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A dominance relation enhanced branch-and-bound task allocation

Yung-Cheng Ma *, Chung-Ping Chung

Department of Computer Science and Information Engineering, National Chiao-Tung University, 1001 Ta Hsueh Road, Hsinchu 30050, Taiwan, ROC

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Abstract

We investigate the task allocation problem of allocating a parallel program on parallel processors with non-uniform communication latencies. A branch-and-bound algorithm with a dominance relation is proposed to obtain an optimal task assignment. The key observation on deriving the dominance relation is that tasks can be clustered according to communication weights. The dominance relation is effective to prune the search space when the task clustering boundary – a small cut – is met. The proposed algorithm is compared to the A*-algorithm for task allocation. Experiment shows that our proposed algorithm achieves a speed-up ranging from 1.02 to 1.68, depending on the degree of task clustering and parallelism. This shows the effectiveness of the proposed dominance relation.

Keywords: Task allocation; Branch-and-bound algorithm; Dominance relation

1. Introduction

Advances in hardware and software technologies have led to the use of parallel and distributed computing systems. To execute a parallel program efficiently, a scheduler should distribute the parallel program tasks to processors such that (1) the processor loads are balanced, and (2) the inter-processor communication time is minimized. This raises the task allocation problem.

We investigate the problem of allocating a parallel program on a multiprocessor with non-uniform inter-processor communication latencies. A parallel program is modeled as a node- and edge- weighted undirected graph, called a task graph. The task allocation problem becomes a problem of mapping the set of tasks to the set of processors such that the completion time is minimized, considering both processor load and communication overhead.

A set of work has been done for the task allocation problem. The task allocation problem has been shown to be NP-complete (Garey and Johnson, 1979) and a set of heuristics have been proposed (Lo, 1988; Bowen et al., 1992; Woodside and Monforton, 1993; Hui and Chanson, 1997). A drawback on these heuristics is the poor quality on the assignment found. On the other hand, (Richard et al., 1982; Shen and Tsai, 1985; Sinclair, 1987; Peng and Shin, 1993; Hou and Shin, 1997; Lee and Shin, 1997) proposed state-space searching methods with differences in the problem formulation for various applications and machine configurations. The state-space searching approach finds an optimal assignment at the cost of un-tractable time and space complexity.

In this paper, we propose a branch-and-bound algorithm with a dominance relation to obtain an optimal assignment. We follow the problem formulation defined in (Shen and Tsai, 1985) to investigate the task allocation problem. The key idea to the efficient task allocation is that a dominance relation is proposed to reduce the time- and space-complexity required for obtaining an optimal assignment. We compare the performance with A*-algorithm (Shen and Tsai, 1985) to demonstrate the effectiveness of the dominance relation.

The key observation on deriving the dominance relation is that tasks in the task graph can be clustered into groups according to communication weights. A group may consist of tasks suitable to be placed in the same processor or in the same subnet in a hierarchical architecture. A small cut in the task graph implies the boundary of a task group in the clustering. The dominance relation prunes task assignments violating the clustering on tasks according to the cut.

This paper is organized as follows. In Section 2, we formulate the optimization problem and model it as a state-space searching problem. We then present the

* Corresponding author. Tel.: +886-3-5712121; fax: +886-3-5724176.
E-mail address: ycma@csie.nctu.edu.tw (Y.-C. Ma).

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dominance relation in Section 3 and derive the task allocation algorithm using the dominance relation in Section 4. The experiment results are shown in Section 5. Finally, a conclusion is given in Section 6.

2. Problem modeling

In this section, we present how the task allocation problem is formulated and transformed into state-space searching problem. This section defines the terminologies used in this paper and gives the framework of our proposed task allocation algorithm.

2.1. Formulating task allocation problem

We follow (Shen and Tsai, 1985; Hui, 1997; Bowen et al., 1992) to formulate the task allocation problem for mapping parallel processes to processors. This model assumes that there are little or no precedence relationships and synchronization requirements so that processor idleness is negligible. Contentions on communication links are also ignored.

The optimization problem is formulated as follows. The input to a task allocation algorithm is a *task graph* $G(T,E,e,c)$ and a *machine configuration* $M(P,d)$. The output, called a *complete assignment*, is a mapping that maps the set of tasks T to the set of processors P. An *optimal assignment* is a complete assignment with minimum *cost*. The cost of an assignment is the *turn-around time* of the last processor finishing its execution. To find an optimal assignment, the branch-and-bound algorithm will go through several *partial assignments*, where only a subset of the tasks has been assigned. We define the above terminology to formulate the task allocation problem.

A parallel program is represented as a *task graph* $G(T,E,e,c)$. The vertex set of the task graph is the set of tasks $T=\{t_0,t_1,\ldots,t_{n-1}\}$. Each task $t_i \in T$ represents a program module. The edge set E of the task graph represents communication between tasks. Two tasks t_i and t_j are connected by an edge if t_i communicates with t_j. For each task $t_i \in T$, a weight $e(t_i)$ is associated with it to represent the execution time of the task t_i. For each edge $(t_i,t_j) \in E$, a weight $c(t_i,t_j)$ is given to represent the amount of data transferred between tasks t_i and t_j.

An example task graph is depicted in Fig. 1. Each vertex is a task and the number on each task is the execution weight $e(t_i)$ for the task t_i. Associated with the number on edge (t_i,t_j) is the communication weight $c(t_i,t_j)$. Throughout this article, we will use this task graph to demonstrate the idea behind our algorithm.

The *machine configuration* is represented as $M(P,d)$. $P=\{p_0,p_1,\ldots,p_{m-1}\}$ is the set of all processors. For each pair of processors $p_k,p_l \in P, k \neq l$, a distance $d(p_k,p_l)$ is associated to represent the latency of transferring one unit of data between p_k and p_l. If two tasks t_i and t_j are assigned to different processors p_k and p_l, respectively, the time required for task t_i to communicate with t_j is estimated to be $c(t_i,t_j)*d(p_k,p_l)$. The communication time between two tasks within the same processor is assumed to be zero.

