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Additive multiplicative increasing functions on nonnegative square matrices and multidigraphs ☆

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Abstract

It is known that if f is a multiplicative increasing function on $\mathscr{N}$, then either $f(n)=0$ for all $n\in\mathscr{N}$ or $f(n)=n^{\alpha}$ for some $\alpha\geqslant 0$. It is very natural to ask if there are similar results in other algebraic systems. In this paper, we first study the multiplicative increasing functions over nonnegative square matrices with respect to tensor product and then restrict our result to multidigraphs and loopless multidigraphs. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

A function f from the set of natural number $\mathscr{N}$ into the set of real number $\mathscr{R}$ is *additive* if $f(m+n)=f(m)+f(n)$ for all $m,n\in\mathscr{N}$, f is *multiplicative* if $f(m\cdot n)=f(m)\cdot f(n)$ for all $m,n\in\mathscr{N}$, and f is *increasing* if $f(m)\leqslant f(n)$ whenever $m\leqslant n$. The following theorem can easily be obtained.

Theorem 1.1. *If f is a multiplicative increasing on $\mathscr{N}$, then either $f(n)=0$ for all $n\in\mathscr{N}$ or $f(n)=n^{\alpha}$ for some $\alpha\geqslant 0$.*

From mathematical point of view, the above theorem is very good. It classifies all multiplicative increasing function on $\mathscr{N}$. We also observe that all multiplicative increasing functions are generated by additive multiplicative functions. It is very natural to study similar results on other algebraic systems.

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On the set of all undirected graphs, we can consider the relation 'subgraphs' as the partial order. The 'disjoint union' of two graphs as the *addition* of two graphs. As for the product of graphs, there are two important products, weak product and strong product, defined on graphs. Thus, we can study the (additive) multiplicative increasing graph functions with respect these two products. Some interesting (additive) multiplicative increasing graph functions with respect to these two products are discussed in literature [4,5,7–11]. Yet the classification of all (additive) multiplicative increasing graph functions with respect to these two products is still open. It is observed that each graph is represented by its adjacency matrix. The adjacency matrix of an (undirected simple) graph is a symmetric $(0,1)$-square matrix with diagonal 0. The adjacency matrix of the weak product of two graphs is actually the tensor product of the two corresponding adjacency matrices. However, the adjacency matrix of the strong product of two graphs can also be viewed as the tensor product of the two corresponding adjacency matrices if we set to 1 for each diagonal entry of the adjacency matrix. In this paper, we first extend our previous result to multiplicative increasing functions on nonnegative square matrices and then restrict it to multidigraphs and loopless multidigraphs. Therefore, we will review previous results on multiplicative increasing graph functions on weak product in Section 2 and on strong product in Section 3. All graphs are assumed to be undirected simple graphs in these sections. In Section 4, we point out the relationship among weak product, strong product, and tensor product. Then we discuss the multiplicative increasing functions over nonnegative square matrices in Section 5. Finally, we discuss the multiplicative increasing functions on multidigraphs and loopless multidigraphs in Sections 6 and 7.

2. MI functions for weak product

Most of the graph definitions used in this paper are standard (see, e.g., [1]). An (*undirected simple*) *graph* $G=(V,E)$ consists of a finite set (*vertices*) V and a subset (*edges*) E of $\{[u,v] \mid u \neq v, [u,v]$ is an unordered pair of elements of $V\}$. Let $\mathscr{G}$ be the set of all graphs. For $S \subseteq V$, we use $G|_S$ for the subgraph of G induces by S. We use P_n to denote the *path graph* with n vertices, and C_n to denote the *cycle graph* with n vertices. A *clique* in a graph G is a complete subgraph of G. The size of the largest clique in G is the *clique number* of G, denoted by $\omega(G)$.

Let $G=(X,E)$ and $H=(Y,F)$ be two graphs. The *sum* of G and H is defined as the disjoint union of G and H. Varying from [1], the *weak product* of G and H is defined as the graph $G \times H = (Z,K)$, where vertex set $Z = X \times Y$, the Cartesian product of X and Y, and edge set $K = \{[(x_1,y_1),(x_2,y_2)] \mid [x_1,x_2] \in E$ and $[y_1,y_2] \in F\}$. Let $[k]G$ denote $G+G+\cdots+G$ (k times) and $G^{[k]}$ denote $G \times G \times \cdots \times G$ (k times). For example, $K_{1,2}^{[2]} = K_{2,2} + K_{1,4}$.

Let $G=(X,E)$ and $H=(Y,F)$ be two graphs. A function ϕ from X into Y is a *homomorphism* from G into H if $[u,v] \in E$ implies $[\phi(u),\phi(v)] \in F$. A graph $\hat{G}$ is a *homomorphic image* of another graph G if there exists a homomorphism $\phi: G \to \hat{G}$

which is onto and for every $[\hat{u}_1, \hat{u}_2] \in E(\hat{G})$ there exists $[u_1, u_2] \in E(G)$ such that $\phi(u_i) = \hat{u}_i$, $i = 1, 2$. The *chromatic number* of G, $\chi(G)$, is the smallest integer m such that K_m is a homomorphic image of G. A graph G is *primary* if for every homomorphic image $\hat{G}$ of G there exists a positive integer k such that G is a subgraph of $\hat{G}^{[k]}$.

Let f be a real-valued function defined on $\mathscr{G}$. The function f is *additive* if $f(G+H) = f(G) + f(H)$ for any $G, H \in \mathscr{G}$, and f is *pseudo-additive* if $f(G+H) = f(G) + f(H)$ for any $G, H \in \mathscr{G}$ such that $f(G) \neq 0$ and $f(H) \neq 0$. The function f is *multiplicative* if $f(G \times H) = f(G) \cdot f(H)$ for any $G, H \in \mathscr{G}$, and f is *pseudo-multiplicative* if $f(G \times H) = f(G) \cdot f(H)$ for any $G, H \in \mathscr{G}$ such that $f(G) \neq 0$ and $f(H) \neq 0$. The function f is *increasing* if $f(G) \leqslant f(H)$ when G is a subgraph of H. A graph function f is MI if it is multiplicative and increasing. A graph function f is AMI if it is additive, multiplicative, and increasing. A graph function f is PAMI if it is pseudo-additive, pseudo-multiplicative, and increasing. Obviously, if a graph function f is AMI then f is PAMI. Note that MI is closed under taking the nonnegative power, finite product, and pointwise convergence. Let $S \subseteq$ MI. We use $\langle S \rangle$ to denote the set of functions obtained by taking nonnegative power, finite product, and pointwise convergence from elements of S. In other words, the following functions are elements in $\langle S \rangle$:

1. f^{α}, $\alpha \geqslant 0$ and $f \in S$.
2. $\prod_{i=1}^{k} f_i^{\alpha_i}$, $\alpha_i \geqslant 0$ and $f_i \in S$.
3. $\lim_{m \to \infty} f_m$, where f_m is of type (1) or (2).

2.1. Homomorphism functions

For a fixed graph G, we can define a function h_G from $\mathscr{G}$ into $\mathscr{R}$ by setting $h_G(H)$ to be the number of homomorphisms from G into H. The following theorem is proved in [4,5].

Theorem 2.1. *h_G is* MI *for any graph G. Moreover, h_G is* AMI *if G is connected.*

For example, let $f_1(G)$ be defined as the number of vertices of G. Obviously, $f_1 = h_{K_1}$. Moreover, it is proved in [4] that $f = h_{K_1}^{\alpha}$ for some $\alpha \geqslant 0$ if f is an MI function with $f(K_1) \neq 0$. Let $f_2(G)$ be defined as $2|E(G)|$. It can be checked that $f_2 = h_{K_2}$. Let $f_3(G)$ be defined as $\max\{\deg(v) \mid v \in G\}$. It is proved in [4] that $f_3 = \lim_{m\to\infty} h_{K_{1,m}}^{1/m}$. Let $f_4(G)$ be defined as $\max\{|\lambda| \mid \lambda$ is an eigenvalue of $A(G)\}$. It is proved in [4] that $f_4 = \lim_{m\to\infty} h_{P_m}^{1/m}$.

2.2. Generalized homomorphism functions

For any graph H and integer $m \in \mathscr{N}$, let H_m be the induced subgraph of H such that $x \in V(H_m)$ if and only if x is in an m-clique of H. For any graph function f, we can define another graph function f_m by $f_m(H) = f(H_m)$ for any graph H. It can be observed that $(H+K)_m = H_m + K_m$ and $(H \times K)_m = H_m \times K_m$. The following theorem is proved in [4,5]:

Theorem 2.2. *For any positive integer m,* f_m *is additive* (*multiplicative, increasing*) *if f is additive* (*multiplicative, increasing*).

Obviously, $\langle \{h_G \mid G \in \mathscr{G}\}\rangle \subseteq \langle \{(h_G)_m \mid G \in \mathscr{G}, m \geqslant 1\}\rangle$. It is proved in [4,5] that $(h_{K_1})_2 \notin \langle \{h_G \mid G \in \mathscr{G}\}\rangle$.

2.3. The $\phi_{G,S}$ functions

Let $G=(V,E)$ be a graph and $\emptyset \neq S = \{s_1, s_2, \ldots, s_k\} \subseteq V$. We define $\phi_{G,S} : \mathscr{G} \to \mathscr{R}$ by $\phi_{G,S}(H) = |\{(f(s_1), f(s_2), \ldots, f(s_k)) | f$ is a homomorphism from G into $H\}|$. The following theorem is proved in [7].

Theorem 2.3. $\phi_{G,S}$ *is* MI *for every graph* $G=(V,E)$ *and* $\emptyset \neq S \subseteq V$. *Moreover,* $\phi_{G,S}$ *is* AMI *if G is connected.*

For any $x_i \in V(G)$ and any $m \in \mathscr{N}$, we define a new set $T(x_i) = \{z_{i,j} \mid 2 \leqslant j \leqslant m\}$ such that $T(x_i) \cap T(x_j) = \emptyset$ if $x_i \neq x_j$. Then, we construct a graph $G(m)$ to be the smallest graph such that (1) $V(G(m)) = V(G) \cup (\bigcup_{x_i \in V(G)} T(x_i))$, (2) $G(m)|_{V(G)}$ is isomorphic to G, and (3) $G(m)|_{T(x_i) \cup \{x_i\}}$ is isomorphic to K_m for every $x_i \in V(G)$. It is proved in [7] that any $(h_G)_m$ can be written as $\phi_{G(m), V(G)}$. Hence, $\langle \{(h_G)_m \mid G \in \mathscr{G}, m \in \mathscr{N}\}\rangle \subseteq \langle \{\phi_{G,S} \mid G \in \mathscr{G}, \emptyset \neq S \subseteq V(G)\}\rangle$. Moreover, it is proved in [7] that $\phi_{W_5,\{o\}} \notin \langle \{(h_G)_m | G \in \mathscr{G}, m \in \mathscr{N}\}\rangle$ where W_5 is the 5-wheel graph with $\{o\}$ as its center vertex.

2.4. The δ function

Let G be a bipartite graph with bipartition (A,B). If G is connected bipartite, such a partition is unique, we say G is of (r,s) *type* if $|A| = r$ and $|B| = s$. For an arbitrary bipartite graph G with connected components $C_1, C_2, \ldots, C_m$, we say G is of $\sum_{i=1}^{n}(r_i, s_i)$ *type* if C_i is of (r_i, s_i) type for every i. Let θ be the function defined on the set of bipartite graphs by setting $\theta(G) = 2(\sum_{i=1}^{m}(r_i \times s_i)^{1/2})$ where G is of $\sum_{i=1}^{m}(r_i, s_i)$ type. For any graph G, it can be checked that $G \times K_{1,1}$ is bipartite. We define $\delta : \mathscr{G} \to \mathscr{R}$ by $\delta(G) = \frac{1}{2}\theta(G \times K_{1,1})$. It is proved in [4,5] that δ is an AMI function which is not generated by functions in Section 2.3.

2.5. The graph capacity functions

For a fixed graph G, the *G-matching function*, γ_G, assigns any graph $H \in \mathscr{G}$ to the maximum integer k such that $[k]G$ is a subgraph of H. The *graph capacity function* for G, $P_G : \mathscr{G} \to \mathscr{R}$, is defined as $P_G(H) = \lim_{m\to\infty}[\gamma_G(H^{[m]})]^{1/m}$. Different graphs G

may have different graph capacity functions. In [8], capacity functions of all graphs are classified into AMI, PAMI but not AMI, and none of the above cases.

