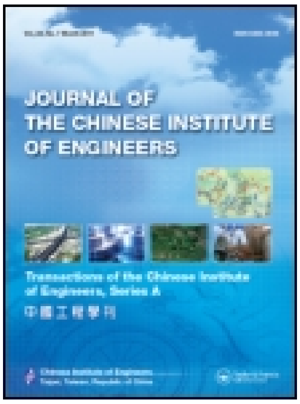

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Capability indices for processes with asymmetric tolerances

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CAPABILITY INDICES FOR PROCESSES WITH ASYMMETRIC TOLERANCES

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Key Words: process capability indices, process yield, process centering, target value.

ABSTRACT

Process capability indices (PCIs) for processes with symmetric tolerances have received substantial research attention. But, PCIs for processes with asymmetric tolerances have been comparatively neglected. Recently, Boyles (1994) reviewed the existing PCI literature and proposed several new indices to handle processes with asymmetric tolerances. In this paper we analyze PCIs based on various process characteristics, then introduce a new class of capability indices to handle processes with asymmetric tolerances. The proposed new indices are compared with existing PCIs in terms of process yield, process centering, and process characteristic related to loss functions. The results indicate that the new indices are superior to the existing capability indices, and provide greater accuracy in current applications using PCIs to measure process potential and performance.

I. INTRODUCTION

Process capability indices (PCIs), whose purpose is to provide a numerical measure on whether a production process is capable of producing items meeting the quality requirement preset by the customers, have received substantial attention in the quality control and statistical literature. Examples include Kane (1986), Chan, Cheng and Spiring (1988), Choi and Owen (1990), Boyles (1991), Pearn, Kotz and Johnson (1992), Franklin and Wasserman (1992), Johnson (1992), Kushler and Hurley (1992), Boyles (1994), Vannman (1995), Pearn and Chen (1996), and many others. Most research work, however, has focused on developing and investigating PCIs for processes with symmetric tolerances. A process is said to have a symmetric tolerance if the target value T is the midpoint of the specification interval (LSL, USL). That is, $T=M=(USL+LSL)/2$, where USL and LSL are the upper and the lower specification limits.

For processes with symmetric tolerances, several capability indices have been proposed to provide unitless measures of process potential and performance. These include C_p, C_{pk}, C_{pm}, and C_{pmk} (see Kane (1986), Chan, Cheng and Spiring (1988),

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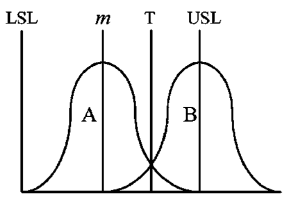

Fig. 1 Process A and B both have C_{pm}=0.55. But the expected proportions non-conforming are 0.27% for A and 50% for B

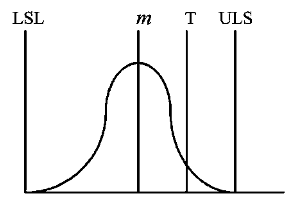

Fig. 2 The process has index values $C^*_{pk}=C^*_{pmk}=0$. But the expected proportion non-comforming is no greater than 0.27%

and Pearn, Kotz and Johnson (1992)). A superstructure containing these four basic indices may be written as (see Vännman (1995)):

$$C_p(u, v) = \frac{d - u|\mu - M|}{3\sqrt{\sigma^2 + v(\mu - T)^2}} \quad (1)$$

Where μ is the process mean, σ is the process standard deviation, $d=(USL-LSL)/2$, $M=(USL+LSL)/2$, T is the target value, and $u, v \geq 0$. It is easy to verify that $C_p(0, 0)=C_p$, $C_p(1, 0)=C_{pk}$, $C_p(0, 1)=C_{pm}$, and $C_p(1, 1)=C_{pmk}$.

As noted by Boyles (1991), C_p and C_{pk} are yield-based indices which are independent of the target T, which may fail to account for process centering (the ability to cluster around the target) with symmetric tolerances, but have an even greater problem with asymmetric tolerances: process yield is maximized (for fixed σ) by $\mu=M$, but $T \neq M$. In this case, process yield and centering are conflicting criteria. For C_{pm}, Pearn, Kotz and Johnson (1993) considered the following example (see Fig. 1) with asymmetric tolerance (LSL, T, USL), where $T=\{3(USL)+(LSL)\}/4$, and $\sigma=d/3$. Then, for processes A and B with $\mu_A=T-d/2=M$ and $\mu_B=T+d/2=USL$ both have the index value of C_{pm}=0.555 and equal degrees of clustering around the target (as $|\mu-T|=d/2$ for both processes A and B). However, the expected proportions nonconforming are approximately 0.27% for process A and 50% for process B. Clearly, C_{pm} inconsistently measures process capability in this case and is inappropriate for asymmetric cases. These problems call for a need to generalize the four basic indices to cover cases with asymmetric tolerances so that positive use of PCIs can be continued.

II. EXISTING PCIś FOR ASYMMETRIC TOLERANCES

There are several generalizations of Eq. (1) proposed to handle processes with asymmetric tolerances, which overcome some problems of C_{pk} and C_{pm}. The first generalization proposed for processes with asymmetric tolerances shifts one of the two specification limits, so that the new (shifted) specification limits are symmetric to the target value (see Kane (1986), and Chan, Cheng and Spiring (1988)). That is, the generalization replaces the true specification limits ($T-D_l$, $T+D_u$) with the new symmetric limits (unjustified sometimes) $T \pm d^*$, where $d^*=\min\{D_l, D_m\}$, $D_u=USL-T$ and $D_l=T-LSL$, then applies the standard definitions of C_p, C_{pk}, C_{pm}, and C_{pmk}. With this generalization, the indices defined in (1) can be rewritten as the following:

$$C^*_p(u, v) = \frac{d^* - u|\mu - T|}{3\sqrt{\sigma^2 + v(\mu - T)^2}} \quad (2)$$