A machine configuration example is depicted in Fig. 2. We take the hierarchical architecture as an example. The machine consists of two subnets. It takes 5 units of time to transfer a unit of data for two processors in the same subnet and 20 units for two processors in different subnets. Throughout this paper, we will use the hierarchical architecture to demonstrate the idea of our task allocation algorithm. However, our proposed algorithm

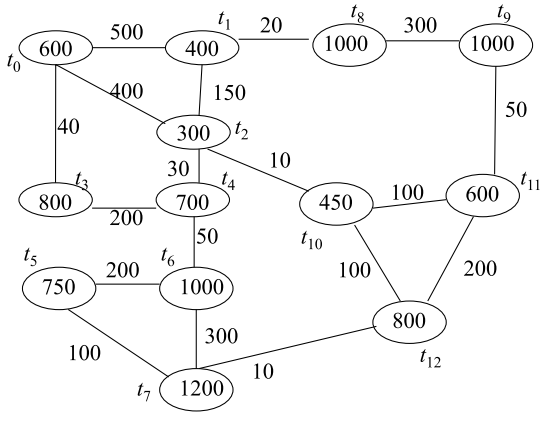


Fig. 1. Example of a task graph.

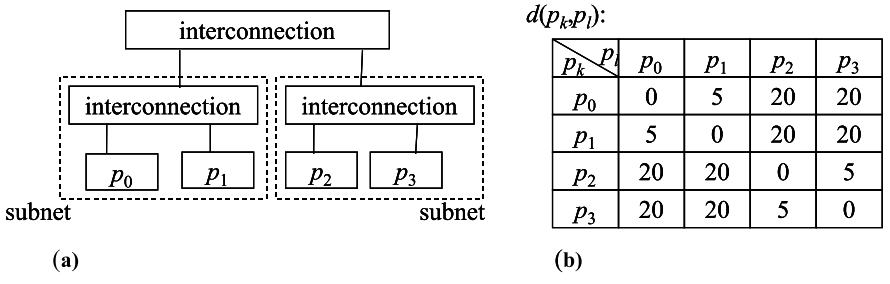


Fig. 2. Example of a machine configuration: (a) the hierarchical architecture; (b) the distance matrix $(d(p_k,P_l))$.

can also be applied to other machine configurations with non-uniform distances between processors.

A *complete assignment* A_c is a mapping that maps the set of tasks T to the set of processors P. To find a complete assignment, our task allocation algorithm will examine several *partial assignments*. A *partial assignment* A is a mapping that maps Q, a proper subset of T, to the set of processors P.

The *turn-around time* of processor p_k, denoted $TA_k(A)$, under a partial/complete assignment A is defined to be the time to execute all tasks assigned to p_k plus the time that these tasks communicate with other tasks not assigned to p_k. That is,

$$TA_k(A) = \sum_{t_i:A(t_i)=p_k} e(t_i) + \sum_{t_i:A(t_i)=p_k} \sum_{t_j:A(t_j)\neq p_k} c(t_i,t_j) * d(p_k, A(t_j)). \quad (1)$$

The *cost* of a partial/complete assignment is the turn-around time of the last processor finishing its execution

$$cost(A) = \max_{\text{processor } p_k} TA_k(A). \quad (2)$$

An *optimal assignment* A_{opt} is a complete assignment with minimum cost

$$cost(A_{opt}) = \min\{cost(A_c) | A_c \text{ is a complete assignment}\}. \quad (3)$$

2.2. Transforming to the state-space searching problem – A*-algorithm

We solve the task allocation problem by state-space searching with a dominance relation. Shen and Tsai (Shen and Tsai, 1985) proposed a state-space searching algorithm without dominance relation to solve the task allocation problem. This state-space search method is known as the A*-algorithm (Hart et al., 1968), which has been proven to guarantee the optimality of the solution obtained. Based on the A*-algorithm, we apply a dominance relation to reduce the number of states to be traversed. In our experiment, this A*-algorithm will be used as a baseline for comparison with our branch-and-bound algorithm.

As illustrated in Fig. 3, the *state-space tree* represents all possible task assignments. We use an $(n+1)$-level m-ary tree to enumerate all possibilities of assigning n tasks to m processors. In the literature of branch-and-bound method, a node in the state-space tree is called a *branching state*. In this study, a branching state represents either a partial or a complete assignment, depending on whether the branching state is an internal node or a leaf node in the state-space tree. In the remaining of this article, we will use the terms branching states and partial/complete assignments interchangeably.

The traversal proceeds as follows. During the traversal, an *active set* (Kohler and Steiglitz, 1974) (also called the *open set* in some literature (Hart et al., 1968)), denoted *ActiveSet*, is used to keep track of all partial/complete assignments that have been explored but not visited. In each iteration during the traversal, the following operations are performed:

Step 1. Remove a partial/complete assignment A_v from *ActiveSet* and visit A_v.

Step 2. If A_v is a complete assignment, terminate the traversal and return A_v as the output.

Step 3. Generate children of A_v in the state-space tree.

Step 4. Put each child node of A_v not pruned by the dominance relation into *ActiveSet*.

For simplicity, we use $ActiveSet^{(k)}$ to denote the contents of the *ActiveSet* at the beginning of the kth iteration, and $A_v^{(k)}$ to denote the partial/complete assignment visited in the kth iteration.

We follow the approach in Shen and Tsai (1985) to determine the traverse order. For each partial/complete assignment A, a lower-bound (denoted $L(A)$) on all complete assignments extended from A (or A itself in case that A is a complete assignment) is estimated. In each iteration during the traversal, the partial/complete assignment A_v with minimum $L(\cdot)$ is removed from *ActiveSet* and visited. $L(A)$ is computed according to the *additional cost* of assigning tasks not assigned in A.