A *digraph* $G=(V,E)$ consists of a finite set V and a subset of $\{(u,v)\,|\,(u,v)$ is an ordered pair of element of $V\}$. The *homomorphism digraph*, $G^*=(V^*,E^*)$ of G is the directed graph with $V^*=V$ and $(a,b)\in E^*$ if there is a homomorphism ϕ from G into itself such that $\phi(a)=b$. Obviously, $(v,v)\in E^*$ for every $v\in V$. Let S be any subset of V. The *out-neighborhood* of S is the set $\Gamma(S)=\{y\,|\,(x,y)\in E^*$ with $x\in S\}$. Thus, $S\subseteq\Gamma(S)$ for every $S\subseteq V$. A nonempty subset S of V is called a *closed set* of G if (1) $\Gamma(S)\subseteq S$ and (2) there is no proper subset S' of S such that $\Gamma(S')\subseteq S'$. It is easy to see that there exists a closed set for every graph. It is proved in [8] that P_G is PAMI if and only if G contains exactly one closed set.

Let G be a graph with exactly one closed set S. A nonempty subset C of S is called a *core* if (1) there exists a homomorphism $\phi:G\to G$ satisfies $\phi(S)=C$; and (2) there is no proper subset C' of C such that there exists a homomorphism $\phi':G\to G$ satisfying $\phi'(S)=C'$. Obviously, such a core C exists. It is proved in [8] that P_G is AMI if (1) $P_G=P_{G|_C}$, where C is a core in the unique closed set in G; and (2) $G|_C$ is primary. Complete graphs, odd cycles, and the Petersen graph are examples of graphs whose capacity functions are AMI. Again, some AMI capacity functions are not generated by functions in previous subsections.

2.6. The $f_{G,S}$ functions

We can combine the concept behind Theorem 2.2 and the $\phi_{G,S}$ functions to build a new family of (additive) multiplicative increasing functions. Let G be a graph and S be a nonempty subset of $V(G)$. For any graph H, we define $H_{G,S}$ to be the induced subgraph of H such that any $y\in V(H_{G,S})$ if and only if there exists a homomorphism ϕ from G into H such that $\phi(x)=y$ for some $x\in S$. For any graph function f, we can define another graph function $f_{G,S}$ by $f_{G,S}(H)=f(H_{G,S})$ for any graph H. It follows from Theorem 2.3 that $H_{G,S}+K_{G,S}=(H+K)_{G,S}$ and $H_{G,S}\times K_{G,S}=(H\times K)_{G,S}$. Thus, we have the following theorem.

Theorem 2.4. *For any graph G and any nonempty subset S of $V(G)$, $f_{G,S}$ is additive (multiplicative, increasing) if f is additive (multiplicative, increasing).*

2.7. Hedetniemi conjecture and MI functions

A family of graphs, I, is called a *hereditary ideal* if (1) the subgraph H of any graph $G\in I$ belongs to I, and (2) $G\times H\in I$ for any $G\in I$ and $H\in\mathcal{G}$. For example, $\Omega_n=\{G\,|\,\omega(G)\leqslant n\}$ is a hereditary ideal for any positive integer n. Given a hereditary ideal I, a positive integer k, and a graph G, an *I-coloring* of G is a function $\pi:V(G)\to\{1,2,\ldots,k\}$ such that the induced subgraph $G|_{\langle\pi^{-1}(i)\rangle}$ of G is in I for every i. The *I-chromatic number* of G, $\chi(G:I)$ is the least k for which G has an I-coloring. Note that $\chi(G)=\chi(G:\Omega_1)$. It is proved in [3] that

$\chi(G \times H : I) \leqslant \min\{\chi(G : I), \chi(H : I)\}$ for any hereditary ideal I and any $G, H \in \mathscr{G}$. In particular, $\chi(G \times H : \Omega_n) = \min\{\chi(G : \Omega_n), \chi(H : \Omega_n)\}$ holds if $n \geqslant 2$. The statement $\chi(G \times H : \Omega_1) = \min\{\chi(G : \Omega_1), \chi(H : \Omega_1)\}$ holds for all G and H is equivalent to the famous Hedetniemi conjecture [3]. Harary and Hsu [2] generalize the Hedetniemi conjecture into the statement $\chi(G \times H : I) = \min\{\chi(G : I), \chi(H : I)\}$ for any I, G and H.

Let n be a positive integer and I be a hereditary ideal. For any graph $G = (V, E)$, we define $G_{[I,n]}$ to be the graph G if $\chi(G : I) \geqslant n$, and $G_{[I,n]}$ to be the empty graph if oth-erwise. Then for any graph function f, we define $f_{[I,n]}$ by setting $f_{[I,n]}(G) = f(G_{[I,n]})$. Let y be a vertex in G. We use $C(y : G)$ denote the connected component of G containing y. Let $V(G'_{[I,n]}) = \{y \mid \chi(C(y : G) : I) \geqslant n\}$. Then we use $G'_{[I,n]}$ to denote the induced subgraph $G|_{V(G'_{[I,n]})}$. Given any graph function f, we define $f'_{[I,n]}$ by setting $f'_{[I,n]}(G) = f(G'_{[I,n]})$. For example, let f_5 be defined as the size of the largest nonbipartite connected component. Then f_5 can be both expressed as $(h_{K_1})'_{[\Omega_1,3]}$ and $\lim_{n\to\infty} \phi_{C_{2n+1},\{x_{2n+1}\}}$ where x_{2n+1} is any vertex in the odd cycle C_{2n+1}. It is easy to obtain the following theorem.

Theorem 2.5. *The following statements are equivalent*:

(1) $\chi((G \times H) : I) = \min\{\chi(G : I), \chi(H : I)\}$ *for any* G, H *and* I.
(2) $G_{[I,n]} \times H_{[I,n]} = (G \times H)_{[I,n]}$ *for any integer n and any hereditary ideal I*.
(3) $G'_{[I,n]} + H'_{[I,n]} = (G + H)'_{[I,n]}$ *and* $G'_{[I,n]} \times H'_{[I,n]} = (G \times H)'_{[I,n]}$ *for any integer n and any hereditary ideal I*.
(4) *For any integer n and any hereditary ideal I*, $f_{[I,n]}$ *is* MI *if* f *is* MI.
(5) *For any integer n and any hereditary ideal I*, $f'_{[I,n]}$ *is* AMI *if* f *is* AMI.
(6) $(h_{K_1})_{[I:n]}$ *is* MI *for any integer n and any hereditary ideal I*.
(7) $(h_{K_1})'_{[I:n]}$ *is* AMI *for any integer n and any hereditary ideal I*.

With the above theorem, we notice that to classify MI functions is at least as difficult as to solve the Hedetniemi conjecture.

3. MI functions for strong product

Let $G = (X, E)$, $H = (Y, F)$ be two graphs. The *strong product* of G and H is the graph $G \cdot H = (Z, K)$ where $Z = X \times Y$ and $K = \{[(x_1, y_1), (x_2, y_2)] \mid ([x_1, x_2] \in E$ and $[y_1, y_2] \in F)$ or $(x_1 = x_2$ and $[y_1, y_2] \in F)$ or $([x_1, x_2] \in E$ and $y_1 = y_2)\}$. With this strong product, the terminology of *strong multiplicative increasing graph function* (SMI) and *strong additive multiplicative increasing graph function* (SAMI) can be similarly defined.

Let $G = (X, E)$, $H = (Y, F)$ be two graphs. A map $\Psi : X \to Y$ is called a *strong homomorphism* from G into H if $[x_1, x_2] \in E$ implies $[\Psi(x_1), \Psi(x_2)] \in F$ or $\Psi(x_1) = \Psi(x_2)$. For a fixed graph G, we can define $\hbar_G$ as a function from $\mathscr{G}$ into $\mathscr{R}$ such that $\hbar_G(H)$ equals the number of strong homomorphisms from G into H. Again $\hbar_G$ is SMI for any graph G and $\hbar_G$ is SAMI if G is connected. The clique number of G, $\omega(G)$, can be proved to be $\lim_{n\to\infty}(\hbar_{K_n})^{1/n}(G)$.

Let G be a graph and $S=\{s_1,s_2,\ldots,s_k\}$ be a nonempty subset of $V(G)$. Similarly, we can define $\Psi_{G,S}:\mathscr{G}\to\mathscr{R}$ by $\Psi_{G,S}(H)=|\{(f(s_1),f(s_2),\ldots,f(s_k))\,|\,f$ is a strong homomorphism from G into $H\}|$. Again, $\Psi_{G,S}$ is MI for any $G=(V,E)$ and $\emptyset\neq S\subseteq V$ and $\Psi_{G,S}$ is AMI if G is connected. Let $T_{(n)}$ be the graph obtained from the star graph S_n ($\cong K_{1,n}$) by replacing each edge with a path of length n. Let $P_{(n)}$ be the pendant vertices of $T_{(n)}$. It is proved in [6,10] that $f_6=\lim_{n\to\infty}\Psi^{1/n}_{T_{(n)},P_{(n)}}$ where $f_6(G)$ is the number of vertices in the largest connected component of G. Moreover, f_6 is not in $\langle\{\hbar_G\,|\,G\in\mathscr{G}\}\rangle$.

4. Tensor product

Let $A=(a_{i,j})_{m\times m}$, $B=(b_{k,l})_{n\times n}$ be two matrices. The *direct sum* of A and B is the $(m+n)\times(m+n)$ matrix $A\oplus B=(c_{r,s})$ where $c_{r,s}=a_{r,s}$ if $1\leqslant r,\ s\leqslant m$; $c_{r,s}=b_{r-m,s-m}$ if $m<r,\ s\leqslant m+n$; and $c_{r,s}=0$ if otherwise. The *tensor product* of A and B is the $mn\times mn$ matrix $A\otimes B=(d_{(i,k),(j,l)})$ where $d_{(i,k),(j,l)}=a_{i,j}\cdot b_{k,l}$.

Let G be a graph with its adjacency matrix $A(G)$ and H be a graph with its adjacency matrix $A(H)$. It is easy to see that $A(G+H)$ is actually $A(G)\oplus A(H)$ and $A(G\times H)$ is actually $A(G)\otimes A(H)$. Let $\mathscr{MU}$ denote the set $\{(a_{i,j})_{n\times n}\,|\,n$ is a nonnegative integer, $a_{i,j}\in\{0,1\}$, $a_{i,j}=a_{j,i}$ for $1\leqslant i,\ j\leqslant n$, and $a_{i,i}=0$ for $1\leqslant i\leqslant n\}$. Note that any undirected simple graph is uniquely determined, up to isomorphism, by its adjacency matrix. We may assign a partial ordering '$\leqslant$' on $\mathscr{MU}$ by assigning $M_1\leqslant M_2$ if and only if the corresponding graph for M_1 is a subgraph of that for M_2. Then $(\mathscr{MU},\oplus,\otimes,\leqslant)$ forms an algebraic system. Obviously, the study of (A)MI functions for weak product is equivalent to study of (A)MI functions on $\mathscr{MU}$.

However, the strong product can also be viewed as the tensor product. Let $A'(G)$ is obtained from $A(G)$ be reassigning 1 to every diagonal entry. It is easy to see that $A'(G+H)$ is $A'(G)\oplus A'(H)$ and $A'(G\cdot H)$ is $A'(G)\otimes A'(H)$. Again the study of (A)MI functions for strong product is equivalent to the study of (A)MI functions on the set of square matrices, symmetric $(0,1)$-matrices with all diagonal entries to be 1. In other words, for any (undirected simple) graph G we construct a new (nonsimple) graph G' by assign a selfloop at each vertex of G. Then we define the adjacency matrix of G', $A(G')$, to be $A'(G)$. Thus, the strong product on $\{G\,|\,G\in\mathscr{G}\}$ is translated into the tensor product on $\{A'(G)\,|\,G\in\mathscr{G}\}$.

In the following sections, we are going to investigate the (A)MI functions on nonnegative square matrices and multidigraphs.

