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RESEARCH NOTE

Monotone Routing in Multirate Rearrangeable Clos Networks

Xiao-Dong Hu[2]

Institute of Applied Mathematics, Chinese Academy of Sciences, Beijing 100080, China; and Department of Computer Science, City University of Hong Kong, Hong Kong
E-mail: xdhu@amath3.amt.ac.ch

Xiao-Hua Jia

Department of Computer Science, City University of Hong Kong, Hong Kong
E-mail: jia@cs.city.edu.hk

Ding-Zhu Du[1, 2]

Department of Computer Science, University of Minnesota, Minneapolis, Minnesota 55455; and Institute of Applied Mathematics, Chinese Academy of Sciences, Beijing 100080, China
E-mail: dzd@cs.umn.edu

and

Frank K. Hwang

Department of Applied Mathematics, Chiao Tung University, Hsinchu, Taiwan
E-mail: fhwang@math.nctu.edu.tw

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In this paper, we study monotone routing in the symmetric three-stage Clos network with general bandwidth, and propose a new approach to analyze the multirate rearrangeability. For networks with small size switches, we show that monotone routing is better than the previous methods.

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1. INTRODUCTION

The symmetric three-stage Clos network $C(n, m, r)$ has been widely used in them design of telecommunication networks [1, 4]. $C(n, m, r)$ consists of r $(n \times m)$-crossbars (switches) in the first (or input) stage, m $(r \times r)$-crossbars in the second (or central) stage, and r $(m \times n)$-crossbars in the third (or output) stage. Every crossbar in the first stage has an outlet connected to an inlet of every crossbar in the second stage and every crossbar in the second stage has an outlet connected to an inlet of every crossbar in the third stage. There are rn inlets in total in the first stage, called *inputs*, and rn outlets in total in the third stage, called *outputs*. A $C(2, 3, 4)$ is illustrated in Fig. 1.

A Clos network $C(n, m, r)$ in classical circuit switching is *rearrangeable* if it can route every matching between inputs and outputs.

Since Melen and Turner [7] initiated the study on nonblocking properties in multirate interconnection networks, it has become one of the most important research topics in ATM networks with applications in computer networks, telecommunications, and Internets. In the multirate Clos network $C(n, m, r)$, the switch is more powerful. Each switch can realize an edge-weighted bipartite graph between inlets and outlets with the property that the total weight of edges at each inlet and outlet is at most one. Each edge still corresponds to a *connection* (*call* or *request*).

A connection in the multirate network is denoted by a triple (i, j, w), where i and j are the input and output of the connection, respectively, while w is the *weight* of the connection representing the bandwidth required by the connection. A *route* is a path in the network joining an input switch (a switch in the first stage) to an output switch (a switch in the third stage). A route r *realizes* a connection (i, j, w) if the switch with input i and the switch with output j are connected by r with capacity w. Each link in the network is assumed to have unit capacity (after normalization). Thus the weight of any connection is in the interval $[0, 1]$.

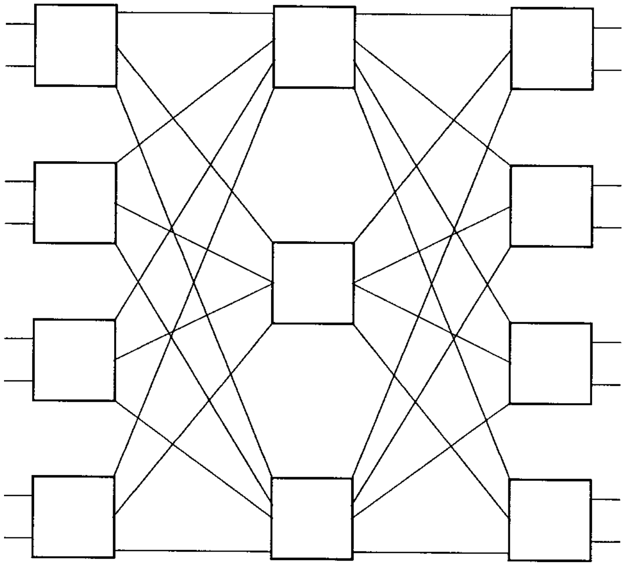

FIG. 1. Symmetric three-stage Clos network $C(2, 3, 4)$.