Given a partial assignment A assigning tasks $Q \subseteq T$, we define $AC_k(t_j \rightarrow p_l, A)$ to reflect the *additional cost* on processor p_k if task t_j is assigned to processor p_l:

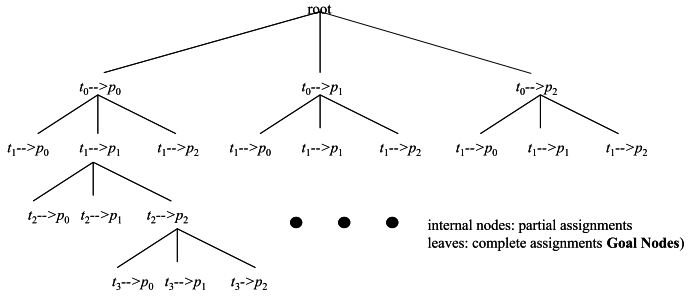


Fig. 3. State-space tree.

$$AC_k(t_j \to p_k, A) = e(t_j) + \sum_{t_i:A(t_i)\neq p_k} c(t_i, t_j) * d(p_k, A(t_i)) \quad \text{if } p_k = p_l, \tag{4}$$

$$AC_k(t_j \to p_l, A) = \sum_{t_i:A(t_i)=p_k} c(t_i, t_j) * d(p_k, p_l) \quad \text{if } p_k \neq p_l. \tag{5}$$

For a partial assignment A, the *cost lower-bound* $L(A)$ for all complete assignments extended from A is estimated to be

$$L(A) \equiv \max_{\text{processor } p_k} \left(\mathrm{TA}_k(A) + \sum_{t_i:\text{not assigned in } A} \left(\min_{\text{prcoessor } p_l} AC_k(t_i \to p_l, A) \right) \right). \tag{6}$$

Without dominance relations (Kohler and Steiglitz, 1974), the method presented so far is known as A*-algorithm (Hart et al., 1968), which was originally proposed by Shen and Tsai (1985) for task allocation. The A-algorithm traverses all partial assignments with $L(\cdot)$ less than the optimal cost. In the remaining sections, we improve the algorithm by adding a dominance relation to reduce the number of states traversed.

3. Dominance relation for space pruning

We propose a *dominance relation* to prune the search space. We pick two partial assignments A_1 and A_2 in which the same set of tasks has been assigned. Suppose $cost(A_1) \leqslant cost(A_2)$. We call A_1 the winner and A_2 the loser. Let $A'_{1-\text{best}}$ and $A'_{2-\text{best}}$ be the complete assignments with a minimum cost in the sub-tree below A_1 and A_2, respectively. We want to be able to check whether it is possible that the winner–loser relationship will be reversed, that is, $cost(A'_{1-\text{best}}) \geqslant cost(A'_{2-\text{best}})$. Our proposed dominance relation claims that what may reverse the winner–loser relationship is the cut the weights of edges between assigned and un-assigned tasks in the task graph.

3.1. Formalization of dominance relation

Definition 1 (*Dominance relation*). Let A_1 and A_2 be two partial assignments. We say A_1 dominates A_2 if we can guarantee that $cost(A'_{1-\text{best}}) \leqslant cost(A'_{2-\text{best}})$ where $A'_{1-\text{best}}$ and $A'_{2-\text{best}}$ are complete assignments with minimum cost extended from A_1 and A_2, respectively.

The inference rule we use to derive a dominance relation is as follows. We omitted the proof since it is a direct consequence from Definition 1.

Corollary 1 (Inference rule for deriving the dominance relation). *Let A_1 and A_2 be two partial assignments. A_1 dominates A_2 if for any complete assignment A'_2 extended from A_2, there exists a complete assignment A'_1 extended from A_1 such that $\mathrm{TA}_k(A'_2) - \mathrm{TA}_k(A'_1) \geqslant 0$ for each processor p_k.*

The idea to derive a dominance relation is depicted in Fig. 4. The assignments A_1, A_2, A'_1, and A'_2 concerned in Corollary 1 are shown in Fig. 4(a), where $S = T - Q$. A'_1 and A'_2 are chosen such that A_1 and A_2 have the same future extension. We rewrite the turn-around time equation according to the task classification shown in Fig. 4(b). In addition to $\mathrm{TA}_k(A_2) - \mathrm{TA}_k(A_1)$, the communication time between assigned and to-be-assigned tasks in $A_1(A_2)$ also contribute to $\mathrm{TA}_k(A'_2) - \mathrm{TA}_k(A'_1)$. This gives a lower bound estimation on $\mathrm{TA}_k(A'_2) - \mathrm{TA}_k(A'_1)$. The proposed dominance relation checks whether A_2 can be pruned according to the estimated *turn-around time difference lower-bound*.

We introduce the following notations:

- $Execution(R) = \sum_{t_i \in R} e(t_i)$ where R is a set of tasks.
- $Communication(R_1, R_2) = \sum_{t_i \in R_1} \sum_{t_j \in R_2} c(t_i, t_j) * d(A'_a(t_i), A'_a(t_j))$, where R_1 and R_2 are sets of tasks.

Following the classification on tasks shown in Fig. 4(b), we rewrite the turn-around time equation in the following lemma. The proof is omitted since it is a trivial computation from the turn-around time formula.