5. MI functions on nonnegative square matrices

A square matrix $M=(m_{ij})_{u\times u}$ is *nonnegative* if $m_{ij}\geqslant 0$ for $1\leqslant i,\ j\leqslant u$. For any $\alpha\geqslant 0$, we use (α) to denote the 1×1 matrix with α at its only entry. Let $\mathscr{M}$ denote the set of all nonnegative square matrices. In the following, all the matrices we discuss are matrices in $\mathscr{M}$. A *network* W is a digraph $G=(V,A)$ together with a nonnegative weight function w defined on A. We can associate a matrix $M=(m_{ij})_{u\times u}$ with a directed graph $G[M]$ and a network $W[M]$. $G[M]$ is the digraph with the vertex set $\{1,2,\dots,u\}$ and an arc joining from i to j if and only if $m_{i,j}>0$. Hence $G[M]$ has a loop at vertex i if $m_{ii}>0$. $W[M]$ is the digraph $G[M]$ together with a weight function that assigns m_{ij} to the arc (i,j) of $G[M]$ if $m_{ij}>0$; i.e., $w(i,j)=m_{ij}$. A digraph $G[M]$ is said to be *strongly connected* if for each vertex u of $G[M]$ there exists a directed path from u to any other vertex of $G[M]$. We say that digraph $G[M]$ is *weakly connected* if, when we remove the orientation from the arcs of $G[M]$, a connected graph or multigraph remains.

Let $R=(r_{ij})_{m\times m}$ and $T=(t_{kl})_{n\times n}$ be two matrices. We say that R is a *submatrix* of T, denote by $R\subseteq T$, if there is a one to one function f from $\{1,2,\dots,m\}$ to $\{1,2,\dots,n\}$ such that $r_{ij}\leqslant t_{f(i)f(j)}$ for every $r_{ij}\in R$. Two $n\times n$ matrices R and T are *isomorphic*, denoted by $R\cong T$, if there is a one to one mapping f from $\{1,2,\dots,n\}$ to itself such that $r_{ij}=t_{f(i)f(j)}$. In other words, $R\cong T$ if and only if $W(R)$ is isomorphic to $W(T)$. Hence, we use matrix and network interchangeably. For $M\in\mathscr{M}$, $[k]M$ denote $M\oplus M\oplus\cdots\oplus M$ (k times) and $M^{[k]}$ denote $M\otimes M\otimes\cdots\otimes M$ (k times), where $\oplus$ and $\otimes$ are direct sum and tensor product, respectively. We use kM for the scalar matrix multiplication. With these direct sum and tensor product, the terminology of *additive*, *pseudo-additive*, *multiplicative*, *pseudo-multiplicative* and *increasing* can be similarly defined. Obviously, the set of MI functions on nonnegative square matrices is closed under taking the nonnegative power, finite product, and pointwise convergence.

5.1. Homomorphism functions

Let $M=(m_{ij})_{u\times u}$ be a matrix and $W(M)$ be the corresponding network with the vertex set $V(M)=\{1,2,\dots,u\}$. Let $N=(n_{ij})_{v\times v}$ be another matrix and $W(N)$ be the corresponding network with $V(N)=\{1,2,\dots,v\}$. A function ϕ from $\{1,2,\dots,u\}$ to $\{1,2,\dots,v\}$ is a *homomorphism* from M to N if ϕ is an arc preserving function from $G[M]$ to $G[N]$. A matrix $\hat{M}$ is a *homomorphic image* of another matrix M if there exists a homomorphism $\phi: M\to\hat{M}$ which is onto and if for every $[\hat{u}_1,\hat{u}_2]\in E(G[\hat{M}])$ there exists $[u_1,u_2]\in E(G[M])$ such that $\phi(u_i)=\hat{u}_i$, $i=1,2$. The *weight* of a homomorphism ϕ is defined as $\omega(\phi)=\prod_{(i,j)\in E(G[M])}(n_{\phi(i)\phi(j)})^{m_{ij}}$. For a fixed matrix M, we can define the function h_M from $\mathscr{M}$ to $\mathscr{R}$ by

$$h_M(N)=\sum_{\substack{\phi \text{ is a homomorphism}\\ \text{from } M \text{ to } N}}\omega(\phi).$$

Similarly, we have the following theorem.

Theorem 5.1. h_M *is* MI *for any* $M\in\mathscr{M}$. *Moreover*, h_M *is* AMI *if* $G[M]$ *is weakly connected.*

Let $f_7 : \mathscr{M} \to \mathscr{R}$ be defined as $f_7((a_{ij})_{n\times n}) = \sum_{i,j} a_{ij}$. It can be shown that $f_7 = h_M$ where

$$M = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Let $f_8 : \mathscr{M} \to \mathscr{R}$ be defined as $f_8((a_{ij})_{n\times n}) = \max\left\{\sum_{j=1}^{n} a_{ij} \mid 1 \leqslant i \leqslant n\right\}$. It can be shown that $f_8 = \lim_{k\to\infty}(h_{M_k})^{1/k}$ where $M_k = (m_{ij}^k)_{k\times k}$ with $m_{ij}^k = 1$ if $i = 1$ and $j \geqslant 2$ and $m_{ij}^k = 0$ if otherwise.

Let $f_9 : \mathscr{M} \to \mathscr{R}$ be defined as $f_9((a_{ij})_{n\times n}) = \max\{|\lambda| \mid \lambda$ is an eigenvalue of $(a_{ij})\}$. It can be shown that $f_9 = \lim_{k\to\infty}(h_{M_k})^{1/k}$ where $M_k = (m_{ij}^k)_{k\times k}$ with $m_{ij}^k = 1$ if $j = i+1$ and $m_{ij}^k = 0$ if otherwise. Obviously, f_7 (f_8 and f_9, respectively) are generalizations of f_2 (f_3 and f_4, respectively) in Section 2.1.

5.2. The δ function

A matrix M is called *directed bipartite matrix* if $G[M]$ is a bipartite digraph with bipartition A and B such that any arc of $G[M]$ is directed from A to B. If $G[M]$ is weakly connected, such a partition is unique. We say that M is of (r, s) *type* if $|A| = r$ and $|B| = s$. For a directed bipartite matrix with weakly connected components $C_1, C_2, \ldots, C_m$ for $G[M]$, we say M is of $\sum_{i=1}^{m}(r_i, s_i)$ *type* where C_i is of (r_i, s_i) *type* for every i. Let $\mathscr{DB}$ denote the set of all directed bipartite matrices. We can define $\theta^* : \mathscr{DB} \to \mathscr{R}$ by assigning $\theta^*(M) = \sum_{i=1}^{m}(r_i \times s_i)^{1/2}$, where M is of $\sum_{i=1}^{m}(r_i, s_i)$ *type*. For any matrix M, it can be checked that

$$M \otimes \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

is directed bipartite matrix. Thus, we can define $\delta^* : \mathscr{M} \to \mathscr{R}$ by

$$\delta^*(M) = \theta^*\left(M \otimes \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right).$$

It is not hard to verify that δ^* is an AMI function. Obviously, δ^* is a generalization of the function δ in Section 2.4.

5.3. The matrix capacity functions

For any matrices M and N, let $\gamma_M(N)$ denote the maximum integer k such that $[k]M \subseteq N$. As before, we would like to know the behavior of $\lim_{m\to\infty}[\gamma_M(N^{[m]})]^{1/m}$. However, let

$$M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and

$$N = \begin{pmatrix} 0 & 2 \\ 0.5 & 0 \end{pmatrix}.$$

It can be checked that $\gamma_M(N^{[m]}) = 0$ if m is odd and $\gamma_M(N^{[m]}) = 2^{m-1}$ if m is even. Hence $\lim_{m\to\infty}[\gamma_M(N^{[m]})]^{1/m}$ may not exist. Since $0 \leqslant \gamma_M(N^{[m]}) \leqslant v^m$ where N is an $v \times v$ matrices, $\overline{\lim}_{m\to\infty}[\gamma_G(H^m)]^{1/m}$ exists. Thus we can define matrix capacity function $P_M(N)$ as $P_M(N) = \overline{\lim}_{m\to\infty}[\gamma_M(N^{[m]})]^{1/m}$. Obviously, $P_{(0)}(N) = v$ where N is an $v \times v$ matrix and $P_N \geqslant P_M$ if N is a submatrix of M.

5.3.1. Basic properties of matrix capacity functions

Theorem 5.2. *If P_M is (pseudo-)additive, then P_M is (pseudo-)multiplicative.*

Proof. Since $P_M(N^{[2]}) = \overline{\lim}_{n\to\infty}[\gamma_M(N^{[2n]})]^{1/n} = \overline{\lim}_{n\to\infty}([\gamma_M(N^{[2n]})]^{1/2n})^2 = P_M^2(N)$, we have $P_M((A \oplus B)^{[2]}) = P_M^2(A \oplus B)$ for any $A, B \in \mathscr{M}$. Then

$$\begin{aligned} P_M((A \oplus B)^{[2]}) &= P_M^2(A \oplus B) \\ &= (P_M(A) + P_M(B))^2 \\ &= P_M^2(A) + 2P_M(A)P_M(B) + P_M^2(B). \end{aligned} \tag{1}$$

However

$$\begin{aligned} P_M((A \oplus B)^{[2]}) &= P_M(A^{[2]}) + 2P_M(A \otimes B) + P_M(B^{[2]}) \\ &= P_M^2(A) + 2P_M(A \otimes B) + P_M^2(B). \end{aligned} \tag{2}$$

Comparing (1) and (2), we obtain $P_M(A \otimes B) = P_M(A)P_M(B)$. The theorem is proved. □

Let N be a matrix with $V(N) = \{y_1, y_2, \ldots, y_v\}$ and let $z = (z_1, z_2, \ldots, z_m)$ be a vertex in $N^{[m]}$ for some positive integer m. For $1 \leqslant i \leqslant v$, we set $a_i(z) = |\{z_j \mid z_j = y_i, 1 \leqslant j \leqslant m\}|/m$. The *distribution* of z, $d(z)$, is defined to be $(a_1(z), a_2(z), \ldots, a_v(z))$. Let $D(N) = \{(a_1, a_2, \ldots, a_v) \mid a_i \geqslant 0, \sum_{i=1}^{v} a_i = 1\}$. Obviously, $d(z) \in D(N)$ for any vertex $z \in V(N^{[m]})$. Let $S(m, d(z)) = \{y \mid y \in V(N^{[m]}), d(y) = d(z)\}$.

Let M be a matrix with $V(M) = \{x_1, x_2, \ldots, x_u\}$ and N be another matrix. We use $\mathrm{Ind}[M : N, m]$ to denote the set of $[z_1, z_2, \ldots, z_u]$ such that the subnetwork induced by $\{z_1, z_2, \ldots, z_u\}$ in $N^{[m]}$ contains a copy of M with z_i corresponding to x_i for every i.

Assume that $[z_1, z_2, \ldots, z_u] \in \mathrm{Ind}[M : N, m]$ and $z_i = (z_{i1}, z_{i2}, \ldots, z_{im})$ with $z_{ij} \in V(N)$ for $1 \leqslant i \leqslant u$. For $1 \leqslant k_1, k_2, \ldots, k_u \leqslant v$, let $\gamma^{m,M}_{k_1,k_2,\ldots,k_u}(z_1, z_2, \ldots, z_u)$ denote $|\{j \mid z_{ij} = y_{k_i}, 1 \leqslant i \leqslant u, 1 \leqslant j \leqslant m\}|/m$. Thus, $a_t(z_i) = \sum_{1 \leqslant k_1,k_2,\ldots,k_u \leqslant v}^{k_i = t} \gamma^{m,M}_{k_1,k_2,\ldots,k_u}(z_1, z_2, \ldots, z_u)$ for $1 \leqslant i \leqslant u$ and $1 \leqslant t \leqslant v$. We define $k : \mathrm{Ind}[M : N, m] \to \mathscr{R}$ by assigning

$$k([z_1, z_2, \ldots, z_u]) = \min_{1 \leqslant i \leqslant u} \left\{ \binom{m}{a_{i_1}m, a_{i_2}m, \ldots, a_{i_v}m}^{1/m} \mid d(z_i) = (a_{i_1}, a_{i_2}, \ldots, a_{i_v}) \right\}.$$

Since $\mathrm{Ind}[M : N, m]$ is finite, we can set $g^m_M(N)$ to be $\max\{k([z_1, z_2, \ldots, z_u]) \mid [z_1, z_2, \ldots, z_u] \in \mathrm{Ind}[M : N, m]\}$ if $\mathrm{Ind}[M : N, m] \neq \emptyset$ and 0 if otherwise.