A set of connections is *compatible* if the sum of weights of all connections from any input and to any output are at most one. A *request frame* is a set of compatible connections. A *configuration* is a set of routes, and it is *compatible* if the total weight of routes passing through every link is at most one. A request frame is said to be *realizable* if there exists a compatible configuration which contains routes realizing all connections in the request frame. A multirate network is said to be (*multirate*) *rearrangeable* if every request frame is realizable.

In classical circuit switching, all connections are assumed to have the same rate one. Namely, a network is said to be rearrangeable in classical circuit switching if every compatible request frame of connections with weight one is realizable. It is well known [1] that the symmetric three-stage Clos network $C(n, m, r)$ is rearrangeable in circuit switching if and only if $m \geqslant n$. Now, since multirate is involved, more crossbars are needed in the center stage to reach the rearrangeability. Chung and Ross [3] showed that if $m \geqslant 2n-1$, the symmetric three-stage Clos network $C(n, m, r)$ is multirate rearrangeable when each connection has weight chosen from a given set $\{1, p\}$. After proving this result (Corollary 3), they stated that "It would be of interest to show that Corollary 3 holds for the general discrete bandwidth case with K distinct rates." For an easy reference, we call it the Chung–Ross conjecture.

Chung–Ross conjecture. If $m \geqslant 2n-1$, the symmetric three-stage Clos network $C(n, m, r)$ is multirate rearrangeable when each connection has weight chosen from a given finite set $\{p_1, p_2, ..., p_k\}$ where $1 \geqslant p_1 > p_2 > \cdots > p_k > 0$ and p_i is an integer multiple of p_k for $1 \leqslant i \leqslant k-1$.

Melen and Turner [7] gave a routing algorithm CAP and with CAP, it can be shown that the multirate three stage Clos network $C(n, 2n-1, r)$ is rearrangeable when each connection has a weight at most 1/2. Using this fact, Lin *et al.* [6] recently showed that the Chung–Ross conjecture is true for a restricted discrete bandwidth case where each connection has a weight chosen from a set $\{p_1, p_2, ..., p_k\}$ such that $1 \geqslant p_1 > p_2 > \cdots > p_k > 0$ and p_i is an integer multiple of p_{i+1} for $2 \leqslant i \leqslant k-1$. In fact, the Chung–Ross conjecture seems to be true not only in the discrete bandwidth case but also in the continuous bandwidth case. By using some coloring and partition arguments from graph theory, Du *et al.* [5] proved that $C(n, m, r)$ for $m \geqslant 41n/16$ is multirate rearrangeable in the general bandwidth case.

In this paper, we consider multirate rearrangeability in the Clos network $C(n, m, r)$ with arbitrary rates. We study a monotone routing and establish a relation between the multirate rearrangeability under monotone routing and a system of linear inequalities. From investigating the system, we obtain some properties of the monotone routing and improve some best known results for small n.

2. MAIN RESULTS

We study a simple routing method as follows.

Monotone routing. Sort all requests in weight-nonincreasing order and realize each request one by one whenever a connection can be found.

Define a linear system $I(n, k)$ with $k \geqslant n+1$, consisting of $k+n$ inequalities as follows.

$$\begin{cases} x_1^1 + x_2^1 + \cdots + x_n^1 + x_1^0 > 1 \\ x_1^2 + x_2^2 + \cdots + x_n^2 + x_1^0 > 1 \\ \vdots \quad \vdots \quad \ddots \quad \vdots \quad \vdots \quad \vdots \\ x_1^n + x_2^n + \cdots + x_n^n + x_1^0 < 1 \\ \vdots \quad \vdots \quad \ddots \quad \vdots \quad \vdots \quad \vdots \\ x_1^k + x_2^k + \cdots + x_n^k + x_1^0 > 1 \end{cases} \tag{1}$$