Lemma 1 (Reformulating the turn-around time). *Let A_a be a partial assignment and A'_a be a complete assignment extended from A_a. Q is the set of tasks assigned in A_a and S is the set of tasks not assigned in A_a. Then*

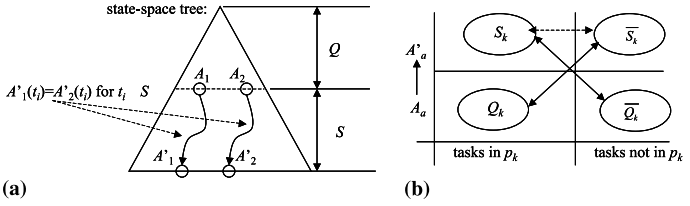


Fig. 4. Idea behind deriving the dominance relation: (a) selection of partial/complete assignments; (b) classifications on tasks.

$$\begin{aligned}\mathrm{TA}_k(A'_a) &= \mathrm{TA}_k(A_a) + \mathit{Execution}(S_k(A_a)) \\ &\quad + \mathit{Communication}\left(Q_k(A_a), \overline{S_k(A_a)}\right) \\ &\quad + \mathit{Communication}\left(\overline{Q_k(A_a)}, S_k(A_a)\right) \\ &\quad + \mathit{Communication}\left(S_k(A_a), \overline{S_k(A_a)}\right), \end{aligned} \tag{7}$$

where

- $Q_k(A_a) = \{t_i \in Q | A_a(t_i) = p_k\}$ and $\overline{Q_k(A_a)} = Q - Q_k(A_a)$
- $S_k(A'_a) = \{t_i \in S | A'_a(t_i) = p_k\}$ and $\overline{S_k(A'_a)} = S - S_k(A_a)$

Before stating the dominance relation, we state the *turn-around time difference lower-bound* $\mathrm{TADL}_k(A_1, A_2)$. Let A_1 and A_2 be two partial assignments with the same set of tasks Q, and $S = T - Q$. $\mathrm{TADL}_k(A_1, A_2)$ is a lower bound on $\mathrm{TA}_k(A'_2) - \mathrm{TA}_k(A'_1)$, where A'_1 and A'_2 are any complete assignments extend from A_1 and A_2, respectively, such that $A'_1(t_i) = A'_2(t_i)$ for each task $t_i \in S$. $\mathrm{TADL}_k(A_1, A_2)$ is estimated to be

$$\begin{aligned}\mathrm{TADL}_k(A_1, A_2) &\equiv \mathrm{TA}_k(A_2) - \mathrm{TA}_k(A_1) \\ &\quad + \sum_{t_i \in S} \left(\min_{p_l \in P} (AC_k(t_i \to p_l, A_2) - AC_k(t_i \to p_l, A_1)) \right). \end{aligned} \tag{8}$$

We then check whether A_2 can be pruned or not by computing $\mathrm{TADL}_k(A_1, A_2)$ for each processor p_k. If $\mathrm{TADL}_k(A_1, A_2)$ is greater than or equal to zero for each processor p_k, it indicates that $\mathrm{TA}_k(A'_2) - \mathrm{TA}_k(A'_1) \geqslant 0$ for each processor p_k and hence we can prune A_2. This is stated in the following theorem.

Theorem 1 (Dominance relation for space pruning). *Let A_1 and A_2 be two partial assignments containing the same set of tasks. If $\mathrm{TADL}_k(A_1, A_2) \geqslant 0$ for each processor p_k, then A_1 dominates A_2.*

Proof. To draw a dominance relation by Corollary 1, we pick the complete assignment A'_1 extended from A_1 such that $A'_1(t_i) = A'_2(t_i)$ for each $t_i \in S$. The pattern is depicted in Fig. 4(a). We want to show that $\mathrm{TA}_k(A'_2) - \mathrm{TA}_k(A'_1) \geqslant 0$ for each p_k.

We decompose both $\mathrm{TA}_k(A'_2)$ and $\mathrm{TA}_k(A'_1)$ as stated in Lemma 1. Since $A'_1(t_i) = A'_2(t_i)$ for each $t_i \in S$, we have

- $\mathit{Execution}\left(S_k\left(A'_2\right)\right) - \mathit{Execution}\left(S_k\left(A'_1\right)\right) = 0$, and
- $\mathit{Communication}(S_k(A'_2), \overline{S_k(A'_2)}) - \mathit{Communication}(S_k(A'_1), \overline{S_k(A'_1)}) = 0$.

Hence, we have

$$\begin{aligned}\mathrm{TA}_k(A'_2) - \mathrm{TA}_k(A'_1) &= \mathrm{TA}_k(A_2) - \mathrm{TA}_k(A_1) \\ &\quad + \left(\mathit{Communication}\left(S_k\left(A'_2\right), \overline{Q_k(A_2)}\right)\right. \\ &\qquad \left. - \mathit{Communication}\left(S_k\left(A'_1\right), \overline{Q_k(A_1)}\right)\right) \\ &\quad + \left(\mathit{Communication}\left(\overline{S_k(A'_2)}, Q_k(A_2)\right)\right. \\ &\qquad \left. - \mathit{Communication}\left(\overline{S_k(A'_1)}, Q_k(A_1)\right)\right) \end{aligned}$$

Procedure DominateTest(A_2,A_1)
input:
– A_2, A_1: two partial assignments assigning the same set of tasks
output:
– returns True if A_2 can be pruned, otherwise returns False
method:
1) prune ♦ True
2) **for** each processor p_k **do**
if $TADL_k(A_1, A_2)$<0 **then**
prune ♦ False
break
3) **return** prune

Fig. 5. Examining partial assignments using the dominance relation.

$$\begin{aligned} &= \mathrm{TA}_k(A_2) - \mathrm{TA}_k(A_1) + \sum_{t_i \in S} \left(AC_k\left(t_i \to A'_2(t_i), A_2\right)\right. \\ &\quad \left. - AC_k\left(t_i \to A'_2(t_i), A_1\right)\right). \end{aligned} \tag{9}$$

Taking a lower bound on the turn-around time difference, we have

$$\begin{aligned}&\mathrm{TA}_k(A'_2) - \mathrm{TA}_k(A'_1)\mathrm{TA}_k(A_2) - \mathrm{TA}_k(A_1) \\ &\quad + \sum_{t_i \in S} \min_{p_l \in P} (AC_k(t_i \to p_l, A_2) - AC_k(t_i \to p_l, A_1)). \end{aligned}$$