Theorem 5.3. $P_M(N) = \overline{\lim}_{m\to\infty} g_M^m(N)$

Proof. There are at most $C(m+v-1,m)$ different distributions in $V(N^{[m]})$. Hence there are at most m^{uv} different $(d(\boldsymbol{z}_1), d(\boldsymbol{z}_2),\ldots,d(\boldsymbol{z}_u))$ with $[\boldsymbol{z}_1,\boldsymbol{z}_2,\ldots,\boldsymbol{z}_u]\in \mathrm{Ind}[M:N,m]$. Let $\mathscr{W}$ be a set of disjoint copies of M in $V(N^{[m]})$ such that $|\mathscr{W}| = \gamma_M(N^{[m]})$. $\mathscr{W}$ can be written as $\mathscr{W} = \{[\boldsymbol{z}_1,\boldsymbol{z}_2,\ldots,\boldsymbol{z}_u] \mid [\boldsymbol{z}_1,\boldsymbol{z}_2,\ldots,\boldsymbol{z}_u]\in \mathrm{Ind}[M:N,m]\}$. We defined an equivalence relation on $\mathscr{W}$ as $[\boldsymbol{z}_1,\boldsymbol{z}_2,\ldots,\boldsymbol{z}_u]\sim[\boldsymbol{y}_1,\boldsymbol{y}_2,\ldots,\boldsymbol{y}_u]$ if and only if $d(\boldsymbol{z}_i)=d(\boldsymbol{y}_i)$ for $1\leqslant i\leqslant u$. By the Pigeonhole Principle, there exists $[\boldsymbol{x}_1,\boldsymbol{x}_2,\ldots,\boldsymbol{x}_u]\in \mathrm{Ind}[M:N,m]$ such that $\mathscr{K} = \{[\boldsymbol{z}_1,\boldsymbol{z}_2,\ \ldots,\boldsymbol{z}_u] \mid\ d(\boldsymbol{z}_i) = d(\boldsymbol{x}_i) \text{ for } 1\leqslant i\leqslant u\}$ with

$$|\mathscr{K}| \geqslant \frac{1}{C(m+v-1,m)}|\mathscr{W}| = \frac{1}{C(m+v-1,m)}\gamma_M(N^{[m]}).$$

Therefore

$$\frac{1}{C(m+v-1,m)}\gamma_M(N^{[m]}) \leqslant |\mathscr{K}| \leqslant \min_{1\leqslant i\leqslant u}\{|S(m,\boldsymbol{a}_i)| \mid \boldsymbol{a}_i = d(\boldsymbol{x}_i)\}.$$

Hence

$$\begin{aligned}\gamma_M(N^{[m]}) &\leqslant C(m+v-1,m)\min_{1\leqslant i\leqslant u}\{|S(m,\boldsymbol{a}_i)| \mid \boldsymbol{a}_i = d(\boldsymbol{x}_i)\}\\ &= C(m+v-1,m)\min_{1\leqslant i\leqslant u}\left\{\left.\binom{m}{a_{i_1}m, a_{i_2}m,\ldots,a_{i_v}m}\right| \boldsymbol{a}_i = d(\boldsymbol{x}_i)\right\}.\end{aligned}$$

Thus

$$\begin{aligned}[\gamma_M(N^{[m]})]^{1/m} &\leqslant [C(m+v-1,m)]^{1/m}\min_{1\leqslant i\leqslant u}\left\{\left.\binom{m}{a_{i_1}m, a_{i_2}m,\ldots,a_{i_v}m}^{1/m}\right| \boldsymbol{a}_i = d(\boldsymbol{x}_i)\right\}\\ &\leqslant [C(m+v-1,m)]^{1/m} g_M^m(N).\end{aligned}$$

Hence

$$\begin{aligned}P_M(N) &= \overline{\lim_{m\to\infty}}\,[\gamma_M(N^{[m]})]^{1/m}\\ &\leqslant \overline{\lim_{m\to\infty}}\,([C(m+v-1,m)]^{1/m} g_M^m(N))\\ &= \overline{\lim_{m\to\infty}}\, g_M^m(N). \qquad (3)\end{aligned}$$

Assume that $[\boldsymbol{z}_1,\boldsymbol{z}_2,\ldots,\boldsymbol{z}_u]\in \mathrm{Ind}[M:N,m]$. Let $d(\boldsymbol{z}_i) = (a_{i1}, a_{i2},\ldots,a_{iv})$ for $1\leqslant i\leqslant u$ and j be the index such that

$$\binom{m}{a_{j_1}m, a_{j_2}m,\ldots,a_{j_v}m} \leqslant \binom{m}{a_{i_1}m, a_{i_2}m,\ldots,a_{i_v}m}$$

for every $1\leqslant i\leqslant u$. For any $\boldsymbol{y}\in S(m,d(\boldsymbol{z}_j))$, there exists a permutation $\pi_{\boldsymbol{y}}\in S_m$, the symmetric group on m letters, such that $\pi_{\boldsymbol{y}}(\boldsymbol{z}_j) = \boldsymbol{y}$. Thus $[\pi_{\boldsymbol{y}}(\boldsymbol{z}_1),\pi_{\boldsymbol{y}}(\boldsymbol{z}_2),\ldots,\pi_{\boldsymbol{y}}(\boldsymbol{z}_u)]\in \mathrm{Ind}[M:N,m]$. Let $A_{\boldsymbol{y}}$ denote the copy of M induced by $\{\pi_{\boldsymbol{y}}(\boldsymbol{z}_1),\pi_{\boldsymbol{y}}(\boldsymbol{z}_2),\ldots,\pi_{\boldsymbol{y}}(\boldsymbol{z}_u)\}$ and let $\mathscr{A}$ denote the union of $\{A_{\boldsymbol{y}} \mid \boldsymbol{y} \in S(m, d(\boldsymbol{z}_j))\}$. Then we repeatedly find an $A_{\boldsymbol{y}}$ in $\mathscr{A}$ and delete those $A_{\boldsymbol{y}'}$ which are adjacent to $A_{\boldsymbol{y}}$ until $\mathscr{A}$ is empty. We get at least

$$\frac{1}{u(u-1)+1}|S(m, d(\boldsymbol{z}_j))|$$

disjoint $A_{\boldsymbol{y}}$ in $\mathscr{A}$. Thus

$$\begin{aligned}\gamma_M(N^{[m]}) &\geqslant \gamma_M(\mathscr{A}) \geqslant \frac{1}{u(u-1)+1}|S(m, d(\boldsymbol{z}_j))| \\ &= \frac{1}{u^2-u+1}\binom{m}{a_{j_1}m, a_{j_2}m, \ldots, a_{j_v}m} \\ &= \frac{1}{u^2-u+1}\min_{1\leqslant i\leqslant u}\left\{\binom{m}{a_{i_1}m, a_{i_2}m, \ldots, a_{i_v}m} \,\middle|\, d(\boldsymbol{z}_i) = (a_{i_1}, a_{i_2}, \ldots, a_{i_v})\right\}.\end{aligned}$$

Therefore

$$P_M(N) = \overline{\lim_{m\to\infty}}\,[\gamma_M(N^{[m]})]^{1/m} \geqslant \overline{\lim_{m\to\infty}}\, g_M^m(N). \tag{4}$$

Combining (3) and (4), $P_M(N) = \overline{\lim}_{m\to\infty} g_M^m(N)$. □

5.3.2. Properties of digraph capacity functions

For further discussion on $P_M(N)$, we assume that M is a $(0,1)$-matrix. In other words, $M = A(G)$ for some digraph G. Since $A(G)$ is uniquely determined by G up to isomorphism, we write G for $A(G)$.

Theorem 5.4. (1) *$P_G(G) \geqslant 1$. Moreover, $P_{G^{[k]}} = P_G$ for any positive integer k.* (2) *$P_G = P_H$ if and only if there exist some $n, t, m \in \mathscr{N}$ such that $H^{[n]} \subseteq G^{[t]} \subseteq H^{[m]}$.*

Proof. (1) Assume that $V(G) = \{x_1, x_2, \ldots, x_u\}$. Let $\boldsymbol{x}_i^k = (x_i, \ldots, x_i)$ with x_i repeated k times. Obviously, $[\boldsymbol{x}_1^k, \boldsymbol{x}_2^k, \ldots, \boldsymbol{x}_u^k] \in \mathrm{Ind}[G : G, k]$. Hence $\gamma_G(G^{[k]}) \geqslant 1$ and $P_G(G) \geqslant 1$. Since G is a subdigraph of $G^{[k]}$, $P_{G^{[k]}} \leqslant P_G$. However, $[\gamma_G(H^{[n]})]G \subseteq H^{[n]}$ for any digraph H and integer n. Thus, $[\gamma_G^k(H^{[n]})]G^{[k]} \subseteq H^{[kn]}$. Therefore, $[\gamma_G^k(H^{[m/k]})]G^{[k]} \subseteq H^{[m]}$ and we get

$$\begin{aligned}P_{G^{[k]}}(H) &= \overline{\lim_{m\to\infty}}\,[\gamma_{G^{[k]}}(H^{[m]})]^{1/m} \geqslant \overline{\lim_{m\to\infty}}\,[\gamma_{G^{[k]}}([\gamma_G^k(H^{[m/k]})]G^{[k]})]^{1/m} \\ &= \overline{\lim_{m\to\infty}}\,[\gamma_G^k(H^{[m/k]})]^{1/m} = \overline{\lim_{m/k\to\infty}}\,[\gamma_G(H^{[m/k]})]^{k/m} \\ &= P_G(H).\end{aligned}$$

Thus $P_{G^{[k]}} = P_G$.

(2) Statement (2) is a direct consequence of statement (1). □

Theorem 5.5. *$P_G \geqslant P_H$ if and only if for any two distinct u and v of G there exists a homomorphism $\phi : G \to H$ such that $\phi(u) \neq \phi(v)$.*

Proof. Let $V(G)=\{x_1,x_2,\dots,x_u\}$ and $V(H)=\{y_1,y_2,\dots,y_v\}$. Since $P_H(H)>0$, we have $P_G(H)>0$. There exists some $[\boldsymbol{x}_1,\boldsymbol{x}_2,\dots,\boldsymbol{x}_u]\in \mathrm{Ind}[G:H,t]$. Let $\boldsymbol{x}_i=(y_{i1},y_{i2},\dots,y_{it})$ for $1\leqslant i\leqslant u$. We define φ_k by $\varphi_k(x_i)=y_{ik}$ for $1\leqslant k\leqslant t$. obviously, φ_k is a homomorphism from G into H. Now, give any two distinct vertices x_i and x_j, there exists some s such that $y_{is}\neq y_{js}$. Therefore, $\varphi_s(x_i)\neq\varphi_s(x_j)$. On the other hand, let $\{\varphi_1,\varphi_2,\dots,\varphi_t\}$ be the set of all homomorphisms from G into H. Define a function $\phi:G\to H^{[t]}$ by $\phi(x)=(\varphi_1(x),\varphi_2(x),\dots,\varphi_t(x))$. It is easy to check that ϕ is a homomorphism. Let x and y be any two distinct vertices of G, there exists a homomorphism φ_k such that $\varphi_k(x)\neq\varphi_k(y)$. Hence ϕ is one to one. We get $G\subseteq H^{[t]}$ and $P_G\geqslant P_{H^{[t]}}$. By Theorem 5.4, $P_G\geqslant P_H$. □

Theorem 5.6. *Assume that $M\in\mathscr{M}$ and $\hat{G}$ is any digraph which is a homomorphic image of $G[M]$ with $P_{\hat{G}}\geqslant P_M$. Then $P_M(N)=P_{\hat{G}}(N)$ if N is any matrix such that $P_M(N)\neq 0$. Furthermore P_M is* PAMI *if $P_{\hat{G}}$ is* PAMI.