$$\begin{cases} x_1^0 + x_1^1 + x_1^2 + \cdots + x_1^n + \cdots + x_1^k \leqslant 1 \\ \qquad x_1^1 + x_2^2 + \cdots + x_2^n + \cdots + x_2^k \leqslant 1 \\ \qquad x_3^1 + x_3^2 + \cdots + x_3^n + \cdots + x_3^k \leqslant 1 \\ \qquad \vdots \quad \vdots \quad \ddots \quad \vdots \quad \ddots \quad \vdots \quad \vdots \\ \qquad x_n^1 + x_n^2 + \cdots + x_n^n + \cdots + x_n^k \leqslant 1 \end{cases} \tag{2}$$

where real variables $x_1^0 \geqslant 0$ and $x_i^j \geqslant 0$ for $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant k$ satisfy the following constraint.

$$x_i^j(x_i^j - x_1^0) \geqslant 0, \qquad \text{for} \quad 1 \leqslant i \leqslant n, \qquad 1 \leqslant j \leqslant k \tag{3}$$

The above inequality (3) means that if $x_i^j > 0$, then $x_i^j \geqslant x_1^0$ for $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant k$.

The relationship between system $I(n, k)$ and the multirate rearrangeability of the Clos network $C(m, n, r)$ is shown in the following lemma.

LEMMA 2.1. *If $I(n, k)$ has no solution, then $C(m, 2k-1, r)$ is multirate rearrangeable under monotone routing.*

Proof. By contradiction, suppose $C(m, 2k-1, r)$ is not multirate rearrangeable under monotone routing. Consider the first request which cannot be routed, and assume it is from input switch I to output switch J with weight w. Then for each center switch H, either the link from input switch I to H has a load greater than $1-w$ or the link from H to output switch J has a load greater than $1-w$. Therefore, either there exist k center switches such that every link from I to them has a load greater than $1-w$ or there exist k center switches such that every link from them to J has a load greater than $1-w$. Without loss of generality, assume the former case occurs. Note that I has n inlets, and without loss of generality, assume the request (I, J, w) is from the first inlet. Set x_j^i to be the weight of request from the jth inlet through the i-th center switch. Then we have

$$\sum_{j=1}^{n} x_j^i > 1 - w, \qquad \text{for} \quad i = 1, 2, ..., k.$$

By substituting w with x_1^0 we obtain the inequalities (1). In addition, the inequalities (2) are satisfied under the constraint of connection capacity, and the inequalities (3) are satisfied, because monotone routing is applied. Therefore, system $I(n, k)$ has a solution, a contradiction. ▮

Now we study the necessary and sufficient conditions that system $I(n, k)$ has a solution.

LEMMA 2.2. *If $I(n, k)$ has a solution, then* $\frac{1}{3} \geqslant x_1^0 > \frac{k-n}{k-1}$.

Proof. Summing up all inequalities in (1) will lead to $(k-1)\, x_1^0 > k-n$. Now consider the left hand side of the inequality. Suppose, by contradiction, $x_1^0 > \frac{1}{3}$. Then there are at most two nonzero variables in $\{x_i^j : 1 \leqslant j \leqslant k\}$, for each i, $2 \leqslant i \leqslant n$, and at most one nonzero variable in $\{x_1^j : 1 \leqslant j \leqslant k\}$. Without loss of generality, we assume $x_1^1 > 0$ and $x_2^1 > 0$. Moreover, we can further assume that $x_2^2 > 0$, $x_3^2 > 0$, and $x_4^3 > 0$, $x_4^3 > 0$, ..., and so on. It is clear that under the constraint (2) at most $n-1$ inequalities in (1) can be satisfied at the same time, a contradiction. ▮

COROLLARY 2.1. $I(n, n + \lfloor \frac{n}{2} \rfloor)$ *has no solution.*

Proof. It follows immediately from Lemma 2.2. ▮

LEMMA 2.3. *If* $x_1^0 \leqslant \frac{k-n}{k-2}$, *then*

(a) *each inequality in* (1) *has at least two nonzero variables, and*

(b) *each set* $\{x_i^j : 1 \leqslant j \leqslant k\}$, *for* $1 \leqslant i \leqslant n$, *has at least two non zero variables.*