This approach yields the following generalized indices C^*_p, C^*_{pk}, C^*_{pm}, and C^*_{pmk}. Unfortunately, these generalized indices can understate process capability by restricting the process to a proper subset of the actual specification range, as observed by Boyles (1994). For example, consider a process with mean $\mu=T-d/2=M$, and standard deviation $\sigma=d/3$, where the target value T = {3(USL) + (LSL)}/4 (see Fig. 2). Then, we have $C^*_{pk}=C^*_{pmk}=0$. The expected proportions nonconforming, however, are approximately 0.27%. Both indices C^*_{pk} and C^*_{pmk} severely understate process capability in this case. It is clear that if $D_u= D_l$, then the production tolerance becomes symmetric and the generalized indices defined in (2) reduce to those basic ones defined in (1).

Another generalization proposed for processes with asymmetric tolerances shifts both specification limits (see Fig. 3) to obtain one that is symmetric (Kushler and Hurley (1992), and Franklin and Wasserman (1992)). That is, the generalization replaces the true specification limits ($T-D_l$, $T+D_u$) with the new symmetric limits (unjustified sometimes) $T \pm (D_l+D_u)/2$, then applies the standard definitions of C_p, C_{pk}, C_{pm}, and C_{pmk}. With this generalization, the indices defined in (1) can be rewritten as the following:

$$C'_p(u, v) = \frac{d - u|\mu - T|}{3\sqrt{\sigma^2 + v(\mu - T)^2}} \quad (3)$$

This approach yields the generalized indices C'_p,

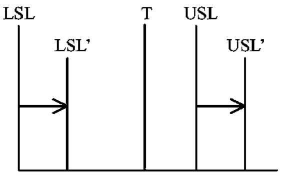

Fig. 3 $C'_p(u, v)$ shifts the given specification limits (LSL, USL) to the new specification limits (LSL, USL) which are symmetric to the target T

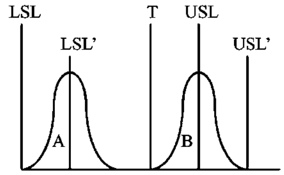

Fig. 4 Process A has C'_{pk}=0, and C'_{pmk}=0. Process B has C'_{pk}=1, and C'_{pm}=0.32. But, the expected proportions non-conforming are 0.135% for A, and 50% for B

C'_{pk}, C'_{pm}, and C'_{pmk}, which can either under-or overstate process capability, depending on the position of μ relative to T as noted by Boyles (1994). For example, consider the following two processes with $\mu_A=T-d$, $\mu_B=T+d/2=USL$, $\sigma=d/6$, and $T=\{3(USL)+(LSL)\}/4$ (see Fig. 4). Then, for process A we have $C'_{pk}=C'_{pmk}=0$, and for process B we have $C'_{pk}=1.0$ and $C'_{pmk}=0.32$. For both indices C'_{pk} and C'_{pmk}, the values given to process B are higher than those given to process A. But, the expected proportion nonconforming, of process B, is approximately 50%, which is significantly greater than that (approximately 0.135%) of process A. Obviously, both indices C'_{pk} and C'_{pmk} understate or overstate process capability in this case. We note that if $D_u=D_l$. Then the specification tolerance becomes symmetric and the generalized indices defined in (3) reduce to those basic ones defined in (1).

To overcome the problems with asymmetric tolerances, Boyles (1994) Defined a smooth function $S(x, y)=\Phi^{-1}\{\Phi(x)/2+\Phi(y)/2\}/3$, where $\Phi(x)$ is the cumulative function of the standard Normal distribution. Based on this smooth function, Boyles (1994) considered a new index S_{pk} generalized from C_{pk}. The index is defined as $S_{pk}=S((USL-\mu)/\sigma, (\mu-LSL)/\sigma$. We note that given $S_{pk}=c$, we can calculate the process yield as % yield=$\Phi((USL-\mu)/\sigma)-\Phi((LSL-\mu)/\sigma)=2\Phi(3c)-1$ for arbitrary values of c. Therefore, S_{pk} represents the actual process yield unlike C_{pk} which is only approximately related to process yield (Boyles (1994)). Extending this generalization to the index C_{pmk}, we obtain $S_{pmk}=S((USL-\mu)/\tau, (\mu-LSL)/\tau)$, where $\tau=[\sigma^2+(\mu-T)^2]^{-1/2}$ (see Boyles (1994)). A superstructure for this generalization may be written as the following:

$$S_p(v) = S(\frac{USL-\mu)}{\sqrt{\sigma^2+v(\mu-T)^2}}, \frac{\mu-LSL}{\sqrt{\sigma^2+v(\mu-T)^2}}) \quad (4)$$

Where $v \geq 0$. It is easy to verify that $S_p(0)=S_{pk}$, and $S_p(1)=S_{pmk}$.

In a recent paper, Johnson, Kotz and Pearn (1994) proposed a flexible Capability index called C_{jkp} to handle non-normal populations. Since the index C_{jkp} can be expressed as $(3\sqrt{2})^{-1}$ min $\{D_u/\tau_u, D_l/\tau_l\}$, where $(\tau_u)^2=\sigma^2\{(1-\Phi(\zeta))(1+\zeta^2)-\zeta\phi(\zeta)\}$, $(\tau_l)^2=\sigma^2\{\Phi(\zeta)(1+\zeta^2)+\zeta\phi(\zeta)\}$ with $\phi(\bullet)$ representing the density function of the standard Normal distribution and $\zeta=(\mu-T)/\sigma$, the corresponding Boyles' generalization for the C_{jkp} index then becomes $S_{jkp}=\Phi^{-1}\{\Phi(D_u/\sqrt{2}\,\tau_u)/2+\Phi(D_l/\sqrt{2}\,\tau_l)/2\}/3$. In addition to the above generalizations, Boyles (1994) also considered the following two indices: $C^{☆}_{m}=\{3(\lambda_1)^{1/2}\}^{-1}$, and $C^{+}_{pm}=\{3(\lambda_r)^{1/2}\}^{-1}$ for asymmetric tolerances, where $\lambda_1=(\tau_u/D_u)^2+(\tau_u/D_u)^2$, $\lambda_r=2\lambda_1/(1+\min\{(r)^2, (r)^{-2}\})$, and $r=D_l/D_u$. It is clear that if r=1 (or equivalently, $D_u=D_l$), then we have $\lambda_r=\lambda_1$, and both generalizations $C^{☆}_{m}$ and C^{+}_{pm} reduce to the basic index C_{pm}.