Proof. Let $V(G[M])=\{x_1,x_2,\dots,x_u\}$, $V(\hat{G})=\{y_1,y_2,\dots,y_v\}$, and let ϕ be a homomorphism from $G[M]$ onto $\hat{G}$. Obviously, there are $\gamma_{\hat{G}}(N^{[m]})$ disjoint $\hat{G}$'s in $N^{[m]}$ for every m. Let $\hat{G}_1,\hat{G}_2,\dots,\hat{G}_{\gamma_{\hat{G}}(N^{[m]})}$ be such disjoint $\hat{G}$'s in $N^{[m]}$ and let $V(\hat{G}_i)=\{\boldsymbol{y}_{i,y_1},\boldsymbol{y}_{i,y_2},\dots,\boldsymbol{y}_{i,y_v}\}$ with $\boldsymbol{y}_{i,y_j}$ corresponding to y_j. We notice that $w(\boldsymbol{y}_{i,y_j},\boldsymbol{y}_{i,y_k})\geqslant 1$ for every $(y_j,y_k)\in E(\hat{G})$, $1\leqslant i\leqslant\gamma_{\hat{G}}(N^{[m]})$. Since $P_M(N)\neq 0$, there exists some $[\boldsymbol{x}_1,\boldsymbol{x}_2,\dots,\boldsymbol{x}_u]\in\mathrm{Ind}[M:N,t]$. We set $\boldsymbol{z}_{ij}$ as $(\boldsymbol{x}_i,\boldsymbol{y}_{j,\phi(x_i)})$ for $1\leqslant i\leqslant u$ and $1\leqslant j\leqslant\gamma_{\hat{G}}(N^{[m]})$. Then $w(\boldsymbol{z}_{ij},\boldsymbol{z}_{kj})=w(\boldsymbol{x}_i,\boldsymbol{x}_k)w(\boldsymbol{y}_{j,\phi(x_i)},\boldsymbol{y}_{j,\phi(x_k)})\geqslant w(\boldsymbol{x}_i,\boldsymbol{x}_k)\geqslant w(x_i,x_k)$ for every $(x_i,x_k)\in E(G[M])$. Thus, $[\boldsymbol{z}_{1j},\boldsymbol{z}_{2j},\dots,\boldsymbol{z}_{uj}]\in\mathrm{Ind}[M:N,m+t]$. Let M_j denote the copy of M induced by $\{\boldsymbol{z}_{1j},\boldsymbol{z}_{2j},\dots,\boldsymbol{z}_{uj}\}$ for $1\leqslant j\leqslant\gamma_{\hat{G}}(N^{[m]})$. Then $M_1,M_2,\dots,M_{\gamma_{\hat{G}}(N^{[m]})}$ are mutually disjoint, because $\hat{G}_1,\hat{G}_2,\dots,\hat{G}_{\gamma\hat{G}(N^{[m]})}$ are mutually disjoint. Thus $\gamma_M(N^{[m+t]})\geqslant\gamma_{\hat{G}}(N^{[m]})$. Therefore,

$$P_{\hat{G}}(N)=\overline{\lim_{m\to\infty}}\,[\gamma_{\hat{G}}(N^{[m]})]^{1/m}\leqslant\overline{\lim_{m\to\infty}}\,[\gamma_M(N^{[m+t]})]^{1/m}$$

$$=\overline{\lim_{m\to\infty}}\,[\gamma_M(N^{[m+t]})]^{1/(m+t)}=P_M(N).$$

Since $P_{\hat{G}}(N)\geqslant P_M(N)$, we have $P_M(N)=P_{\hat{G}}(N)$. □

Theorem 5.7. *$P_M(N)\leqslant P_{G[M]}(N)$ for any matrices $M,N\in\mathscr{M}$. Moreover, $P_M(N)=P_{G[M]}(N)$* if *$P_M(N)>0$.*

Proof. Let $V(G[M])=\{x_1,x_2,\dots,x_u\}$ and $V(G[N])=\{y_1,y_2,\dots,y_v\}$. By Theorem 5.3, $P_M(N)=\overline{\lim}_{m\to\infty}g_M^m(N)$. Therefore, there exists an infinite subsequence of integers $\{m_t|t\in\mathscr{N}\}$ such that $P_M(N)=\lim_{t\to\infty}g_M^{m_t}(N)$.

Suppose that $P_M(N)>0$. Then there exists $[\boldsymbol{z}_1^t,\boldsymbol{z}_2^t,\dots,\boldsymbol{z}_u^t]\in\mathrm{Ind}[M:N,m_t]$ such that $g_M^{m_t}(N)=k([\boldsymbol{z}_1^t,\ \boldsymbol{z}_2^t,\dots,\boldsymbol{z}_u^t])$ if t is sufficiently large. Let $\boldsymbol{z}_i^t=(y_{i1},y_{i2},\dots,y_{im_t})$ with $y_{i,j}\in V(G[N])$ for $1\leqslant i\leqslant u$. We have $w(\boldsymbol{z}_i^t,\boldsymbol{z}_j^t)\geqslant w(x_i,x_j)$ for every $(x_i,x_j)\in E(G[M])$. We claim that $w(\boldsymbol{z}_i^t,\boldsymbol{z}_j^t)\geqslant 1$ for every $(x_i,x_j)\in E(G[M])$ if t is sufficiently large. First we observed that

$$w(z_i^t, z_j^t) = \prod_{l=1}^{m_t} w(y_{il}, y_{jl}) = \prod_{(y_{k_i}, y_{k_j}) \in X} \left[(w(y_{k_i}, y_{k_j}))^{\gamma_{k_1,k_2,\dots,k_u}^{m_t,M}(z_1,z_2,\dots,z_u)} \right]^{m_t}$$

where X is the set, not multiset, $\{(y_{il}, y_{jl}) | 1 \leqslant l \leqslant m_t\}$.

Suppose that $w(z_i^t, z_j^t)$ is not greater than 1 for every $(x_i, x_j) \in E(G[M])$ if t is sufficiently large. Then $\lim_{t\to\infty} \prod_{(y_{k_i}, y_{k_j}) \in X} \left[(w(y_{k_i}, y_{k_j}))^{\gamma_{k_1,k_2,\dots,k_u}^{m_t,M}(z_1,z_2,\dots,z_u)} \right]^{m_t}$ either is 0 or does not exist. Thus $P_M(N) \neq \lim_{t\to\infty} g_M^{m_t}(N)$ and we get a contradiction. Therefore, $w(z_i^t, z_j^t) \geqslant 1$ for every $(x_i, x_j) \in E(G[M])$ if t is sufficiently large. Thus $[z_1^t, z_2^t, \dots, z_u^t] \in \mathrm{Ind}[G[M] : N, m_t]$ if t is sufficiently large. We can conclude that $P_M(N) \leqslant P_{G[M]}(N)$. It follows from Theorem 5.6 that $P_M(N) = P_{G[M]}(N)$ if $P_M(N) > 0$. □

Corollary 5.1. *$P_M(A \oplus B) \neq 0$ if $P_M(A) \neq 0$ and $P_M(B) \neq 0$.*

Proof. Assume that $V(G[M]) = \{x_1, x_2, \dots, x_u\}$, $P_M(A) \neq 0$ and $P_M(B) \neq 0$. There exists an infinite subsequence of integers $\{m_t \mid t \in \mathcal{N}\}$ with $[\boldsymbol{x}_1^t, \boldsymbol{x}_2^t, \dots, \boldsymbol{x}_u^t] \in \mathrm{Ind}[M : A, m_t]$. From the proof of Theorem 5.7, we notice that $w(\boldsymbol{x}_i^t, \boldsymbol{x}_j^t) \geqslant 1$ for all $(x_i, x_j) \in E(G[M])$ if t is sufficiently large. Similarly, there exists an infinite subsequence of integers $\{n_t \mid t \in \mathcal{N}\}$ with $[\boldsymbol{y}_1^t, \boldsymbol{y}_2^t, \dots, \boldsymbol{y}_u^t] \in \mathrm{Ind}[M : B, n_t]$. Moreover, $w(\boldsymbol{y}_i^t, \boldsymbol{y}_j^t) \geqslant 1$ for all $(x_i, x_j) \in E(G[M])$ if t is sufficiently large. We set $z_i^t = (\boldsymbol{x}_i^t, \boldsymbol{y}_i^t)$ for $1 \leqslant i \leqslant u$. Obviously, $[z_1^t, z_2^t, \dots, z_u^t] \in \mathrm{Ind}[M : A \oplus B, m_t + n_t]$ if t is sufficiently large. Hence $P_M(A \oplus B) \neq 0$. square

Lemma 5.1. *Assume that G is a vertex transitive digraph and $[z_1, z_2, \dots, z_u] \in \mathrm{Ind}[G : N, m]$. Then there exists some integer k with $[z_1', z_2', \dots, z_u'] \in \mathrm{Ind}[G : N, k]$ such that $d(z_i') = \sum_{j=1}^u d(z_j)/u$ for every $1 \leqslant i \leqslant u$.*

Proof. Let $V(G) = \{x_1, x_2, \dots, x_u\}$. Let $T(G) = \{\pi_1, \pi_2, \dots, \pi_t\}$ be the automorphism group for G and $T_{ij}(G)$ be $\{\pi | \pi \in T(G)$ and $\pi(x_i) = x_j\}$. Since G is vertex transitive, $|T_{ij}(G)| = |T_{rs}(G)|$ for any $1 \leqslant i, j, r, s \leqslant u$. Let $z_i = (z_{i_1}, z_{i_2}, \dots, z_{i_m})$ for $1 \leqslant i \leqslant u$. We set $\pi_j(z_i) = (\pi_j(z_{i_1}), \pi_j(z_{i_2}), \dots, \pi_j(z_{i_m}))$ for $1 \leqslant j \leqslant t$. Then we set $z_i' = (\pi_1(z_i), \pi_2(z_i), \dots, \pi_t(z_i))$ for $1 \leqslant i \leqslant u$. Obviously, $[z_1', z_2', \dots, z_u'] \in \mathrm{Ind}[G : N, tm]$ and $d(z_i') = \sum_{j=1}^u d(z_j)/u$. The lemma is proved. □

Let $i_G(N) = \{d(z) |$ there exists $[z_1, z_2, \dots, z_u] \in \mathrm{Ind}[G : N, m]$ such that $d(z_i) = d(z)$ for $1 \leqslant i \leqslant u\}$. We use $I_G(N)$ to denote the closure of $i_G(N)$; i.e., $i_G(N)$ and its accumulation points. Let $\mathscr{H} : D(N) \to \mathscr{R}$ be the function defined by $\mathscr{H}(\boldsymbol{a}) = \prod_{i=1}^v a_i^{-a_i}$ where $\boldsymbol{a} = (a_1, a_2, \dots, a_v)$. Note that $\log_v \mathscr{H}$ is the entropy function. Hence the function $\mathscr{H}$ satisfies

(1) $\lim_{m\to\infty} \binom{m}{a_{i_1} m, a_{i_2} m, \dots, a_{i_v} m}^{1/m} = \mathscr{H}(\boldsymbol{a})$, where $a_i m \in \mathscr{I}$ for every i; and
(2) $\mathscr{H}(\sum_{i=1}^u \boldsymbol{a}_i/u) \geqslant \min\{\mathscr{H}(\boldsymbol{a}_i) \mid i = 1, 2, \dots, u\}$.

With Theorem 5.3, we have the following theorem.

Theorem 5.8. *For any* $N \in \mathscr{M}$, $P_G(N) = \max_{\boldsymbol{a}\in I_G(N)} \mathscr{H}(\boldsymbol{a})$ *if* G *is a vertex transitive digraph.*

Theorem 5.9. P_G *is* PAMI *if* G *is vertex transitive.*

Proof. Let G be a vertex transitive digraph with $V(G) = \{x_1, x_2, \ldots, x_u\}$. Consider any two matrices A and B with $|V(A)| = v$ and $|V(B)| = w$. By Theorem 5.8, $P_G(A) = g = \mathscr{H}(\boldsymbol{a})$ for some $\boldsymbol{a} = (a_1, a_2, \ldots, a_v) \in I_G(A)$ and $P_G(B) = h = \mathscr{H}(\boldsymbol{b})$ for some $\boldsymbol{b} = (b_1, b_2, \ldots, b_w) \in I_G(B)$.