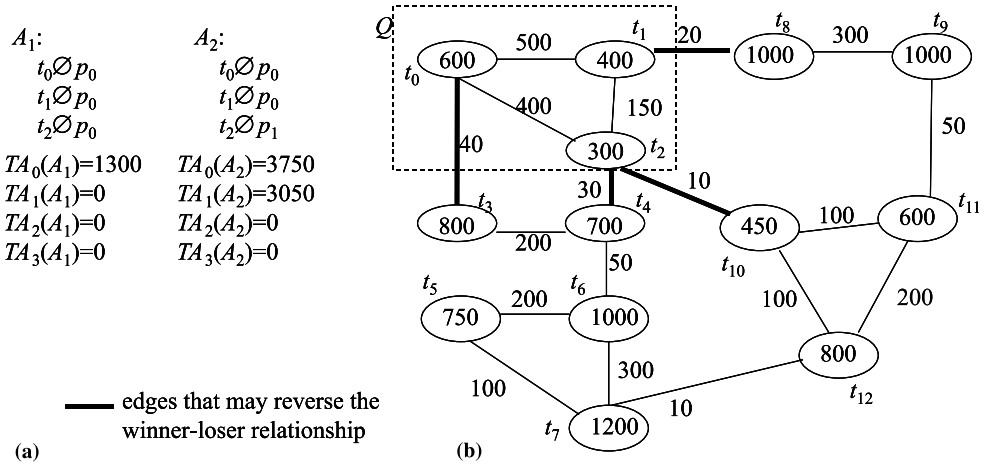
The right-hand side of above inequality is the $\mathrm{TADL}_k(A_1, A_2)$ defined previously. Hence if $\mathrm{TADL}_k(A_1, A_2) \geqslant 0$ for each p_k, it implies A_1 dominates A_2. □

The procedure to detect whether a partial assignment dominates another is depicted in Fig. 5. The procedure computes $\mathrm{TADL}_k(A_1, A_2)$ for each processor to determine whether A_2 can be pruned or not.

3.2. Example of the dominance relation

We use the task graph in Fig. 1 and the machine configuration in Fig. 2 to illustrate the idea of the dominance relation given in Theorem 1. The partial assignments concerned are A_1 and A_2 shown in Fig. 6(a). A_1 is the winner and A_2 is the loser in this comparison. We apply Theorem 1 to guarantee that the winner–loser relationship will not be reversed.

Assuming that A_1 and A_2 have the same future extension, the weights of the bolded edges in Fig. 6(b) are the only factors that may reverse the winner–loser relationship. Following the decomposition of turn-around time stated in Lemma 1, execution time of tasks $\{t_3, t_4, \ldots, t_{12}\}$ and communication time between these tasks will not reverse the winner–loser relationship. Fig. 6(c) shows the effects on processor p_0 for all possibilities of extending A_1 and A_2. Theorem 1 states that assignments of tasks t_4 and t_{10} are the only possible causes to reverse the winner–loser relationship since they both communicate with task t_2 – the primary



$AC_0(t_i \varnothing p_l, A_1)$:

t_i \ p_l	p_0	p_1	p_2	p_3
t_3	800	200	800	800
t_4	700	150	600	600
t_5	750	0	0	0
t_6	1000	0	0	0
t_7	1200	0	0	0
t_8	1000	100	400	400
t_9	1000	0	0	0
t_{10}	450	50	200	200
t_{11}	600	0	0	0
t_{12}	800	0	0	0

$AC_0(t_i \varnothing p_l, A_2)$:

t_i \ p_l	p_0	p_1	p_2	p_3
t_3	800	200	800	800
t_4	850	0	0	0
t_5	750	0	0	0
t_6	1000	0	0	0
t_7	1200	0	0	0
t_8	1000	100	400	400
t_9	1000	0	0	0
t_{10}	500	0	0	0
t_{11}	600	0	0	0
t_{12}	800	0	0	0

(c) $TA_0(A'_2) - TA_0(A'_1)$ 3750-1300 + (-600) due to t_4 + (-200) due to t_{10} 0

Fig. 6. Example to illustrate the dominance relation: (a) partial assignments in consideration; (b) the task graph; (c) effects on p_0 for all possible extensions.

difference between A_1 and A_2. The sum of communication weights between tasks $\{t_2, t_4\}$ and $\{t_2, t_{10}\}$ is small compared to the turn-around time lost in A_2. This means that it is impossible for A_2 to win back what it has lost to A_1 on processor p_0. Similar check applied to other processors also leads to the same conclusion. It indicates that the traversal of the subtree rooted at A_2 can be ignored. This is the primary idea of the dominance relation.

4. Task allocation algorithm

We now present the task allocation algorithm. We present how a good enumeration order is obtained in Section 4.1. In Section 4.2, the branch-and-bound algorithm along with the complexity analysis will be presented. We prove that an optimal assignment will be found by the branch-and-bound algorithm in Section 4.3.

4.1. Preprocessing stage to determine the task enumerating order

The enumeration order plays an important role on the performance of the branch-and-bound algorithm. Our proposed dominance relation is effective when a small cut between assigned and unassigned tasks is met. To exploit the effectiveness of the dominance relation before space overflow, the tasks should be enumerated in an order such that heavily communicated tasks will be enumerated consecutively.