Boyles (1994) analyzed these six capability indices, C_{pmk}, S_{pmk}, C_{jpk}, S_{jpk}, $C^{☆}_{m}$, and C^{+}_{pm}, and provided a comparison in order to assess their accuracy in measuring process potential and performance. The comparison is based on several process characteristics including (a) process yield, and (b) process centering. Boyles (1994) pointed out that:

(A) C_{pmk} and S_{pmk} are superior to the other four indices in terms of process yield. S_{pmk} is closely reated to actual process yield, while C_{pmk} is only related to approximate process yield. Thus, C_{pmk} may be viewed as an approximation to S_{pmk};

(B) $C^{☆}_{m}$, C_{jpk} and S_{jpk}, provide no protection at all with respect to process yield, and therefore should not be considered further;

(C) S_{pmk} guarantees levels of process yield conventionally associated with given index levels, c, across all r valued, while C^{+}_{pm} provides such guarantees only for ranges of (r, c); C^{+}_{pm} places bounds on μ proportional to tolerance, while the bounds given to S_{pmk} are disproportionately sharp on the "long" side of the specification ;

Boyles (1994) concluded that S_{pmk} is well-calibrated with respect to process yield, and is most appropriate for general use.

We point out, however, that for fixed standard deviation σ, these six indices (including the most appropriate index S_{pmk}) obtain their maximal values

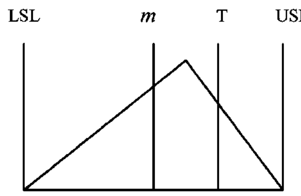

Fig. 5 For fixed σ, the existing indices obtain maximal values not at T, but at some μ^* that is between the target T and midpoint m

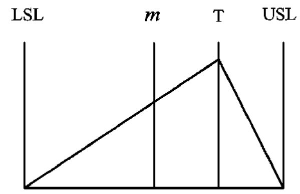

Fig. 6 For fixed σ, the new indices obtain maximal values at T, and give same index values to processes A, and B, satisfying $(\mu_A-T)/D_u=(T-\mu_B)/D_l$

not at $\mu=T$, but at some μ^* which is between the target value T and $M=(USL+LSL)/2$ (see Fig. 5). The value of μ^* relative to T and M reflects the compromise established by each of the six indices between process centering and process yield. Consequently, these six indices may conflict or show inconsistent results in terms of process yield, process centering, and other process characteristics, and thus, reflect process capability inaccurately. For example, consider the following case with asymmetric tolerance $(LSL, T, USL)=(26, 50, 58)$. Assume we have two processes A and B with $\mu_A=49$, $\mu_B=50$ respectively, and standard deviation $\sigma_A=\sigma_B=5.33$. It is easy to verify that the index values of C_{pmk}, S_{pmk}, C_{jpk}, S_{jpk}, $C_m^{☆}$, and C_{pm}^{+} for process A are higher than those for process B in this case. While process B is on-target, process A is off-target. None of the six indices discussed in Boyles (1994) reflect process capability accurately enough in this case.

III. NEW PCIS′ FOR ASYMMETRIC TOLERANCES

In this section, we consider a new class of generalized capability indices. The design of the new PCIs is based on the following criteria used in Chio and Owen (1990), Pearn, Kotz and Johnson (1992), and Boyles (1994) in analyzing and comparing the existing capability indices: (a) process yield, (b) process centering, and (c) a process characteristic related to loss functions. The new indices may be defined as:

$$C''_p(u,v)=\frac{d^*-uF^*}{3\sqrt{\sigma^2+vF^2}} \qquad (5)$$

Where $F=\max\{d(\mu-T)/D_u,\ d(\mu-T)/D_l\}$, $F^*=\max\{d^*(\mu-T)/D_u,\ d(\mu-T)/D_l\}$, and $u, v\geq 0$. This generalization yields the following new indices $C''_p(0,0)=C''_p$, $C''_p(1,0)=C''_{pk}$, $C''_p(0,1)=C''_{pm}$, and $C''_p(1,1)=C''_{pmk}$. We note that if $T=M$ (tolerance is symmetric), then $F=F^*=|\mu-T|$ and generalized indices $C''_p(u,v)$ reduce to the basic indices $C_p(u,v)$ defined in Eq. (1). Further, if $\mu=T$ (process is on target), then $C''_{pmk}=C''_{pm}=C''_{pk}=C''_p=d^*/3\sigma$. But, in general, the relationships among the four new indices C''_p, C''_{pk}, C''_{pm}, and C''_{pmk} can be established as the following:

$$C''_{pk}=C''_p(1-k),$$

$$C''_{pm}=C''_p(1+dK/\sigma]^2)^{1/2},$$

$$C''_{pmk}=C''_p(1-k)(1+[dK/\sigma]^2)^{1/2},$$

Where $K=\max\{(\mu-T)/D_u,\ d(T-\mu)/D_l\}=F^*/d^*$. Thus, in developing the new indices we have replaced $|\mu-T|$ with F^*, and $(\mu-T)^2$ with F^2 in (2). This ensures that the new indices $C''_p(u,v)$ obtain the maximal values at $\mu=T$ regardless of whether the tolerances are symmetric or asymmetric.