Obviously, there exits a sequence $\{\boldsymbol{a}_i\}_{i=1}^{\infty}$ in $i_G(A)$ and a sequence $\{\boldsymbol{b}_i\}_{i=1}^{\infty}$ in $i_G(B)$ such that $\lim_{i\to\infty} \boldsymbol{a}_i = \boldsymbol{a}$ and $\lim_{i\to\infty} \boldsymbol{b}_i = \boldsymbol{b}$. Hence, there exists some $[\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_u] \in \mathrm{Ind}[G : A, m]$ with $d(\boldsymbol{x}_j) = \boldsymbol{a}_i$ for $1 \leqslant j \leqslant u$ and some $[\boldsymbol{y}_1, \boldsymbol{y}_2, \ldots, \boldsymbol{y}_u] \in \mathrm{Ind}[G : B, l]$ with $d(\boldsymbol{y}_j) = \boldsymbol{b}_i$ for $1 \leqslant j \leqslant u$. Since g and h are real numbers, there exists a sequence of rational numbers $\{g_i\}_{i=1}^{\infty}$ and $\{h_i\}_{i=1}^{\infty}$ such that $\lim_{i\to\infty} g_i = g$ and $\lim_{i\to\infty} h_i = h$. Thus for every i we can choose an integer t such that $p = tg_i/(g_i + h_i)$ and $q = th_i/(g_i + h_i)$ are integers. Let $\boldsymbol{z}_j = (\boldsymbol{x}_j, \boldsymbol{x}_j, \ldots, \boldsymbol{x}_j, \boldsymbol{y}_j, \boldsymbol{y}_j, \ldots, \boldsymbol{y}_j)$, each $\boldsymbol{x}_j$ repeats pl times and $\boldsymbol{y}_j$ repeats qm times, for $1 \leqslant j \leqslant u$. We can easily check that $[\boldsymbol{z}_1, \boldsymbol{z}_2, \ldots, \boldsymbol{z}_u] \in \mathrm{Ind}[G : A \oplus B, tlm]$ and

$$d(\boldsymbol{z}_j) = \left(\frac{g_i}{g_i + h_i}\boldsymbol{a}_i, \frac{h_i}{g_i + h_i}\boldsymbol{b}_i\right) \quad \text{for } 1 \leqslant j \leqslant u.$$

Thus

$$\left(\frac{g_i}{g_i + h_i}\boldsymbol{a}_i, \frac{h_i}{g_i + h_i}\boldsymbol{b}_i\right) \in i_G(A \oplus B).$$

Hence

$$\left(\frac{g}{g + h}\boldsymbol{a}, \frac{h}{g + h}\boldsymbol{b}\right) \in I_G(A \oplus B).$$

Therefore

$$\begin{aligned}
&P_G(A \oplus B)\\
&\quad \geqslant \mathscr{H}\left(\frac{g}{g + h}\boldsymbol{a}, \frac{h}{g + h}\boldsymbol{b}\right) = \prod_{i=1}^{v}\left(\frac{g}{g + h}a_i\right)^{(-g/(g+h))a_i}\\
&\quad \times \prod_{j=1}^{w}\left(\frac{h}{g + h}b_j\right)^{(-h/(g+h))b_j}\\
&\quad = \left(\frac{g}{g + h}\right)^{(-g/(g+h))\sum_{i=1}^{v} a_i}\left(\prod_{i=1}^{v} a_i^{-a_i}\right)^{g/(g+h)}\left(\frac{h}{g + h}\right)^{-h/(g+h)\sum_{j=1}^{w} b_j}
\end{aligned}$$

$$\times \left(\prod_{j=1}^{w} b_j^{-b_j}\right)^{h/(g+h)}$$

$$= \left(\frac{g}{g+h}\right)^{-g/(g+h)} (g)^{g/(g+h)} \left(\frac{h}{g+h}\right)^{-h/(g+h)} (h)^{h/(g+h)}$$

$$= g + h$$

$$= P_G(A) + P_G(B). \tag{5}$$

On the other hand, let $\boldsymbol{c}=(\boldsymbol{a},\boldsymbol{b})=(a_1,a_2,\ldots,a_v,b_1,b_2,\ldots,b_w)$ be a vector in $I_G(A\oplus B)$ such that $P_G(A\oplus B)=\mathscr{H}(\boldsymbol{c})$. Let $p=\sum_{i=1}^{v} a_i$ and $q=\sum_{j=1}^{w} b_j$. Obviously, $p+q=1$ and there exists a sequence in $i_G(A\oplus B)$, $\{(\boldsymbol{a}_i,\boldsymbol{b}_i)\}_{i=1}^{\infty}$, such that $\lim_{i\to\infty}(\boldsymbol{a}_i,\boldsymbol{b}_i)=(\boldsymbol{a},\boldsymbol{b})$. Assume that $\boldsymbol{a}_i=(a_{i,1},a_{i,2},\ldots,a_{i,v})$ and $\boldsymbol{b}_i=(b_{i,1},b_{i,2},\ldots,b_{i,w})$. Let $p_i=\sum_{j=1}^{v} a_{i,j}$ for every i. Then $\lim_{i\to\infty} p_i=\lim_{i\to\infty}\sum_{j=1}^{v} a_{i,j}=\sum_{j=1}^{v}\lim_{i\to\infty} a_{i,j}=\sum_{j=1}^{v} a_j=p$. Similarly, let $q_i=\sum_{j=1}^{w} b_{i,j}$ for every i, we have $\lim_{i\to\infty} q_i=q$.

Since $(\boldsymbol{a}_i,\boldsymbol{b}_i)\in i_G(A\oplus B)$, there exists some $[\boldsymbol{z}_1,\boldsymbol{z}_2,\ldots,\boldsymbol{z}_u]\in \mathrm{Ind}[G:A\oplus B,m]$ with $d(\boldsymbol{z}_j)=(\boldsymbol{a}_i,\boldsymbol{b}_i)$ for $1\leqslant j\leqslant u$. Without loss of generality, we may assume $\boldsymbol{z}_j=(z_{j,1},z_{j,2},\ldots,z_{j,n},\ z_{j,n+1},\ldots,z_{j,m})$ with $z_{j,k}\in V(A)$ if and only if $1\leqslant k\leqslant n$.

Let $\boldsymbol{x}_j=(z_{j,1},z_{j,2},\ldots,z_{j,n})$. Then $d(\boldsymbol{x}_j)=\boldsymbol{a}_i/p_i$ for $1\leqslant j\leqslant u$. If $[\boldsymbol{x}_1,\boldsymbol{x}_2,\ldots,\boldsymbol{x}_u]\in \mathrm{Ind}[G:A,n]$, then $\boldsymbol{a}_i/p_i\in i_G(A)$. Otherwise, $\{\boldsymbol{x}_1,\boldsymbol{x}_2,\ldots,\boldsymbol{x}_u\}$ form a homomorphic image of G. Since $P_G(A)\neq 0$, there exists some $[\boldsymbol{w}_1,\boldsymbol{w}_2,\ldots,\boldsymbol{w}_u]\in \mathrm{Ind}[G:A,r]$ and $d(\boldsymbol{w}_j)=\boldsymbol{d}$ for some distribution $\boldsymbol{d}$. Let s be an integer. We define $\boldsymbol{y}_j=(\boldsymbol{w}_j,\boldsymbol{x}_j,\ldots,\boldsymbol{x}_j)$ with each $\boldsymbol{x}_j$ repeated s times. Then $[\boldsymbol{y}_1,\boldsymbol{y}_2,\ldots,\boldsymbol{y}_u]\in \mathrm{Ind}[G:A,r+sn]$ and $d(\boldsymbol{y}_j)=(\boldsymbol{d}+(s\boldsymbol{a}_i)/p_i)/(s+1)$. Since $\lim_{s\to\infty}(\boldsymbol{d}+(s\boldsymbol{a}_i)/p_i)/(s+1)=\boldsymbol{a}_i/p_i$, $\boldsymbol{a}_i/p_i\in I_G(A)$. Thus $\boldsymbol{a}/p\in I_G(A)$. Similarly, $\boldsymbol{b}/q\in I_G(B)$. Let $k=\mathscr{H}(\boldsymbol{a}/p)$ and $l=\mathscr{H}(\boldsymbol{b}/q)$. Then

$$P_G(A\oplus B)=\mathscr{H}(\boldsymbol{c})=\prod_{i=1}^{v} a_i^{-a_i}\prod_{j=1}^{w} b_j^{-b_j}$$

$$=\prod_{i=1}^{v}\left(p\left(\frac{a_i}{p}\right)\right)^{-p(a_i/p)}\prod_{j=1}^{w}\left(q\left(\frac{b_j}{q}\right)\right)^{-q(b_j/q)}$$

$$=p^{-p}\left[\prod_{i=1}^{v}\left(\frac{a_i}{p}\right)^{-(a_i/p)}\right]^{p} q^{-q}\left[\prod_{j=1}^{w}\left(\frac{b_j}{q}\right)^{-(b_j/q)}\right]^{q}$$

$$=p^{-p}k^{p}q^{-q}l^{q}$$

$$=p^{-p}k^{p}(1-p)^{-(1-p)}l^{(1-p)}.$$

Consider $f(x)=x^{-x}k^{x}(1-x)^{-(1-x)}l^{(1-x)}$. Then

$$f'(x)=f(x)\ln\left(\frac{k(1-x)}{lx}\right).$$

Since $f'(x)=0$ if and only if $x=k/(k+l)$ and $f'(x)>0$ when $x<k/(k+l)$, $f'(x)<0$ when $x>k/(k+l)$. Therefore $f(x)$ has a maximum value at $x=k/(k+1)$. Thus

$$P_G(A\oplus B)\leqslant\left(\frac{k}{k+l}\right)^{-k/(k+l)}(k)^{k/(k+l)}\left(\frac{l}{k+l}\right)^{-l/(k+l)}(l)^{l/(k+l)}$$
$$=k+l\leqslant P_G(A)+P_G(B). \tag{6}$$

From (5) and (6), $P_G(A\oplus B)=P_G(A)+P_G(B)$. Hence P_G is pseudo-additive. Applying Theorem 5.2, P_G is pseudo-multiplicative. Thus P_G is PAMI. □

Let $G=(V,E)$ be a digraph. The *homomorphism digraph* $G^*=(V^*,E^*)$ of G is the directed graph with $V^*=V$ and $(a,b)\in E^*$ if there is a homomorphism ϕ from G into itself such that $\phi(a)=b$. Obviously, $(v,v)\in E^*$ for every $v\in V$. Let S be a subset of V. The *out-neighborhood* of S is the set $\Gamma(S)=\{y|(x,y)\in E^* \text{ with } x\in S\}$. Thus, $S\subseteq\Gamma(S)$ for every $S\subseteq V$. A nonempty subset S of V is called a *closed set* of G if (1) $\Gamma(S)\subseteq S$ and (2) there is no nonempty proper subset S' of S such that $\Gamma(S')\subseteq S'$. Obviously, there exists a closed set for every digraph.

Lemma 5.2. *Suppose that S is a closed set of G and D is a subset of S. The induced directed graph $G^*|_D$ in G^* is a complete digraph.*

Proof. First, we prove that $G^*|_D$ is strongly connected. Suppose that $G^*|_D$ is not strongly connected. Then there exists a nonempty proper subset D' of D such that $\Gamma(D')\cap D\subseteq D'$. Let $X=\{x|x\in S-D$ and there exists a homomorphism $f:G\to G$ such that $f(x)\in D-D'\}$.

Suppose that there exists a homomorphism $g:G\to G$ for which $g(y)\in X$ for some $y\in D'$. Since $g(y)$ is in X, there exists a homomorphism $h:G\to G$ such that $h(g(y))\in D-D'$. Then $h\circ g$ is a homomorphism mapping the element y in D' to an element in $D-D'$. This contradicts $\Gamma(D')\cap D\subseteq D'$. Thus, there is no homomorphism g from G into itself such that $g(y)\in X$ for some $y\in D'$.

It follows from the above discussion that the set $Y=((S-D)-X)\cup D'$ is a proper subset of S such that $\Gamma(Y)\subseteq Y$. This contradicts the fact that S is a closed set. Thus, $G^*|_D$ is strongly connected.

Since the composite of homomorphism functions is again a homomorphism function, $G^*|_D$ forms a complete digraph. □

Corollary 5.2. *For any two different closed sets S_1 and S_2 of digraph G, $S_1\cap S_2=\emptyset$.*

Proof. The proof follows from the fact that $G^*|_S$ is a complete digraph for every closed set of S. □

Lemma 5.3. *Let S be a closed set of the digraph G and f be any homomorphism from G into itself. There is exactly one closed set B of $f(G)$ contained in $S\cap f(G)$. Moreover, $f(S)$ is a subset of B.*

Proof. We prove this lemma through the following steps.

(1) Let s be any element in $S \cap f(G)$ and g be any homomorphism from $f(G)$ into itself. Since S is a closed set, $g(s) \in S \cap f(G)$. Thus, the out-neighborhood of $S \cap f(G)$ in $f(G)^*$ is a subset of $S \cap f(G)$. Thus, there exists at least one closed subset B of $f(G)$ in $S \cap f(G)$.

(2) Let B be any closed set of $f(G)$ in $S \cap f(G)$ and x be any element of B. Obviously, $f|_{f(G)}$ is a homomorphism from $f(G)$ into itself. Since B is a closed set, $f(x) \in B \subseteq S$. Thus, the set $f(S) \cap B$ contains at least the element $y(=f(x))$.