The task enumeration order is determined by applying the max-flow min-cut algorithm recursively to partition the task graph. Each time the max-flow min-cut procedure is applied, the set of tasks is decomposed into two partitions connected by a minimum cut. We repeat the partitioning recursively until each partition contains only one task. The whole partitioning process can be represented by a tree. Each leaf in the tree represents a group containing only one task. The enumeration order

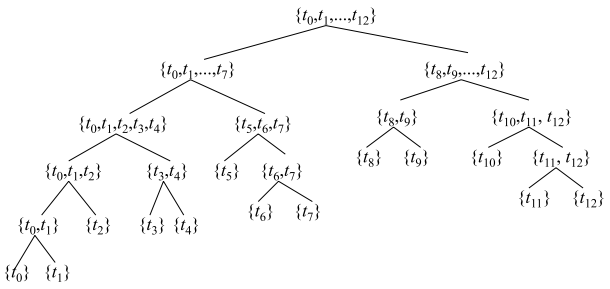


Fig. 7. Determining the task enumeration order.

is thus be determined as a sequence of all leaf nodes by performing depth first traversal. For instance, the partitioning process for the task graph in Fig. 1 is depicted in Fig. 7. Following this result, we obtain the enumeration order that has been used for illustration in previous discussion.

4.2. The optimal branch-and-bound algorithm and its complexity

The branch-and-bound algorithm is shown in Fig. 8. This is based on the A* traversal scheme with addition of the dominance relation for space pruning. The dominance relation is applied to test whether a child of the currently visited partial assignment A_v can be pruned or not (cf. Step 4 in Fig. 8). We pick the child of A_v with minimum $L(\cdot)$ as the *killer* to examining remaining child of A_v (cf. Step 3.3 in Fig. 8). The partial assignment $child[k]$ is pruned and will not be inserted into the *ActiveSet* if the *killer* dominates $child[k]$.

The time complexity encountered by both A*-algorithm and our proposed algorithm to visit a branching state is as follows. Let n be the number of tasks and m be the number of processors. In the implementation, $AC_k(t_j \rightarrow p_l, A')$s for all children of the visited partial assignment A_v are calculated and stored in the *AC-tables* first. Remaining quantities $(\mathrm{TA}, \mathrm{TAL}, L, \mathrm{TADL})$ can then be calculated by looking up the AC-tables. By examining each edge in the task graph and accumulating communication costs into the AC-table, the AC-table for a partial assignment can be established in $O(n^2 * m)$ and AC-tables for all children can be established in $O(n^2 * m^2)$. The quantity $\mathrm{TAL}_k(\cdot)$ can be calculated in $O(m * n)$ and $L(\cdot)$ for a partial assignment can be calculated in $O(m^2 * n)$. The time complexity to compute $L(\cdot)$ for all children is $O(m^3 * n)$. The *ActiveSet* is implemented as a heap and the time complexity to insert all children is $O(m * n * (\log m))$. Both A*-algorithm and our proposed algorithm encounters the $O(n^2 * m^2) + O(m^3 * n)$ time complexity to visit a branching state.

Algorithm BB-Alloc(G,M)
- /* initialization phase */
  - L(root of the state-space tree) ♦ 0
  - *ActiveSet* ♦ {root of the state-space tree}
- repeat the following /* traversal phase */
  1) remove a partial/complete assignment A_v with minimum $L(\ )$ from *ActiveSet*
  2) **if** A_v is a complete assignment **then return** A_v
  3) /* expand A_v and select the killer */
     3.1) let t_j be the last task assigned in A_v
     3.2) **for** each processor p_k **do**
        3.2.1) $child[k]$ ♦ partial assignment extended from A_v by assigning t_{j+1} to p_k
        3.2.2) compute $L(child[k])$
     3.3) *killer* ♦ $child[i]$ where $L(child[i]) = min\ \{L(child[k]) \mid child[k]$ is a child of $A_v\}$
     3.4) insert *killer* into the *ActiveSet*
  4) /* insert each children (except for the killer) of A_v into *ActiveSet* if it cannot be pruned */
     for each processor p_k such that $child[k]$ $killer$ **do**
        4.1) prune ♦ DominateTest($child[k]$,*killer*)
        4.2) **if** prune=False **then** insert $child[k]$ into *ActiveSet*

Fig. 8. The branch-and-bound algorithm to obtain an optimal assignment.

Our proposed algorithm encounters an additional time complexity to test for the dominance relation. In general, the time complexity to compute $\mathrm{TADL}_k(A_1, A_2)$ is $O(m * n)$. But we notice that, in the proposed algorithm, the two partial assignments A_1 and A_2 for testing differ in only the assignment of one task, say t_a. For processor p_k that $A_1(t_a) \neq p_k$ and $A_2(t_a) \neq p_k$, we have $AC_k(t_i \rightarrow p_l, A_1) = AC_k(t_i \rightarrow p_l, A_2)$ for $p_k \neq p_l$. The quantity $\mathrm{TADL}_k(A_1, A_2)$, for $p_k : A_1(t_a) \neq p_k$ and $A_2(t_a) \neq p_k$, can be written as

$$\mathrm{TADL}_k(A_1, A_2) = \mathrm{TA}(A_2) - \mathrm{TA}(A_1) + \sum_{t_i \text{ not assigned}} (AC_k(t_i \rightarrow p_k, A_2) - AC_k(t_i \rightarrow p_k, A_2)).$$

An implementation of procedure DominateTest with time complexity $O(n * m)$ can be achieved and the time complexity to test the dominance relation for all children of A_v is $O(n * m^2)$. The time complexity for the proposed algorithm to visit a branching state is thus $O(n^2 * m^2) + O(m^3 * n) + O(n * m^2) = O(n^2 * m^2) + O(m^3 * n)$. Since the additional complexity is an order lower than the base complexity required, it is worthwhile to spend the additional complexity for reducing number of states traversed.

The space complexity required by the proposed algorithm is the same as that of the A*-algorithm. Besides the *ActiveSet*, the AC-tables for all children of a partial assignment occupy a space of $O(n * m^3)$. Note that AC-tables can be reused among different branching states and the space does not grow with the number of branching states traversed.

4.3. Correctness proof of the proposed branch-and-bound algorithm

We now show that our proposed algorithm returns an optimal assignment. We first show that the traversal procedure reserves some optimal assignments in the future search space. A complete assignment A_c is said to be in the *future search space* of $ActiveSet^{(k)}$ if either $A_c \in ActiveSet^{(k)}$ or there exists a partial assignment

$A_a \in ActiveSet^{(k)}$ such that A_c can be extended from A_a. Provided that some optimal assignments survived in the future search space, we show that the terminate condition implies the optimality of the solution obtained.