For processes with asymmetric tolerances, the corresponding loss function is also asymmetric to T. We take into account the asymmetry of the loss function by adding the factors d^*/D_u and $-d^*/D_u$ to $\mu-T$ according to whether μ is greater, or less, than T. The factors d^*/D_u and $-d^*/D_l$ ensures that if processes A and B with $\mu_A>T$ and $\mu_B<T$ satisfy $(\mu_A-T)/D_u=(T-\mu_B)/D_l$, then the index values given to A and B are the same (Fig. 6). It is easy to verify that if the process is on the specification limits ($\mu=LSL$, or $\mu=USL$), then $C''_{pk}=C''_{pmk}=0$. On the other hand, if $LSL<\mu<USL$, then we have $C''_p(u,v)>0$.

In Figs. 7(A), 7(B), 7(C), we plot contours of $C''_p(u,v)$ (dashed) and S_{pk} (solid) for the standard index values; 1/3, 2/3, 1, 4/3, 5/3, and, 2, with Fig. 7(A) for C''_{pk}, Fig. 7(B) for C''_{pm}, and Fig. 7(C) for C''_{pmk}. In all three cases, we have $C''_p(u,v)<S_{pk}$ for all values of μ. Thus, given a process with $C_p(u,v)=c$ which is $2\{1-\Phi(3c)\}$. Further, given $C''_p(u,v)>c$, we can calculate the bounds on $|\mu-T|$ as:

$$T-\frac{(1-R)D_l}{3c\sqrt{v}+u(1-R)}<\mu<T+\frac{(1-R)D_u}{3c\sqrt{v}+u(1-R)},$$

Where $R=|1-r|/(1+r)$, and $r=D_l/D_u$. Therefore, the bounds on $|\mu-T|$ corresponding to $C''_{pk}>c$ would be $T-D_l<\mu<T+D_u$ (equivalently, $LSL<\mu<USL$). The bounds on $|\mu-T|$ corresponding to $C''_{pm}>c$ would be $T-\{(1-R)/3c\}D_l<\mu<T+\{(1-R)/3c\}D_u$, and the bounds on $|\mu-T|$ corresponding to $C''_{pmk}>c$ would be

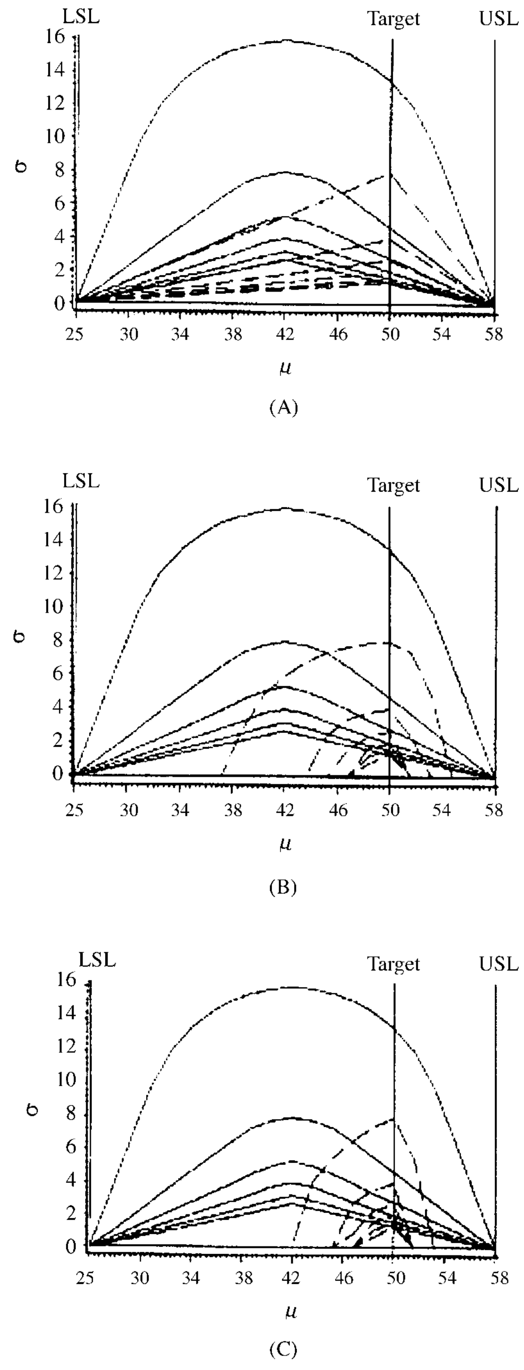

Fig. 7 (A) Contours of C''_{pk} (dashed) and S_{pk} (solid) for the standard values 1/3, 2/3, 1, 4/3, 5/3, and 2 (top to bottom in plot). (B) Contours of C''_{pm} (dashed) and S_{pk} (solid) for the standard values 1/3, 2/3, 1, 4/3, 5/3, and 2 (top to bottom in plot). (C) Contours of C''_{pmk} (dashed) and S_{pk} (solid) for the standard values 1/3, 2/3, 1, 4/3, 5/3, and 2 (top to bottom in plot)

$T-\{(1-R)/(3c+1-R)\}D_l<\mu<T+\{(1-R)/(3c+1-R)\}D_u$.

