cut from a large rectangle to get minimum area. Assortment optimization techniques have been widely

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applied in industries such as in solving cutting problems of rectangle steel bars [1] and in solving a guillotine cutting problem [2]. Chen et al. [3] presented a mixed integer programming for assortment problems. Their method however can only find a feasible solution instead of an optimal solution. Recently, Li and Chang [4] developed a method for finding the optimal solution of the assortment problem. However, Li and Chang's method requires to use numerous 0–1 variables to linearize the polynomial objective function in their models, which would cause heavy computational burden.

This paper proposes another method for finding the optimum of an assortment problem. The major advantage of this method is that it obtains the optimal solution by solving a linear 0–1 problem without adding any extra variables. The computational efficiency in the proposed model can therefore be improved significantly. The numerical examples demonstrate that the proposed method can find the optimal solution. In addition, the computational time of the proposed method is much less than that in current methods.

2. Problem formulation

Given n rectangles with fixed lengths and widths. An assortment optimization problem is to allocate all of these rectangles within an enveloping rectangle, which has minimum area. Denote x and y as the width and the length of the enveloping rectangle ($x > 0, y > 0$), the assortment optimization problem is stated briefly as follows:

Minimize xy

subject to 1. All of n rectangles are non-overlapping.

2. All of n rectangles are within the range of x and y.

3. $0 < \underline{x} \leqslant x \leqslant \bar{x}$ and $0 < \underline{y} \leqslant y \leqslant \bar{y}$ ($\underline{x}$, $\bar{x}$, $\underline{y}$ and $\bar{y}$ are constants).

The related terminologies used in assortment models, referring to Li and Chang [4], are described below (Fig. 1).

(p_i, q_i): Dimension of rectangle i, p_i is the long side and q_i is the short side, p_i and q_i are constants, $i \in J$, J is the set of given rectangles.

(x, y): The top right corner coordinates of the enveloping rectangle, x and y are variables.

x'_i: Distance between center of rectangle i and original point along the x-axis.

y'_i: Distance between center of rectangle i and original point along the y-axis.

s_i: An orientation indicator for rectangle i, $i \in J$. $s_i = 1$ if p_i (the longer dimension of rectangle i) is parallel to the x-axis; $s_i = 0$ if p_i is parallel to the y-axis. Take Fig. 1 for example, $s_1 = 0$ and $s_2 = 1$.

The conditions of non-overlapping between rectangles i and k can be reformulated by introducing two binary variables u_{ik} and v_{ik} as follows (Fig. 2).

Condition 1: $u_{ik} = 0$ and $v_{ik} = 0$ if and only if rectangle i is at the right of rectangle k.

Condition 2: $u_{ik} = 1$ and $v_{ik} = 0$ if and only if rectangle i is at the left of rectangle k.

Condition 3: $u_{ik} = 0$ and $v_{ik} = 1$ if and only if rectangle i is above the rectangle k.

Condition 4: $u_{ik} = 1$ and $v_{ik} = 1$ if and only if rectangle i is below the rectangle k.

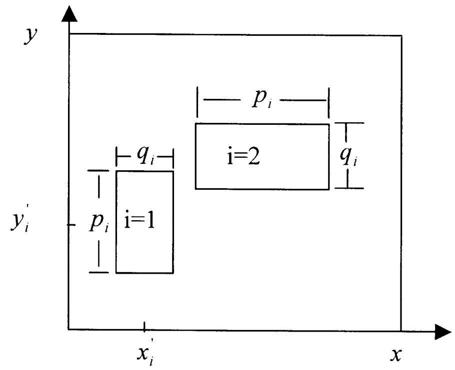

Fig. 1. Graphical illustration of assortment problem.

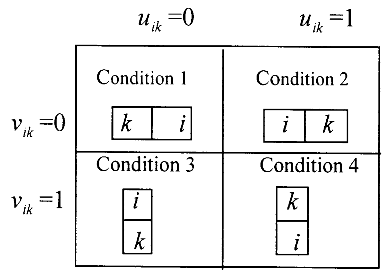

Fig. 2. Graphical illustration of non-overlapping conditions.

The assortment optimization problem can then be formulated as follows, referring to Li and Chang [4].