Proof. (a) Suppose, by contradiction (and without loss of generality), that $x_2^1 \geqslant x_1^0 > 0$ and $x_i^1 = 0$ for $i = 1$ and $3 \leqslant i \leqslant n$. Note that if $x_2^j \neq 0$, then $x_2^j \geqslant x_1^0$, and $x_2^1 + x_2^j \geqslant x_2^1 + x_1^0 > 1$, a contradiction. Hence, $x_2^j = 0$ for $2 \leqslant i \leqslant k$. The sum of nonzero variables in the last $(k-1)$ inequalities of (1) is at most $n - 1 + (k-2)\, x_1^0 \leqslant (n-1) + (k-2) \frac{k-n}{k-2} = k-1$, a contradiction.

(b) Suppose, again by contradiction (and without lose of generality), that there exists $x_{i_0}^1 > 0$ such that $x_{i_0}^j = 0$ for $2 \leqslant j \leqslant k$. Then the sum of nonzero variables in the last $(k-1)$ inequalities of (1) is at most $n - 1 + (k-2)\, x_1^0 \leqslant k-1$, a contradiction. ▮

LEMMA 2.4. $I(4, 5)$ *has no solution.*

Proof. Suppose, by contradiction, that it has a solution. Then by Lemma 2.2, we can assume that $x_1^0 = \frac{1}{4} + \varepsilon$, where $\varepsilon \leqslant \frac{1}{12}$. We can further assume, by Lemma 2.3(b), that $x_1^2 \geqslant x_1^0$ and $x_1^0 \leqslant x_1^1 \leqslant \frac{1}{2}(1 - x_1^0) = \frac{3}{8} - \frac{\varepsilon}{2}$. In the following we consider two cases separately.

Case 1. $x_2^1 \neq 0$ and $x_3^1 = x_4^1 = 0$. In this case, $x_2^1 > \frac{3}{8} - \frac{\varepsilon}{2}$. If there are two nonzero variables in $\{x_2^2, x_3^2, x_4^2\}$, then the sum of nonzero variables in the last three inequalities of (1) is at most $2\frac{7}{8} + \frac{3}{2}\varepsilon \leqslant 3$, a contradiction.

Subcase 1.1. $x_3^2 \neq 0$ and $x_2^2 = x_4^2 = 0$. In this special case, we can assume, by Lemma 2.3, that $x_2^3 \neq 0$ and at least one variable in $\{x_3^3, x_4^3\}$ is nonzero. If both of them are nonzero (note that $x_2^1 + x_2^3 + x_3^2 + x_3^3 + x_4^3 > 1 + 2x_1^0$), then the sum of nonzero variables in the last two inequalities of (1) is less than $3 - (1 + 2x_1^0) + 2x_1^0 = 2$, a contradiction. If $x_3^3 \neq 0$ and $x_4^3 = 0$ (note that $x_2^1 + x_2^3 + x_3^2 + x_3^3 > 2 - 2x_1^0$), then at most one variable in $\{x_2^4, x_3^4, x_2^5, x_3^5\}$ is nonzero. This means that at least one of the last two inequalities of (1) has just one nonzero variable, contradicting Lemma 2.3(a). Thus $x_3^3 \neq 0$ and $x_4^3 \neq 0$. If $x_2^4 = x_2^5 = 0$, then $x_3^4 + x_4^4 + x_3^5 + x_4^5 + 2x_1^0 \leqslant 2 - 2x_1^0 + 2x_1^0 = 2$, a contradiction. Hence, without loss of generality, let $x_2^4 \neq 0$ and $x_4^4 \neq 0$. Since $x_4^3 + x_4^4 > 2 - 2x_1^0 - (1 - \frac{3}{8} + \frac{\varepsilon}{2}) = \frac{7}{8} - \frac{5}{2}\varepsilon \geqslant 1 - x_1^0$, then $x_4^5 = 0$ (and $x_1^5 = x_2^5 = 0$), again contradicting Lemma 2.3(a).

Subcase 1.2. $x_4^2 \neq 0$ and $x_3^2 = x_2^2 = 0$. This special case is the symmetry of Case 1.1.