IV. COMPARISONS

In this section, we compare the new generalizations, $C''_p(u, v)$, with the existing generalizations

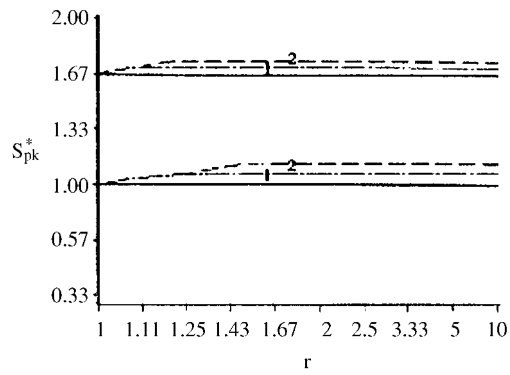

Fig. 8 Process yield in terms of S_{pk} for a continuum of processes with constant values c=1 and c=5/3 for the four indices shown. All the curves are symmetric about $r=D_l/D_u=1$ on a logarithmic scale (curves labeled '1' for C''_{pk}, C''_{pm}, C''_{pmk}, and curves labeled '2' for C_{pmk})

described in section 2. We note that Boyles (1994) has provided a comparison among the existing indices, and made the conclusion that S_{pmk} is the most appropriate index for general use. We will provide the same comparison which is based on the criteria used by Boyles (1994) including (1) process yield, (2) process centering, adding another criterion (3) a process characteristic (relationships to loss function) considered by Choi and Owen (1990). We first focus on the relationships to the yield-based index S_{pk}, and second on process centering (the ability to cluster around the target) in the form of bounds placed on $|\mu-T|$, and last on a process characteristic related to loss functions.

1. Process Yield

Inspection of the contour plots in section 3 of Boyles (1994) reveals that with asymmetric tolerances the existing indices are maximized (for fixed σ) not by μ=T, but by a value μ^* between T and M (see Figs. 4(b), 5(b), 6(b), 6(c), 7(b), and 8(b) in Boyles (1994)), as we indicated earlier. On the other hand, we observed that in Figs. 7(A), 7(B), 7(C), the proposed generalizations $C''_p(u, v)$ are maximal (for fixed σ) by $\mu^*=T$ which occurs when the contours of $C''_p(u, v)$ reach their maximal height at $\sigma=\sigma^*$. Assume the process corresponding to $\mu=\mu^*$ and $\sigma=\sigma^*$ has S_{pk} value denoted as S^*_{pk}, a function of r and c, where $r=D_l/D_u$. Since $C''_p(u, v)<S_{pk}$, (the contours of $C''_p(u, v)$ are undercovered by S_{pk} contours for the same level c), we conclude that if $C''_p(u, v)=c$, then the process yield must be no less than that corresponding to $S_{pk}=c$. It can be easily seen that the condition $C''_p(u, v)$ with $\mu^*=T$ implies $\sigma^*=(d-|M-T|)/3c$.

For the six indices, C_{pmk}, S_{pmk}, C_{jkp}, S_{jkp}, $C^{\star}_{pm}$, and C^+_{pm} discussed in Boyles (1994), a comparison based on process yield in terms of S_{pk} for a continuum

Table 1 Constants for bounds on $|m-T|$ implied by various indices

	k_l	k_u
C''_{pk}	1	1
C''_{pm}	$f(r)/3c$	$f(r)/3c$
C''_{pmk}	$f(r)/(3c+f(r))$	$f(r)/(3c+f(r))$
C_{pmk}	$\min\{1/(3c+1), 1/r(3c-1)\}$	$\min\{1/(3c-1), 1/r(3c+1)\}$
S_{pmk}	$\min\{1/(3c'+1), 1/r(3c'-1)\}$	$\min\{1/(3c'-1), 1/r(3c'+1)\}$
$C^{☆}_{pm}$	$1/(3c)$	$1/(3c)$
C^{+}_{pm}	$1/\{3c[A(r)]^{1/2}\}$	$1/\{3c[A(r)]^{1/2}\}$
C_{jkp}	$1/\{3c(2)^{1/2}\}$	$1/\{3c(2)^{1/2}\}$
S_{jkp}	$1/\{3c'(2)^{1/2}\}$	$1/\{3c'(2)^{1/2}\}$

Note: $f(r)=1-|1-r|/(1+r)>0$, where $r=D_l/D_u$, $A(r)=2/(1+\min\{(r)^2, (r)^{-2}\})$, and $c'=\Phi^{-1}\{2\Phi(3c)-1\}/3$

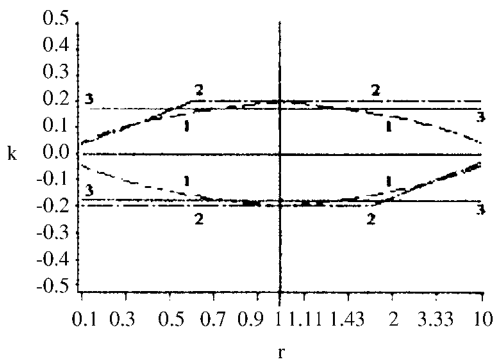

Fig. 9 Bounds placed on $|\mu-T|$ for c=4/3 by the three indices shown. Positive values represent fractions of D_u, negative values represent fractions of D_l, and 0 represents T (curves labeled '1' for C''_{pmk}, curves labeled '2' for C_{pmk}, and curves labeled '3' for C_{jpk}

of processes with constant values c=1 and c=5/3 is provided (see Fig. 9 in Boyles (1994)). Boyles (1994) noted that only C_{pmk} and S_{pmk} can assure that the process yield is at or above nominal index levels for all values of r. In Fig. 8, we plot S^*_{pk} curves for the new generalizations $C''_p(u, v)$ and C_{pmk} at index levels c=1 and c=5/3. It can be seen that the S^*_{pk} curves for $C''_p(u, v)$ are bounded by the straight line c and the S^*_{pk} curve for C_{pmk}. We note that for any level c the S^*_{pk} curve for S_{pmk} is also bounded by the straight line c and the S^*_{pk} curve for C_{pmk} although it is omitted from Fig. 8 (see also Fig. 9 in Boyles (1994)). Thus, $C''_p(u, v)$ (except for (u, v)=(0, 0)) guarantees process yield at or above nominal index levels for all values of r (like C_{pmk} and S_{pmk}).