Assortment problem

$$\textbf{P1}: \quad \text{Minimize} \quad xy \quad \text{subject to} \tag{1}$$

$$(x_i' - x_k') + u_{ik}\bar{x} + v_{ik}\bar{x} \geqslant \tfrac{1}{2}[p_i s_i + q_i(1-s_i) + p_k s_k + q_k(1-s_k)], \quad \forall i,k \in J, \tag{2}$$

$$(x_k' - x_i') + (1-u_{ik})\bar{x} + v_{ik}\bar{x} \geqslant \tfrac{1}{2}[p_i s_i + q_i(1-s_i) + p_k s_k + q_k(1-s_k)], \quad \forall i,k \in J, \tag{3}$$

$$(y_i' - y_k') + u_{ik}\bar{y} + (1-v_{ik})\bar{y} \geqslant \tfrac{1}{2}[p_i(1-s_i) + q_i s_i + p_k(1-s_k) + q_k s_k], \quad \forall i,k \in J, \tag{4}$$

$$(y_k' - y_i') + (1-u_{ik})\bar{y} + (1-v_{ik})\bar{y} \geqslant \tfrac{1}{2}[p_i(1-s_i) + q_i s_i + p_k(1-s_k) + q_k s_k], \quad \forall i,k \in J, \tag{5}$$

$$\bar{x} \geqslant x \geqslant x_i' + \tfrac{1}{2}[p_i s_i + q_i(1-s_i)], \quad \forall i \in J, \tag{6}$$

$$\bar{y} \geqslant y \geqslant y_i' + \tfrac{1}{2}[p_i(1-s_i) + q_i s_i], \quad \forall i \in J, \tag{7}$$

$$x_i' - \tfrac{1}{2}[p_i s_i + q_i(1-s_i)] \geqslant 0, \quad \forall i \in J, \tag{8}$$

$$y_i' - \tfrac{1}{2}[p_i(1-s_i) + q_i s_i] \geqslant 0, \quad \forall i \in J, \tag{9}$$

where u_{ik}, v_{ik}, s_i, s_k are 0–1 variables, and x, y, x_i', x_k', y_i' and y_k' are bounded continuous variables.

Constraints (2)–(5) are non-overlapping conditions and constraints (6)–(9) ensure that all rectangles are within the enveloping rectangle.

Problem (1)–(9) is a mixed 0–1 program with a quadratic objective function, which is difficult to solve to find an optimal solution by the method discussed in Chen et al. [3]. For treating the quadratic term xy in (1), Chen et al. [3] first fix x variable as a constant, then solve the linear mixed 0–1 program to obtain the solution. The solution they find however is only a feasible solution instead of an optimal solution. Li and Chang [4] proposed an approach for solving the problem to obtain an optimum. The basic idea of their method is to approximately substitute x and y continuous variables in (1) by a set of 0–1 variables thus to linearize the product term xy. Problem (1)–(9) can then be reformulated as a linear mixed 0–1 problem which can be solved to reach an optimum within a tolerable error. Li and Chang's model is introduced briefly as follows.

3. Li and Chang approach

Li and Chang [4] substitutes x and y in (1) as follows:

$$x = \bar{\varepsilon}_x \sum_{g=1}^{G} 2^{g-1}\theta_g + \varepsilon_x, \qquad y = \bar{\varepsilon}_y \sum_{h=1}^{H} 2^{h-1}\delta_h + \varepsilon_y,$$

where ε_x and ε_y are small positive variables. $\bar{\varepsilon}_x$ and $\bar{\varepsilon}_y$ are the pre-specified constants which are the upper bounds of ε_x and ε_y, respectively. θ_g and δ_h are 0–1 variables, and G and H are integers which denote the number of required 0–1 variables for representing x or y. The polynomial term xy in (1) is then represented as

$$xy = \bar{\varepsilon}_x \sum_{g=1}^{G} 2^{g-1}\theta_g y + \bar{\varepsilon}_y \sum_{h=1}^{H} 2^{h-1}\delta_h \varepsilon_x + \varepsilon_x \varepsilon_y. \tag{10}$$

A full Li and Chang model is reformulated as a linear mixed 0–1 program below.