(3) Let $z = f(w)$ with $w \in S$ be any element of $f(S)$. By Lemma 5.2, there exists a homomorphism $h : G \to G$ such that $h(y) = w$. Then $f \circ h|_{f(G)}$ is a homomorphism from $f(G)$ into itself such that $f \circ h|_{f(G)}(y) = f(w) = z$. Since B is a closed set, z is an element of B. Thus, $f(S) \subseteq B$.

(4) It follows from Corollary 5.2 that there is exactly one closed set B of $f(G)$ contained in $S \cap f(G)$. □

Let $G = (V, E)$ be a digraph. A nonempty subset C of a closed set S is called a *core* if (1) there exists a homomorphism $\phi : G \to G$ satisfying $\phi(S) = C$ and (2) there is no proper subset C' of C such that there exists a homomorphism $\phi' : G \to G$ satisfying $\phi'(S) = C'$. Again there exists a core for every closed set.

A digraph G is called an *n-core digraph* if G has exactly n closed sets $C_1, C_2, \ldots, C_n$ with $V(G) = C_1 \cup C_2 \cup \cdots \cup C_n$ such that C_i is a core for every i.

Lemma 5.4. *Let G be a digraph with n closed sets. G contains an n-core subdigraph $\hat{G}$ as a homomorphic image of G.*

Proof. We construct a sequence of subdigraphs $G_0, G_1, \ldots, G_k$ as follows:

Let $G_0 = G$. If there is no homomorphism $f : G_0 \to G_0$ such that $f(G_0) \subset G_0$, the sequence terminates. If there exists a homomorphism $f_0 : G_0 \to G_0$ such that $f_0(G_0) \subset G_0$, then set $G_1 = f_0(G_0)$. Let G_i be the newly constructed subdigraph. If there is no homomorphism $f : G_i \to G_i$ such that $f(G_i) \subset G_i$, the sequence terminates. If there exists a homomorphism $f_i : G_i \to G_i$ such that $f_i(G_i) \subset G_i$, then set $G_{i+1} = f_i(G_i)$. Since G is a finite digraph, the sequence terminates at some G_k. Let $f = f_{k-1} \circ f_{k-2} \circ \cdots \circ f_0$. Then, f is a homomorphism from G onto the subdigraph of G, G_k. It follows from Lemma 5.3 that G_k is a subdigraph with n closed sets. Since there is no homomorphism from G_k into a proper subdigraph of itself, G_k is an n-core subdigraph. □

Lemma 5.5. *Let C be a core of the digraph G for some closed set S. The induced subdigraph $G|_C$ is vertex transitive.*

Proof. We prove this lemma through the following steps.

(1) Let ϕ by any homomorphism of G such that $\phi(S) = C$. We claim that the restriction of ϕ on C, $\phi|_C$, is an isomorphism for C. First, we prove that $\phi(C) = C$.

Suppose that $\phi(C) \neq C$. $\phi(C)$ is a proper subset of C. Since $\phi(S)=C$, $\phi^2(S)=\phi(C)$. In other words, $\phi(C)$ is a proper subset of C having a homomorphism ϕ^2 such that $\phi^2(S)=\phi(C)$. This contradicts the fact that C is a core of S. Hence $\phi(C)=C$. Since C is a finite set, ϕ is also one to one from C onto C. Thus, $\phi|_C$ is an isomorphism on C.

(2) From step 1, we know that ϕ_C^{-1} is an isomorphism from C onto itself. Let f be any homomorphism from G into itself. Then $f \circ \phi|_C^{-1}(C) \subseteq S$ because S is a closed set. Therefore $\phi \circ f \circ \phi|_C^{-1}(C) \subseteq C$. We claim that $\phi \circ f \circ \phi|_C^{-1}$ is again an isomorphism on C. Suppose that $\phi \circ f \circ \phi|_C^{-1}$ is not an isomorphism. Then $\phi \circ f \circ \phi|_C^{-1}(C)$ is a proper subset of C. Since $\phi|_C^{-1}(C)=C$, we have $\phi \circ f(C)$ is a proper subset of C. Note that $\phi \circ f \circ \phi(S) = \phi \circ f(C)$. Thus, $\phi \circ f(C)$ is a proper subset of C and $\phi' = \phi \circ f \circ \phi$ is a homomorphism satisfying $\phi'(S) = \phi \circ f(C)$. This contradicts the fact that C is a core. Thus, $\phi \circ f \circ \phi|_C^{-1}$ is an isomorphism on C for every homomorphism $f : G \to G$.

(3) Let a and b be any two vertices of C, Since $\phi|_C^{-1}$ is an isomorphism on C, we can find a' and b' in C such that $\phi(a') = a$ and $\phi(b') = b$. By Lemma 5.2, we know that there exists a homomorphism $f : G \to G$ such that $f(a') = b'$. Then $\phi \circ f \circ \phi|_C^{-1}$ is an isomorphism on C such that $\phi \circ f \circ \phi|_C^{-1}(a) = b$. Thus $G|_C$ is vertex transitive. □

Let G be a digraph with $V(G)=\{x_1,x_2,\ldots,x_u\}$ and let $\boldsymbol{r}=(r_1,r_2,\ldots,r_u)$ be a vector of positive integers. We use $G^{\boldsymbol{r}}$ to denote the digraph with $V(G^{\boldsymbol{r}})=\{x_{i,j} \mid 1 \leqslant i \leqslant u, 1 \leqslant j \leqslant r_i\}$ and $(x_{i,j}, x_{k,l}) \in E(G^{\boldsymbol{r}})$ if and only if $(x_i, x_k) \in E(G)$. Assume that $c_1, c_2, \ldots, c_n$ are positive integers. We use G^{c_1} to denote the digraph $G^{\boldsymbol{r}}$ with $r_i = c_1$ for every i. Moreover, if $V(G) = C_1 \cup C_2$, we use $G^{c_1c_2}$ to denote the digraph $G^{\boldsymbol{r}}$ with c_1 corresponding to every vertex u in C_1 and c_2 corresponding to every vertex v in C_2. Similarly, we can define $G^{c_1c_2\ldots c_n}$.

Lemma 5.6. *$P_G \geqslant P_{\hat{G}^2}$ for any homomorphic image $\hat{G}$ of G if G is a digraph.*

Proof. Since both $V(G)$ and $V(\hat{G}^{\mathbf{2}})$ are finite, the number of homomorphisms from G to $\hat{G}^{\mathbf{2}}$ is finite. Let $\{\phi_1, \phi_2, \ldots, \phi_k\}$ be the set of homomorphism from G to $\hat{G}^{\mathbf{2}}$. We define a function $\phi : G \to (\hat{G}^{\mathbf{2}})^{[k]}$ by setting $\phi(x) = (\phi_1(x), \phi_2(x), \ldots, \phi_k(x))$. Obviously, ϕ is an isomorphism from G into $(\hat{G}^{\mathbf{2}})^{[k]}$. Hence $P_G \geqslant P_{(\hat{G}^2)^{[k]}}$. By Theorem 5.4, $P_G \geqslant P_{\hat{G}^2}$. □

5.3.3. Classification of matrix capacity functions

Lemma 5.7. *Let M be any matrix in $\mathscr{M}$ such that $G[M]$ has at least two closed sets. Then P_M is non-*PAMI.

Proof. We only prove the lemma for the case $G[M]$ which has exactly two closed sets through the following steps.

(1) It follows from Lemma 5.4 that $G[M]$ contains a 2-core subdigraph $\hat{G}$ as a homomorphic image. Since $\hat{G}$ is a subdigraph of $G[M]$, $P_{G[M]} \leqslant P_{\hat{G}}$. By Lemma 5.6, we have $P_{\hat{G}^2} \leqslant P_{G[M]} \leqslant P_{\hat{G}}$.

(2) Let C_1 and C_2 be the two cores of $\hat{G}$. Assume that $|C_1| = c_1$, $|C_2| = c_2$, $r = 2c_2$, and $s = 2c_1$. Let $\tilde{G} = \hat{G}^{rs}$. Since $\hat{G}^{\mathbf{2}}$ is a subdigraph of $\tilde{G}$, we have $P_{\hat{G}^2} \geqslant P_{\tilde{G}}$. By Theorem 5.5, $P_{\hat{G}^2} \leqslant P_{\tilde{G}}$. We have $P_{\hat{G}^2} = P_{\tilde{G}}$.

(3) Let H be any digraph such that $P_{\hat{G}^2}(H) \neq 0$. By step 1, $P_{\hat{G}^2} \leqslant P_{\hat{G}}$. Note that $\hat{G}$ is a homomorphic image of $\hat{G}^{\mathbf{2}}$. By Theorem 5.6, $P_{\hat{G}}(H) = P_{\hat{G}^2}(H)$.

(4) By steps 1, 2 and 3, $P_{G[M]}(H) = P_{\hat{G}}(H) = P_{\hat{G}^2}(H) = P_{\tilde{G}}(H)$ if $P_{\hat{G}^2}(H) \neq 0$.

(5) Let x and y be any two positive integers with $1 < x < y$. Obviously, $\hat{G}^{\mathbf{2}} \subseteq \tilde{G}^{xy}$. We have $P_{\hat{G}^2}(\tilde{G}^{xy}) \neq 0$. By step 4, $P_{G[M]}(\tilde{G}^{xy}) = P_{\hat{G}}(\tilde{G}^{xy}) = P_{\hat{G}^2}(\tilde{G}^{xy}) = P_{\tilde{G}}(\tilde{G}^{xy})$. By Theorem 5.3, $P_{G[M]}(\tilde{G}^{xy}) = xrs/2$, $P_{G[M]}(\tilde{G}^{yx}) = xrs/2$, and $P_{G[M]}(\tilde{G}^{xy}\tilde{G}^{yx}) = xyr^2s^2/4$.

(6) Let $M = (m_{ij})_{u\times u}$. We set $\alpha = \max\{m_{ij} | 1 \leqslant i,\ j \leqslant u\} + 1$. For any digraph H with $|V(H)| = v$, $t_\alpha(H)$ denotes the matrix $(t_{ij})_{v\times v}$ where $t_{ij} = \alpha$ if $(i,j) \in E(H)$ and 0 if otherwise. Obviously, $P_M(t_\alpha(\tilde{G}^{xy})) = P_{G[M]}(t_\alpha(\tilde{G}^{xy})) = P_{G[M]}(\tilde{G}^{xy}) = xrs/2$. Similarly, $P_M(t_\alpha(\tilde{G}^{yx})) = xrs/2$ and $P_M(t_\alpha(\tilde{G}^{xy}) \otimes t_\alpha(\tilde{G}^{yx})) = xyr^2s^2/4$. Hence P_M is not pseudo-multiplicative. By Theorem 5.2, P_M is non-PAMI. □

Theorem 5.10. *Let $M = (m_{ij})_{u\times u}$ be any matrix in $\mathscr{M}$. P_M is* PAMI *if and only if $G[M]$ has exactly one nonempty closed set. Moreover, P_M is* AMI *if and only if $M = (0)$.*

Proof. From Lemma 5.7, P_M is non-PAMI if $G[M]$ has at least two closed sets. Assume that $G[M]$ has exactly one closed set. Let C be a core of $G[M]$ for the closed set S of $G[M]$. Then $P_{G[M]|_C} \geqslant P_{G[M]}$. Obviously, $G[M]|_C$ is a homomorphic image of $G[M]$. By Theorem 5.6, $P_{G[M]}(N) = P_{G[M]|_C}(N)$ if $P_{G[M]}(N) \neq 0$. Assume that N is any matrix such that $P_M(N) \neq 0$. By Theorem 5.7, $P_M(N) = P_{G[M]}(N)$. Hence, $P_M(N) = P_{G[M]|_C}(N)$ if $P_M(N) \neq 0$. Let A and B be matrices with $P_M(A) \neq 0$ and $P_M(B) \neq 0$. By Corollary 5.1, $P_M(A \oplus B) \neq 0$. Thus $P_M(A) = P_{G[M]|_C}(A)$ and $P_M(B) = P_{G[M]|_C}(B)$. Since $G[M]|_C$ is vertex transitive, $P_{G[M]|_C}(A \oplus B) = P_{G[M]|_C}(A) + P_{G[M]|_C}(B)$. However, $P_M(A\oplus B) = P_{G[M]|_C}(A\oplus B)$ because $P_M(A\oplus B) \neq 0$. Hence $P_M(A\oplus B) = P_M(A) + P_M(B)$. Thus P_M is pseudo-additive. By Theorem 5.2, P_M is PAMI. Hence P_M is PAMI if and only if $G[M]$ has exactly one closed set.