2. Process Centering

Process centering is defined as the ability of the process to cluster around the target value T. In most cases, process centering can be measured by the departure of process mean μ from the target value T, $|\mu-T|$. If we impose the condition that the index value is no less than a given level c, then we can calculate the bounds on $|\mu-T|$ for the existing indices as well as the new generalizations $C''_p(u, v)$, which can be expressed in the form:

$$T-k_lD_l<\mu<T+k_uD_u$$

For unitless functions (k_l, k_u) of r and c. The (k_l, k_u) values for the six indices discussed by Boyles (1994) as well as the proposed new generalizations are displayed in Table 1, where $c'=\Phi^{-1}\{2\Phi(3c)-1\}/3<c$. From Table 1, we can see that:

(a) The bounds for S_{pmk} and S_{jkp} are very close to but slightly greater than those for C_{pmk} and C_{jkp} respectively. Thus, C_{pmk} and C_{ikp} are superior to S_{pmk} and S_{jkp} in terms of process centering.

(b) The bounds for C_{pmk} and C_{jkp} are tighter than those for C^*_{pm} and C^+_{pm}. Thus, C_{pmk} and C_{jkp} are superior to C^*_{pm} and C^+_{pm} in terms of process centering.

(c) Since f (r)/(3c+$f(r)$)<1, and $f(r)/(3c+f(r))<f(r)/3c$, the bounds for C''_{pmk} are tighter than those for C''_{pk} and C''_{pm}.

Therefore, in Fig. 9 we only plot $(-k_l, k_u)$ curves (as a function of r) for indices C''_{pmk}, C_{pmk} and C_{jkp} with c=4/3. In Fig. 9, we note that the bound for C''_{pmk} (curves labeled "1") is significantly tighter than that of C_{pmk} (curves labeled "2") for all values of r. The bound for C''_{pmk} is also tighter than that of C_{jkp} (curves labeled "3") except for $r\cong 1$. Therefore, C''_{pmk} is considered to be superior to C_{pmk} and C_{jkp} (and hence superior to S_{pmk}) in terms of process centering.

Table 2 A comparison among the new indices and existing ones for various of μ and fixed σ=8/3, (LSL, T, USL)=(26, 50, 58)

μ	C''_{pk}	C''_{pm}	C''_{pmk}	C_{pmk}	S_{pmk}	$C^{☆}_{pm}$	C^{+}_{pm}	C_{jkp}	S_{jkp}
26	0.000	0.164	0.000	0.000	0.178	0.331	0.247	0.234	0.391
27	0.042	0.171	0.007	0.014	0.188	0.346	0.258	0.244	0.399
28	0.083	0.179	0.015	0.030	0.198	0.361	0.269	0.255	0.407
29	0.125	0.187	0.023	0.047	0.210	0.378	0.282	0.267	0.417
30	0.167	0.196	0.033	0.066	0.233	0.396	0.296	0.280	0.427
31	0.208	0.206	0.043	0.087	0.237	0.417	0.311	0.295	0.439
32	0.250	0.217	0.054	0.110	0.253	0.440	0.328	0.311	0.452
33	0.292	0.229	0.067	0.136	0.272	0.465	0.347	0.329	0.466
34	0.333	0.243	0.081	0.164	0.292	0.493	0.368	0.349	0.483
35	0.375	0.258	0.097	0.197	0.316	0.525	0.391	0.371	0.501
36	0.417	0.275	0.114	0.234	0.343	0.261	0.418	0.397	0.523
37	0.458	0.294	0.135	0.276	0.375	0.603	0.449	0.426	0.548
38	0.500	0.316	0.158	0.325	0.412	0.651	0.485	0.460	0.577
39	0.542	0.342	0.185	0.383	0.455	0.707	0.527	0.500	0.611
40	0.583	0.371	0.217	0.451	0.506	0.773	0.576	0.547	0.652
41	0.625	0.406	0.254	0.533	0.565	0.853	0.635	0.603	0.702
42	0.667	0.447	0.298	0.632	0.632	0.950	0.708	0.672	0.764
43	0.708	0.496	0.351	0.667	0.706	1.072	0.799	0.758	0.843
44	0.750	0.555	0.416	0.711	0.778	1.231	0.917	0.872	0.948
45	0.792	0.625	0.495	0.765	0.845	1.443	1.075	1.029	1.096
46	0.833	0.707	0.589	0.832	0.911	1.719	1.282	1.259	1.316
47	0.875	0.800	0.700	0.914	0.987	2.003	1.493	1.612	1.657
48	0.917	0.894	0.820	1.000	1.086	2.052	1.530	1.947	2.010
49	0.958	0.970	0.930	1.053	1.119	1.739	1.296	1.376	1.482
50	1.000	1.000	1.000	1.000	1.068	1.340	1.000	1.000	1.068
51	0.875	0.800	0.700	0.819	0.899	1.033	0.770	0.755	0.840
52	0.750	0.555	0.416	0.600	0.699	0.817	0.609	0.590	0.691
53	0.625	0.406	0.254	0.415	0.538	0.664	0.495	0.476	0.590
54	0.500	0.316	0.158	0.277	0.425	0.553	0.412	0.394	0.520
55	0.375	0.258	0.097	0.176	0.347	0.469	0.350	0.333	0.470
56	0.250	0.217	0.054	0.102	0.292	0.406	0.302	0.287	0.433
57	0.125	0.187	0.023	0.044	0.254	0.356	0.265	0.252	0.404
58	0.000	0.164	0.000	0.000	0.225	0.316	0.236	0.224	0.383

3. A process Characteristic Related to Loss Functions

In the following, we compare the new generalizations $C''_p(u, v)$ with the six indices discussed in Boyles (1994) based on a process characteristic discussed in Choi and Owen (1990), which is related to loss functions. As we discussed earlier, the new indices $C''_p(u, v)$ obtain the maximal values when the process is on-target ($\mu^*=T$). On the other hand, the six indices C_{pmk}, S_{pmk}, C_{jkp}, S_{jkp}, $C^{☆}_{pm}$, and C^{+}_{pm} obtain the maximal values when the process is off- target ($M<\mu^*<T$). To illustrate this point, we consider the following example with specifications (LSL, T, USL) =(26, 50, 58). Since $D_u=USL-T=8$, and $D_l=T-LST=24$, the process has an asymmetric tolerance.