Case 2. $x_2^1 \neq 0$ and x_3^1 or x_4^1 is nonzero. In this case, we know, from Lemma 2.3(a), that at least one variable in $\{x_2^2, x_3^2, x_4^2\}$ is nonzero. Thus the sum of nonzero variables in the last three inequalities of (1) is at most $3 - 3x_1^0 + 3x_1^0 = 3$, a contradiction. ∎

LEMMA 2.5. *$I(6, 8)$ has no solution.*

Proof. Suppose, by contradiction, that $I(6, 8)$ has a solution. Then due to Lemma 2.3 we can assume that $x_1^0 = \frac{2}{7} + \varepsilon$, where $0 < \varepsilon \leqslant \frac{1}{21}$. According to Lemma 2.3(b), we can further assume (without lose of generality), that $x_1^1 > 0$ and $x_1^2 > 0$. If there are more than two nonzero variables in $\{x_i^j : j = 1, 2, 2 \leqslant i \leqslant 6\}$, then the sum of nonzero variables in the last six inequalities of (1) is at most $5 - 3x_1^0 + 6x_1^0 \leqslant 6$, a contradiction. Hence, for the first two inequalities of (1), each of them has exactly one nonzero variable besides x_1^j, $j = 0, 1, 2$. If there exists an i_0 such that $x_{i_0}^j \neq 0$, $j = 1, 2$, then $x_{i_0}^j = 0$ for $3 \leqslant j \leqslant 8$, since $x_{i_0}^1 + x_{i_0}^2 > 1 - x_1^0$. Hence, the sum of the nonzero variables in the last six inequalities of (1) is at most $4 + 6x_1^0 \leqslant 6$, a contradiction. So without loss of generality, we can assume that

$$x_1^1 > 0, x_2^1 > 0, \qquad \text{and} \qquad x_i^1 = 0, \qquad \text{for} \quad i \neq 1, 2;$$

$$x_1^2 > 0, x_3^2 > 0, \qquad \text{and} \qquad x_i^2 = 0, \qquad \text{for} \quad i \neq 1, 3.$$

If one of the last six inequalities of (1) has four nonzero variables, then for the rest of the five inequalities, the sum of nonzero variables is at most $5 - 5x_1^0 + 5x_1^0 = 5$, a contradiction. Thus, there must exist a nonzero variable x_{i_0} that appears in exactly two inequalities. If $i_0 \neq 2, 3$, then among the last six inequalities of (1) four of them do not have nonzero variable x_{i_0}. Thus the sum of nonzero variables is at most $4 - 4x_1^0 + 4x_1^0 = 4$, a contradiction. Therefore, without loss of generality, we assume $i_0 = 2$, i.e., $x_2^3 \neq 0$ and $x_2^j = 0$ for $j \neq 1, 3$. Note that the sum of nonzero variables in the last five inequalities of (1) is at most $4 - 2x_1^0 + 5x_1^0 \leqslant 5$, a contradiction. ∎

THEOREM 2.1. *$C(n, m, r)$ is multirate rearrangeable under monotone routing when*

(a) $2 \leqslant n \leqslant 4$ *and* $m \geqslant 2n + 1$, *or*

(b) $5 \leqslant n \leqslant 6$ *and* $m \geqslant 2n + 3$.

Proof. It follows from Lemma 1 and above results on system $I(n, k)$. ∎

When comparing this theorem with the best known results (Theorem 5 in [5]), we find that monotone routing shows better performance when $n = 3, 4$ and equal performance when $n = 2, 5, 6$.

3. CONCLUSION

In this paper, we have established a relationship between the rearrangeability of the multirate Clos network $C(n, m, r)$ and a linear system $I(n, k)$ of inequalities through studying monotone routing. This gives an improvement for the upper bound of m for small n. Since small Clos networks are used to be fundamental recursive structure of large networks, this improvement is significant in the hardware optimization of switching networks. For future research, it would be interesting to analyze monotone routing combined with other routings. In addition, as monotone routing does not require any assumption on bandwidth and structures of interconnection networks, the proposed approach used in the Clos network may be extended in other multirate rearrangeable switching networks such as the Beneš network [1] and the Cantor network [2].

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