Model 1:

$$\text{Minimize} \quad \bar{\varepsilon}_x \sum_{g=1}^{G} 2^{g-1} z_g + \bar{\varepsilon}_y \sum_{h=1}^{H} 2^{h-1} u_h \tag{11}$$

subject to

$$z_g \geqslant y + \bar{y}(\theta_g - 1), \quad g = 1, 2, \ldots, G, \tag{12}$$

$$u_h \geqslant \varepsilon_x + \bar{\varepsilon}_x(\delta_h - 1), \quad h = 1, 2, \ldots, H, \tag{13}$$

(2)–(9)

$$z_g \geqslant 0, \quad u_h \geqslant 0, \quad \theta_g, \quad \delta_h \in \{0, 1\}.$$

The major difficulty of Model 1 is that it involves $G + H$ additional 0–1 variables. The smaller the tolerable errors (i.e., ε_x and ε_y), the larger the size of G and H and the longer the CPU time for solving the problem.

4. Proposed method

Denote F as a feasible set of Problem P1, $F = \{(2), (3), (4), \ldots, (9)\}$.
Define a local optimum of Problem P1 below.

Definition 1. A point (x^*, y^*), $(x^*, y^*) \in F$, is a local optimum of Problem P1 if $x^* y^* \leqslant (x^* \pm \varepsilon)(y^* \pm \varepsilon)$ for all $(x^* \pm \varepsilon, y^* \pm \varepsilon) \in F$, where ε is a noticeable small positive value.

Remark 1. By referring to Definition 1, there are eight neighborhood points for a given reference (x^0, y^0) as shown in Fig. 3:

$$(x^0 + \varepsilon, y^0 + \varepsilon),\ (x^0 + \varepsilon, y^0),\ (x^0 + \varepsilon, y^0 - \varepsilon),\ (x^0, y^0 + \varepsilon),\ (x^0, y^0 - \varepsilon),$$

$$(x^0 - \varepsilon, y^0 + \varepsilon),\ (x^0 - \varepsilon, y^0) \text{ and } (x^0 - \varepsilon, y^0 - \varepsilon).$$

Consider a linear mixed 0–1 program below.

P2: Minimize $z = x + y$
subject to $(x, y) \in F,\ x \geqslant y.$
where the constraint $x \geqslant y$ is to denote x as the larger side of the rectangle.

Let the obtained objective value of solving P2 be z^0.
P2 may have multiple optimal solutions (x', y'). What we are interested in is to find one of these solutions, which has minimal xy value.
Consider the following proposition.

Proposition 1. *Let S be a set of solutions of P1.* $S = \{(x, y) \mid x + y = z^0, (x, y) \in F,\ x \geqslant y\}$. *If there is a point* $(x^0, y^0) \in F$ *in which* $x^0 \geqslant x'$ *for all* $(x', y') \in F$, *then* $x^0 y^0 \leqslant x' y'$.

Proof. Since $x^0 \geqslant x'$, there exist a $\theta \geqslant 0$ such that $x' = x^0 - \theta$ and $y' = y^0 + \theta$. Because $x^0 - \theta \geqslant y^0 + \theta$, it is clear that $x^0 - y^0 - 2\theta \geqslant 0$. We then have

$$x' y' = (x^0 - \theta)(y^0 + \theta) = x^0 y^0 + (x^0 - y^0 - \theta) \geqslant x^0 y^0. \quad \square$$

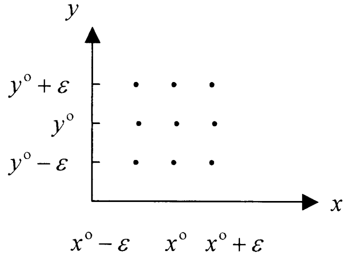

Fig. 3.

A point (x^0, y^0) satisfying Proposition 1 can be obtained by solving following linear mixed 0–1 program.

P3: Maximize x

subject to $(x, y) \in F,\ x \geqslant y,\ x + y = z^0$.

Let the solution be (x^0, y^0).

We then deduce the following theorem.

Theorem 1. *The optimal solution* (x^0, y^0) *of P3 is also a local optimum of P1.*

Proof. Examining the eight neighborhood points of (x^0, y^0) in Remark 1 as follows.

(i) Some of these points may not be the feasible points of P1. Since the minimal value of $x + y$ for $(x, y) \in F$ is $x^0 + y^0 = z^0$, it is clear that three points $(x^0, y^0 - \varepsilon)$,$(x^0 - \varepsilon, y^0)$ and $(x^0 - \varepsilon, y^0 - \varepsilon)$ are not feasible points of P1.