Table 2 displays the values of the six indices discussed in Boyles (1994) as well as the proposed new indices C''_{pk}, C''_{pm}, C''_{pmk} for various values of μ, with fixed standard deviation σ=8/3. We note that in Table 2, C_{pmk}, S_{pmk} are maximized by μ^*=49, and the other four indices C_{pmk}, C^{+}_{pm}, C_{jkp}, and S_{jkp} are maximized by μ^*=48. In all cases, we have $M<\mu^*<T$. On the other hand, the new generalizations $C''_p(u, v)$ are maximized by $\mu^*=50=T$, and the index values are 1.00 for all three new indices C''_{pk}, C''_{pm}, and C''_{pmk}.

Further, the new indices have taken into account the asymmetry of the loss function. Thus, given two processes A and B with $\mu_A>T$ and $\mu_B<T$ satisfying $(\mu_A-T)/D_u=(T-\mu_B)/D_l$, the (new) index values given to A and B are the same. Table 3 is a summary of processes (taken from Table 2) satisfying $(\mu_A-T)/D_u=$

Table 3 The corresponding index values for processes satisfying $(\mu_A-T)/D_\mu=(T-\mu_B)/D_l$

μ	C''_{pk}	C''_{pm}	C''_{pmk}	C_{pmk}	S_{pmk}	$C^{☆}_{pm}$	C^{+}_{pm}	C_{jkp}	S_{jkp}
47	0.875	0.800	0.700	0.914	0.987	2.003	1.493	1.612	1.657
51	0.875	0.800	0.700	0.819	0.899	1.033	0.770	0.755	0.840
44	0.750	0.555	0.416	0.711	0.778	1.231	0.917	0.872	0.948
52	0.750	0.555	0.416	0.600	0.699	0.817	0.609	0.590	0.691
41	0.625	0.406	254	0.533	0.565	0.853	0.635	0.603	0.702
53	0.625	0.406	0.254	0.415	0.538	0.664	0.495	0.476	0.590
38	0.500	0.316	0.158	0.325	0.412	0.651	0.485	0.460	0.577
54	0.500	0.316	0.158	0.277	0.425	0.553	0.412	0.394	0.520
35	0.375	0.258	0.097	0.197	0.316	0.525	0.391	0.371	0.501
55	0.375	0.258	0.097	0.176	0.347	0.469	0.350	0.333	0.470
32	0.250	0.217	0.054	0.110	0.253	0.440	0.328	0.311	0.452
56	0.250	0.217	0.054	0.102	0.292	0.406	0.302	0.287	0.433
29	0.125	0.187	0.023	0.047	0.210	0.378	0.282	0.267	0.417
57	0.125	0.187	0.023	0.044	0.254	0.356	0.265	0.252	0.404
26	0.000	0.164	0.000	0.000	0.178	0.331	0.247	0.234	0.391
58	0.000	0.164	0.000	0.000	0.225	0.316	0.236	0.224	0.383

$(T-\mu_B)/D_l$. For example, consider processes A and B with $\mu_A=51>T$, and $\mu_B=47<T$. Clearly, we have $(\mu_A-T)/D_u=1/8$, and $(T-\mu_B)/D_l=3/24=1/8$. Thus, quality loss for processes A and B are the same. Checking Table 3 for the index values corresponding to $\mu_A=51$ and $\mu_B=47$, we have $C''_{pk}=0.875$, $C''_{pm}=0.800$, and $C''_{pmk}=0.700$ for both processes A and B. On the other hand, the values the other six indices give to process B are considerably higher than those given to process A. In particular, for indices $C^{☆}_{pm}$, C^{+}_{pm}, C_{jkp}, and S_{jkp} the values given to process B are roughly twice those given to process A.

V. ESTIMATION OF $C''_p(u, v)$

To estimate the new indices $C''_p(u, v)$, Pearn and Chen (1995) considered the natural estimators which can be defined as the following:

$$C''_p(u,v)=\frac{d^*-u\hat{F}^*}{3\sqrt{S^2+v\hat{F}^2}},$$

Where $\hat{F}^*=\max\{d^*(\overline{X}-T)/D_u,\ d^*(T-\overline{X})/D_l\}$ and $\hat{F}=\max\{d(\overline{X}-T)/D_u,\ d(T-\overline{X})/D_l\}$ with $\overline{X}=(\sum_{i=1}^{n}X_i)/n$, and $S=\{(n-1)^{-1}\sum_{i=1}^{n}(X_i-\overline{X})^2\}^{1/2}$, the conventional estimators of μ and σ which may be obtained from a process that is demonstrably stable (in-control).