Lemma 2. *During the traversal, there are always some optimal assignments survived in the future search space.*

Proof. If the pruned partial assignment can be extended to be an optimal assignment, the definition of the dominance relation implies that the killer can also be extended to be an optimal assignment. The branch-and-bound algorithm always inserts the killer into the *ActiveSet* (cf. Step 3.4 in Fig. 8) and hence there are always some optimal assignments that survive in the future search space. □

Theorem 2 (Correctness of our proposed algorithm). *Our proposed branch-and-bound algorithm will end up with an optimal assignment.*

Proof. A complete assignment A_c will be removed from the *ActiveSet* in the last iteration during the traversal. The complete assignment returned is this A_c. We want to show that A_c is optimal.

We prove this by contradiction. Suppose A_c is not optimal. Consider the contents of $ActiveSet^{(j)}$ for the last iteration j. Lemma 2 states the existence of an optimal assignment A_{opt} in the future search space of $ActiveSet^{(j)}$. Thus, we have $cost(A_c) > cost(A_{opt})$ since A_{opt} is optimal. Let A_a be the ancestor of A_{opt} (or A_{opt} itself) in $ActiveSet^{(j)}$. By the definition of $L(\cdot), L(A_a) \leqslant cost(A_{opt})$. And hence $L(A_a) \leqslant cost(A_{opt}) < cost(A_c) = L(A_c)$. However, A_c is the one with minimum $L(\cdot)$ in $ActiveSet^{(j)}$. This means $L(A_c) \leqslant L(A_a)$. This produces a contradiction and hence proves this theorem. □

5. Experiments and evaluation

We evaluate the proposed task allocation algorithm by feeding it with several configuration samples generated randomly. The test samples cover different degrees of task clustering and parallelism to test the effectiveness of the dominance relation.

5.1. Test samples generation

We randomly generate a set of task graphs and map the task graphs to selected hierarchical machine architectures. In generating task graphs, the distribution on weights and edge densities are chosen to cover all degrees of clustering on tasks. In selecting the machine configuration, the processor distances are chosen such that the parallelism in optimal assignments ranges from using a few processors to using all processors in the machine. The effectiveness of the dominance relation is tested among various degrees of task clustering and parallelism.

Following the scheme in (Bowen et al., 1992), we generate task graphs by hierarchically combining small sub-graphs. At the lowest level is a set of small complete graphs, each containing 1–4 tasks. The lowest level sub-graphs are then randomly combined to form a middle-level sub-graph. The middle-level sub-graphs are then randomly combined as a final task graph.

Randomly combining sub-graphs are guided by two parameters, the execution-to-communication weight ratio (denoted E/C ratio) and the edge density, defined as follows:

- $$E/C = \frac{\text{Average execution weight among all tasks}}{\text{Average communication weight among all edges}}.$$
- edge density = Probability that two vertices in different sub-graphs are connected by an edge.

In the process of randomly combining sub-graphs, each pair of tasks in different sub-graphs is examined. Whether there is an edge connecting these two tasks is decided according to the edge density. Once an edge is really chosen, the weight on the edge is determined according to E/C ratio.

We denote the attributes of a task graph as a tuple of E/C ratio and an edge density. Combination at each level has its own E/C ratio. For example, a task graph may be generated as follows: (1) selecting sub-graphs with $E/C = 1$ as the lowest level sub-graph, (2) combining lowest level sub-graphs to form a middle level sub-graph with $E/C = 5$ and edge density = 20%, (3) combining middle level sub-graphs to complete a task graph with $E/C = 10$ and edge density = 20%. We denote such a task graph with E/C:(1, 5, 10) and edge density = 20%.

The degree of clustering on tasks is controlled through selecting the E/C ratio and the edge density. The set of tasks can be clearly clustered into groups when (1) the gap on E/C ratio between adjacent levels is large, and (2) the sub-graphs are combined in low edge density. In the experiment, the E/C ratio ranges from 1 to 20 and the edge density varies from 20% to 80%.

Another input for the task allocation program is the machine configuration. The machine configuration for experiments is hierarchical machine similar to Fig. 2(a) but with a larger size and different latency. In the experiment, each machine consists of three subnet, and each subnet consists of three processors. We fix the intra-subnet latency to be one. The inter subnet latency varies from 5 to 20. On mapping the same task graph to different machines, the parallelism in optimal assignments ranges from using processors in only one subnet

to using processors across subnets. The parallelism decreases as the inter subnet latency increases.

5.2. Evaluation metrics

We compare the performance of the proposed task allocation algorithm to A*-algorithm. Let n be the number of tasks and m be the number of processors. According to Section 4.2, we estimate the time for the A*-algorithm to visit a branching state as $C_1 * n^2 * m^2 + C_2 * n * m^3$, where C_1 and C_2 are some constants. Similarly, the time for the proposed algorithm to visit a branching state is $C_1 * n^2 * m^2 + C_2 * n * m^3 + C_3 * n * m^2$ for some constant C_3. Let ST be the number of states traversed by the proposed algorithm and the number of states traversed by the A*-algorithm is R times ST. The speed-up is thus

$$\text{Speed-up} = \frac{(C_1 * n^2 * m^2 + C_2 * n * m^3) * R * ST}{(C_1 * n^2 * m^2 + C_2 * n * m^3 + C_3 * n * m^2) * ST} = \frac{R}{1 + \frac{C_3}{C_1 * n + C_2 * m}}.$$

We estimate the constants on a PC with PentiumPro processor and find that $C_1 \gg C_3$. This is because establishing AC-tables is mainly the multiplication operations and testing the dominance relation is mainly table lookup with the add operations. The speed-up can be approximated by the ratio on traversed states when n and m exceed certain threshold. We thus take the metric to evaluate the effectiveness of the dominance relation as follows.

$$\text{speed-up} \approx R = \frac{\text{number of states traversed by A}*\text{-algorithm}}{\text{number of states traversed by the proposed algorithm}}.$$

5.3. Experiment result

The performance is evaluated using 240 task graphs and three hierarchical machine configurations. The task graphs are generated according to six different E/C tuples and four different edge density values, hence resulting in 24 sets of task graphs. We generate 10 task graphs per set. The three machine configurations differ in the inter subnet latencies, varying from 5 to 20. The combinations of task graphs and machine configurations cover all degree of clustering on tasks and parallelism to test the effectiveness of the dominance relation.