(ii) Since $x^0 = \{\text{Max}\, x \,|\, (x, y) \in F,\ x \geqslant y,\ x + y = z^0\}$, point $(x^0 + \varepsilon, y^0 - \varepsilon)$ is also an infeasible point of P1 $(\Theta x^0 + \varepsilon > x^0)$.

(iii) It is unclear that whether these points $(x^0 + \varepsilon, y^0 + \varepsilon)$, $(x^0 + \varepsilon, y^0)$ and $(x^0, y^0 + \varepsilon)$ are feasible or not for P1. However, since $(x^0 + \varepsilon, y^0 + \varepsilon) > x^0 y^0$, $(x^0 + \varepsilon) y^0 > x^0 y^0$ and $x^0 (y^0 + \varepsilon) > x^0 y^0$, (x^0, y^0) is better than these three points.

(iv) By referring to Proposition 1, we have, $(x^0 - \varepsilon)(y^0 + \varepsilon) > x^0 y^0$.

Since each neighborhood point of (x^0, y^0) is either infeasible or inferior to (x^0, y^0). Therefore, (x^0, y^0) is a local optimum of P1 following Definition 1. □

Table 1
Computational comparison of two models

Problem No.	No. of rectangles	p_i	q_i	CPU time (hh:mm:ss)		Objective value	
				Model 1	Proposed method	Model 1	Proposed method
1	4	24	20	00:05:12	00:00:03	1178	1178
		18	16				
		16	14				
		21	7				
2	5	33	10	> 10:00:00	00:01:18	NA	1518
		30	11				
		25	15				
		18	14				
		18	10				

5. Numerical examples

Consider the following assortment optimization problem adopted from Chen et al. [3]: Some given rectangles are required to be placed within a rectangle which has minimum area. The sizes of pieces of rectangles are given in Table 1 and Figs. 4 and 5. Here we solve the same problem using Chen et al. [3] model, Model 1 and the proposed model by LINGO 5.0 (LINDO SYSTEMS INC., 1998, a common-used optimization package) [5,6] running in a personal computer.

Chen et al.'s model treats Problem 1 by fixing the value of y as $y = 36$, then they solved a nonlinear mixed 0–1 program to obtain a local optimal solution with an objective value that equals 1224. Model 1 solves the problem by specifying $\bar{\varepsilon}_x = \bar{\varepsilon}_y = 0.1$, and obtains the optimal solution which has the objective value 1178. Proposed method solves the problem and obtains the same solution as found by Model 1. Table 1 shows that for the cases with four and five rectangles, the proposed method spends much less time than Model 1 for finding the optimal solution.

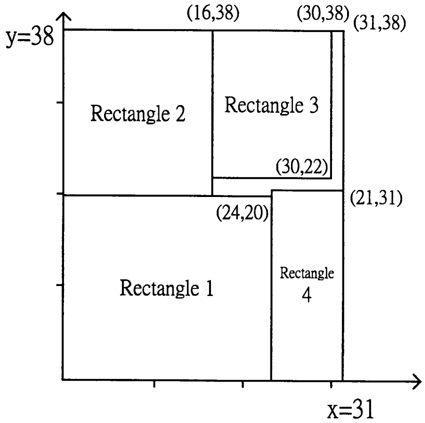

Fig. 4. Result for four rectangles.

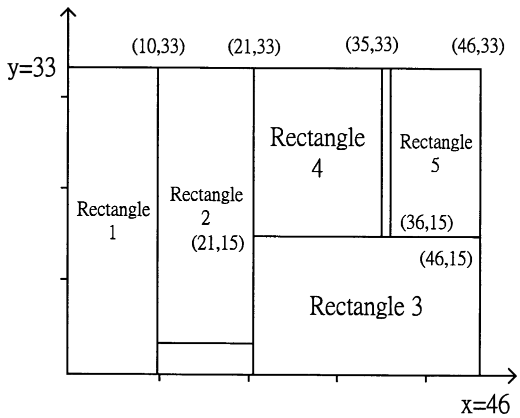

Fig. 5. Result for five rectangles.

6. Conclusions

This paper proposes a new method to solve the assortment problem. The proposed method reformulates the original problem as a linear mixed 0–1 program. Numerical examples demonstrate that the proposed method uses much less CPU time than that in Model 1 for reaching the optimal solution.

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