Let $\alpha = \max\{m_{ij} \,|\, 1 \leqslant i,\ j \leqslant u\} + 1$ and let β be $\min\{m_{ij} \,|\, m_{ij} > 0\}$ if there exists some $m_{ij} > 0$ and 0 otherwise. Suppose that $\beta \geqslant 1$. Obviously, $m_{ij} \leqslant (m_{ij})^k$ for every $1 \leqslant i, j \leqslant u$ and $k \in \mathscr{N}$. Hence $P_M(M) \neq 0$. Since P_M is increasing, $P_M(\alpha M) \neq 0$. Since $m_{ij}/\alpha < 1$ for every $1 \leqslant i, j \leqslant u$, $P_M((1/\alpha)M) = 0$. Then $P_M((1/\alpha)M \otimes \alpha M) = P_M(M^{[2]}) = P_M^2(M) \neq P_M((1/\alpha)M)P_M(\alpha M)$. Hence P_M is not AMI. Suppose that $0 < \beta < 1$. Obviously, $P_M((1/\alpha)M) = 0$ and $P_M((\alpha/\beta)M) \neq 0$. Hence $P_M((\alpha^3/\beta^2)M) \neq 0$. Then $P_M((\alpha^3/\beta^2)M \otimes (1/\alpha)M) = P_M((\alpha/\beta)^2 M^{[2]}) = P_M^2((\alpha/\beta)M) \neq P_M((\alpha^3/\beta^2)M)P_M((1/\alpha)M)$. Hence P_M is not AMI. Finally, suppose $\beta = 0$. Suppose that $u = 1$. Then M is (0). Obviously, $P_M(N) = v$ if N is an $v \times v$ matrix. Moreover, P_M is AMI. Suppose that

$u > 1$. It is obvious that $P_M((1)) = 0$ and $P_M(M) = u$. However $(1) \otimes M = M$. Thus $P_M((1) \otimes M) \neq P_M((1))P_M(M)$. Therefore P_M is not AMI. Hence P_M is AMI if and only if $M = (0)$. $\square$

5.4. *Other* MI *functions*

From the above discussion, we notice that most of MI functions on $\mathscr{M}$ we discussed are generalizations of those MI functions on $\mathscr{G}$. Obviously, we will get other MI functions for $\mathscr{M}$. Furthermore, we can extend Hedetniemi conjecture on $\mathscr{M}$.

6. MI functions on multidigraphs

There are a lot of algebraic subsystems of $(\mathscr{M}, \oplus, \otimes, \leqslant)$. We can study the MI functions on each algebraic subsystem of $\mathscr{M}$. In this paper, we are only interested in two important algebraic subsystems of $\mathscr{M}$, the set of multidigraphs and the set of loopless multidigraphs. Similar technique can be used to discuss other algebraic subsystems of $\mathscr{M}$. In this section, we are going to study the MI functions on the set of all multidigraphs. Note that the set of all multidigraphs corresponds to the subset of all matrices of $\mathscr{M}$ with all nonnegative integer entries. We use $\mathscr{M}_1$ to denote the set of all multidigraphs.

It seems that all the MI functions on $\mathscr{M}$ discussed above can easily transformed into the MI functions on $\mathscr{M}_1$ by restricting its domain. For example, let

$$M = \begin{pmatrix} 0 & 0.5 \\ 0.8 & 0 \end{pmatrix} \quad \text{and} \quad M' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Obviously, $M \notin \mathscr{M}_1$ and $M' \in \mathscr{M}_1$. Yet $h_M|_{\mathscr{M}_1} = h_{M'}|_{\mathscr{M}_1}$. For simplicity, we will use f for $f|_{\mathscr{M}_1}$ for any function f defined on $\mathscr{M}$ in this section.

It is interesting to point out there is a difference as we study the MI functions on $\mathscr{M}_1$. The difference is only on capacity functions. Let $M = (m_{ij})_{u\times u}$ be any matrix in $\mathscr{M}$. We can define another matrix $M^* = (m^*_{ij})_{u\times u}$ by setting $m^*_{ij} = \lceil m_{ij} \rceil$. Obviously, $P_M(H) = P_{M^*}(H)$ if $H \in \mathscr{M}_1$. For this reason, we may assume that M is a multidigraph as we study the MI functions on $\mathscr{M}_1$.

Theorem 6.1. $\lim_{m\to\infty} [\gamma_G(H^{[m]})]^{1/m}$ *exists for any* $G, H \in \mathscr{M}_1$. *Thus, for any* G *and* H *in* $\mathscr{M}_1$, *we have* $P_G(H) = \lim_{m\to\infty} [\gamma_G(H^{[m]})]^{1/m}$.

Proof. Let $V(G) = \{x_1, x_2, \ldots, x_u\}$ and $V(H) = \{y_1, y_2, \ldots, y_v\}$. Assume that $\gamma_G(H^{[k]}) \geqslant 1$ for some integer k. Then there exists $\gamma_G(H^{[k]})$ disjoint copies of G in $H^{[k]}$, say $G_1, G_2, \ldots, G_{\gamma_G(H^{[k]})}$. Let $V(G_i) = \{\boldsymbol{x}_{i1}, \boldsymbol{x}_{i2}, \ldots, \boldsymbol{x}_{iu}\}$ with $\boldsymbol{x}_{ij}$ corresponding to x_j and let $\boldsymbol{x}_{ij} = (y_{ij_1}, y_{ij_2}, \ldots, y_{ij_k})$ with $y_{ij_l} \in V(H)$ for $1 \leqslant i \leqslant \gamma_G(H^{[k]})$, $1 \leqslant j \leqslant u$. We set $\boldsymbol{x}'_{ij} = (y_{ij_1}, \boldsymbol{x}_{ij}) = (y_{ij_1}, y_{ij_1}, y_{ij_2}, \ldots, y_{ij_k})$ for $1 \leqslant i \leqslant \gamma_G(H^{[k]})$ and $1 \leqslant j \leqslant u$. Obviously, each $\{\boldsymbol{x}'_{i1}, \boldsymbol{x}'_{i2}, \ldots, \boldsymbol{x}'_{iu}\}$ induces a copy G'_i of G in $H^{[k+1]}$. Moreover, $G'_1, G'_2, \ldots, G'_{\gamma_G(H^{[k]})}$

are disjoint. Thus $\gamma_G(H^{[k+1]}) \geqslant \gamma_G(H^{[k]})$. Therefore, $\gamma_G(H^{[r]}) \geqslant \gamma_G(H^{[s]})$ if $r \geqslant s$. Let $a_k = \log[\gamma_G(H^{[k]})]$. Obviously, $0 \leqslant a_k/k \leqslant \log|V(H)|$ and $\overline{\lim}_{k\to\infty} a_k/k$ exists, say a^*. Therefore, for every $\varepsilon > 0$ there exists a positive integer $n > k$ such that $a^* - \varepsilon \leqslant a_n/n$. Hence $n(a^* - \varepsilon) \leqslant a_n$. Since $a_m \geqslant a_n$ for every $m \geqslant n$, $n(a^* - \varepsilon) \leqslant a_n \leqslant a_m$. Thus $(n/m)(a^* - \varepsilon) \leqslant a_m/m$ and $\lim_{m\to\infty} a_m/m = a^* = \overline{\lim}_{m\to\infty}(a_m/m)$. Therefore $\lim_{m\to\infty}[\gamma_G(H^{[m]})]^{1/m} = e^{a^*}$. □

Theorem 6.2. *Assume that $G \in \mathscr{M}_1$. Then P_G is* PAMI *if and only if G has exactly one closed set. Moreover, P_G is* AMI *if and only if $A(G) = (0)$ or (1).*

Proof. Using the same argument as on $\mathscr{M}$, we can easily prove that P_G is PAMI if and only if G has exactly one closed set. Suppose that $|V(G)| \geqslant 2$ or $A(G) = (\alpha)$ with $\alpha \geqslant 2$. It is obvious that $(1) \otimes G = G$, $P_G(G) \neq 0$, and $P_G((1)) = 0$. Thus P_G is not AMI.

Now, $A(G)$ can only be (0) or (1). It is easy to see that $P_{(0)}(H)$ is the number of nodes in H and $P_{(1)}(H)$ is the number of nodes in H with a selfloop. Thus $P_{(0)}$ and $P_{(1)}$ are AMI. The theorem is proved. □

Corollary 6.1. *P_G is* PAMI *if G is a multidigraph with a loop.*

7. MI functions on loopless multidigraphs

We use $\mathscr{M}_2$ to denote the set of loopless multidigraphs. Obviously $\mathscr{M}_2$ consists of all matrices of $\mathscr{M}_1$ with 0 at all diagonal entries. Again, all the MI functions on $\mathscr{M}_1$ discussed above can easily be transformed into MI functions on $\mathscr{M}_2$. Again, we use f for $f|_{\mathscr{M}_2}$ in this section. Let G be any multidigraph in $\mathscr{M}_1 - \mathscr{M}_2$. Obviously $P_G(H) = 0$ for any H in $\mathscr{M}_2$. Hence we concentrate on those G in $\mathscr{M}_2$.

Theorem 7.1. *Assume that $G \in \mathscr{M}_2$. Then P_G is* PAMI *if and only if G has exactly one closed set. Moreover, P_G is* AMI *if and only if G has exactly one closed set such that $P_G(H) \neq 0$ for any homomorphic image $H \in \mathscr{M}_2$ of G.*

Proof. Using the same argument as on $\mathscr{M}$, we can easily conclude that P_G is PAMI if and only if G has exactly one closed set. Let H_1 and H_2 be two digraphs in $\mathscr{M}_2$. Let $k = |\{i \mid H_i$ contains a homomorphic image of $G\}|$. Suppose $k = 2$. Then H_i contains a homomorphic image $\hat{H}_i$ of G for $i = 1, 2$. By our assumption, $P_G(\hat{H}_i) \neq 0$ for $i = 1, 2$. Since P_G is increasing, $P_G(H_i) \neq 0$ for $i = 1, 2$. Since P_G is PAMI, $P_G(H_1 \oplus H_2) = P_G(H_1) + P_G(H_2)$. Suppose $k = 1$. Without loss of generality, we may assume that H_1 contain a homomorphic image of G. Hence $P_G(H_1) \neq 0$ and $P_G(H_2) = 0$. Let n be any positive integer. Obviously, $\gamma_G(H_1^{[k]} \otimes H_2^{[n-k]}) = 0$ for any $0 \leqslant k \leqslant n-1$. Hence $\gamma_G((H_1 \oplus H_2)^{[n]}) = \gamma_G(H_1^{[n]})$ for any positive integer n. Thus $P_G(H_1 \oplus H_2) = P_G(H_1) = P_G(H_1) + P_G(H_2)$. Suppose $k = 0$. Obviously $P_G(H_1 \oplus H_2) = P_G(H_1) + P_G(H_2) = 0$. Hence P_G is additive. By Theorem 5.2, P_G is AMI.

On the other hand, suppose that G has a homomorphic image H such that $P_G(H)=0$. It is obvious that $P_G(G) \neq 0$ and $P_G(G \otimes H) \neq 0$. Thus P_G is not AMI. Hence the theorem is proved. □

For example, P_G is AMI in $\mathscr{M}_2$ if G is a directed odd cycle.

Corollary 7.1. *Assume that* $G \in \mathscr{M}_2$ *and* P_G *is* AMI. *Then* G *has no parallel edges.*

With Theorem 7.1, we can classify those loopless multidigraphs G such that P_G is AMI in $\mathscr{M}_2$. However, it is not easy to check all the homomorphic image H of G such that $P_G(H) \neq 0$. It follows from the proof of Theorem 5.10 that we may assume that $G=\tilde{G}|_C$ where C is a core of $\tilde{G}$. With this assumption, G is a connected vertex transitive digraph. For this reason, we say a loopless digraph G is *nice* if it is a connected vertex transitive digraph such that any homomorphism of G is an isomorphism. We say a loopless digraph G is *good* if it is a connected vertex transitive digraph such that $P_G(H) \neq 0$ for any homomorphic image $H \in \mathscr{M}_2$. We have the following conjecture.

Conjecture 1. A digraph is nice if and only if it is good.

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