Fig. 9 shows the experiment results. Experiment results on different machine configurations are depicted in different charts. We take the harmonic mean on the speed-up for each set of ten task graphs generated under the same E/C tuple and edge density. The speed-up ranges from 1.02 to 1.68, depending on the degree of clustering on tasks and parallelism. As expected, the curves show that the dominance relation is effective when the tasks can be clearly clustered into groups and the parallelism becomes large.

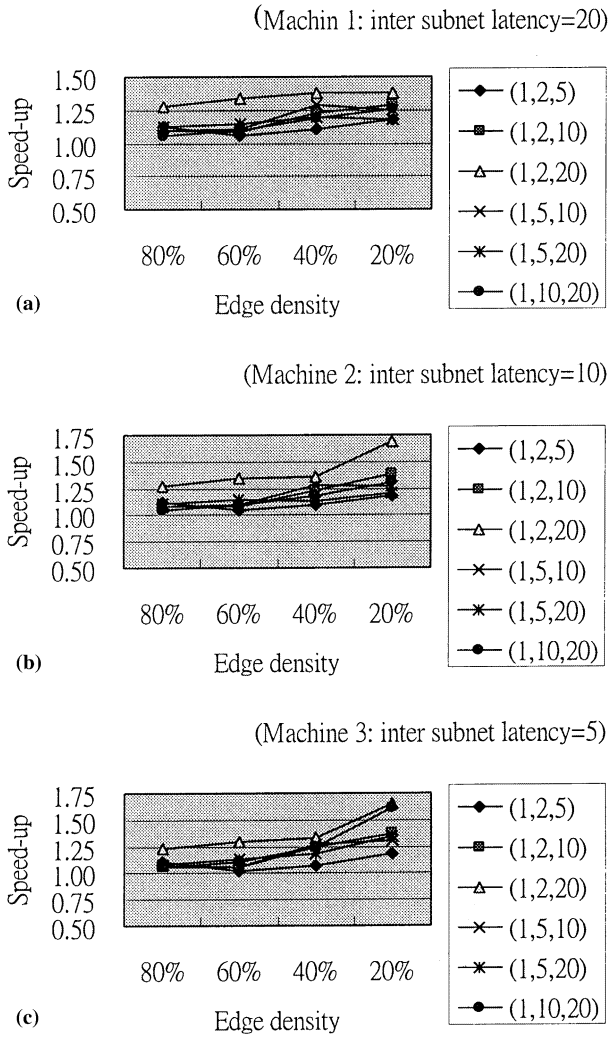


Fig. 9. Performance of the proposed task allocation algorithm: (a) performance comparison in Machine 1; (b) performance comparison in Machine 2; (c) performance comparison in Machine 3.

6. Conclusion

In this paper, we proposed a two-stage task allocation algorithm to obtain an optimal solution for the task allocation problem. The first stage is a recursive partitioning procedure for determining the task enumeration order to exploit the effectiveness of the dominance relation. The second stage is a branch-and-bound algorithm using the dominance relation to prune the search space such that the time and space required can be significantly reduced. We evaluate the performance of our proposed branch-and-bound algorithm by comparing it to the A*-algorithm for task allocation (Shen and Tsai, 1985).

The dominance relation is the key to the efficient task allocation. The key observation is that tasks can be clustered according to the communication weights. When the boundary of task clustering – a small cut – is met, whether a sub-tree in the state-space tree needs further traversal becomes clear. The dominance relation determines whether a partial assignment can be pruned according to the edge weights contributed to the cut.

The proposed task allocation algorithm is evaluated on randomly generated task graphs. Compared to the A*-algorithm for task allocation (Shen and Tsai, 1985), the speed-up of our proposed algorithm ranges from 1.02 to 1.68, depending on whether the tasks can be clearly clustered into groups. This shows the effectiveness of the proposed dominance relation.

7. For further reading

Billionnet et al. (1992), Chou and Chung (1995), Stone (1979), Tom and Murthy (1998).

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Yung-Cheng Ma received the B.S. degree in computer science and information engineering from the National Chiao-Tung University, Hsinchu, Taiwan, Republic of China in 1994. Currently he is pursuing the Ph.D. degree in computer science and information engineering at the National Chiao-Tung University, Hsinchu, Taiwan, Republic of China. His research interests include computer architecture, parallel and distributed systems, and information retrieval.

Chung-Ping Chung received the B.E. degree from the National Cheng-Kung University, Hsinchu, Taiwan, Republic of China in 1976, and the M.E. and Ph.D. degrees from the Texas A&M University in 1981 and 1986, respectively, all in electrical engineering. He was a lecturer in electrical engineering at the Texas A&M University while working towards the Ph.D. degree. Since 1986 he has been with the Department of Computer Science and Information Engineering at the National Chiao-Tung University, Hsinchu, Taiwan, Republic of China, where he is a professor. From 1991 to 1992, he was a visiting associate professor of computer science at the Michigan State University. From 1998, he joined the Computer and Communications Laboratories, Industrial Technology Research Institute, ROC as the Director of the Advanced Technology Center, and then became the Consultant to the General Director. He is expected to return to his teaching position after this three-year assignment. His research interests include computer architecture, parallel processing, and parallelizing compiler.