As an example, we consider the following process with asymmetric specification tolerances $USL=16$, $T=13.5$, and $LSL=10$. Suppose the sample mean $\overline{X}=14$, and the sample standard deviation $S=1$. Then, we can calculate $d=(USL-LSL)/2=3$, $d^*=\min\{D_u, D_l\}=\min\{2.5, 3.5\}=2.5$, $\hat{F}=\max\{d(\overline{X}-T)/D_u, d(T-\overline{X})/D_l\}=0.6$, and $\hat{F}^*=\max\{d^*(\overline{X}-T)/D_u, d^*(T-\overline{X})/D_l\}=0.5$. Thus, we may obtain $\hat{C}''_p(u, v)=(2.5-0.5u)\{3(1+0.36v)^{1/2}\}^{-1}$. By setting $(u, v)=(0, 0)$, $(1, 0)$, $(0, 1)$, $(1, 1)$, we obtain $\hat{C}''_p=0.83$, $\hat{C}''_{pk}=0.67$, $\hat{C}''_{pm}=0.71$, and $\hat{C}''_{pmk}=0.57$.

Pearn and Chen (1995) investigated the statistical properties of the estimators $\hat{C}''_p(u, v)$ and obtained the exact distributions of $\hat{C}''_p(u, v)$ although the derivations were cumbersome. Pearn and Chen (1995) also derived the formulas for the exact r-th moment (about zero) of the estimators $\hat{C}''_p(u, v)$. Expressions of the r-th moment, the expected value, and the variance formulas as well as other inferential properties are as complicated as those which appeared in Vännman and Kotz (1995). Further, Pearn and Chen (1995) showed that in special cases where the specification tolerances are symmetric ($D_u=D_l$), their results are identical to (reduce to) those obtained by Vännman and Kotz (1995).

VI. CONCLUSIONS

In this paper, we first reviewed the existing generalizations of the basic capability indices $C_p(u, v)$ including $C^*_p(u, v)$, $C'_p(u, v)$, $S_p(v)$ and many others, which have been proposed to handle processes with asymmetric tolerances. Then, we introduced a new class of generalizations which we referred to as $C''_p(u, v)$. The new generalizations $C''_p(u, v)$ are developed from the basic indices $C_p(u, v)$ by taking into account the asymmetry of the specification tolerance (loss function).

The proposed new generalizations are compared with existing ones in terms of process yield, process centering (the ability to cluster around the target), and a process characteristic related to loss functions. The results indicate that: (1) the new generalizations, particularly, C''_{pk}, C''_{pm}, and C''_{pmk} guarantee process yield at or above nominal index levels for all given index values (like S_{pmk}, the index recommended by Boyles (1994)), (2) C''_{pmk} is superior to S_{pmk} in terms of process centering, and (3) C''_{pk}, C''_{pm}, and C''_{pmk} obtain the maximal values (for fixed σ) at $\mu^*=T$ (on-target), while the others (including S_{pmk}) obtain the maximal values at some μ^* with $M<\mu^*<T$ (off-target). In practical application, process engineers can set their machine parameter as target value when $C_p(u, v)$ is applied to evaluate process capability. Large $C_p(u, v)$ insures high process yield and small $C_p(u, v)$ indicates the chance of process improvement. Thus, the proposed new generalizations are superior to existing ones, which provide a greater accuracy in current practice of using PCIs to monitor process potential and performance.

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NOMENCLATURE

μ	the process mean
σ	the process standard deviation
USL	upper specifications limit
LSL	lower specifications limit
d	$(USL-LSL)/2$
M	$(USL+LSL)/2$
T	the target value
D_u	$T-LSL$
D_l	$USL-T$
d^*	$\min\{D_l, D_u\}$
$\Phi(x)$	the cumulative function of the standard Normal distribution
S_{pk}	$S((USL-\mu)/\sigma, (\mu-LSL)/\sigma$
S_{pmk}	$S((USL-\mu)/\tau, (\mu-LSL)/\tau)$
τ	$[\sigma^2+(\mu-T)^2]^{-1/2}$
$(\tau_u)^2$	$\sigma^2\{(1-\Phi(\zeta))(1+\zeta^2)-\zeta\phi(\zeta)\}$, $(\tau_l)^2=\sigma^2\{\Phi(\zeta)(1+\zeta^2)+\zeta\phi(\zeta)\}$
$\phi(\bullet)$	representing the density function of the standard Normal distribution
$(\tau_l)^2$	$\sigma^2\{\Phi(\zeta)(1+\zeta^2)+\zeta\phi(\zeta)\}$
ζ	$(\mu-T)/\sigma$
S_{jkp}	$\Phi^{-1}\{\Phi(D_u/\sqrt{2}\,\tau_u)/2+\Phi(D_l/\sqrt{2}\,\tau_l)/2\}/3$
$C^{☆}_{pm}$	$\{3(\lambda_1)^{1/2}\}^{-1}$
C^{+}_{pm}	$\{3(\lambda_r)^{1/2}\}^{-1}$
λ_1	$(\tau_u/D_u)^2+(\tau_l/D_l)^2$
λ_r	$2\lambda_1/(1+\min\{(r)^2, (r)^{-2}\})$
r	D_l/D_u
F	$\max\{d(\mu-T)/D_u, d(T-\mu)/D_l\}$
F^*	$\max\{d^*(\mu-T)/D_u, d^*(\mu-T)/D_l\}$
K	$\max\{(\mu-T)/D_u, dT(-\mu)/D_l\}=F^*/d^*$
R	$\lvert 1-r\rvert/(1+r)$
σ^*	$(d-\lvert M-T\rvert)/3c$
c'	$\Phi^{-1}\{2\Phi(3c)-1\}/3$
$\hat{F}^*$	$\max\{d^*(\overline{X}-T)/D_u, d^*(T-\overline{X})/D_l\}$
$\hat{F}$	$\max\{d(\overline{X}-T)/D_u, d(T-\overline{X})/D_l\}$
$\overline{X}$	$(\sum_{i=1}^{n} X_i)/n$
S	$\{(n-1)^{-1}\sum_{i=1}^{n}(X_i-\overline{X})^2\}^{1/2}$

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非對稱規格區間的製程能力指標

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摘　要

製程能力指標的研究與運用著重於產品的規格是對稱的情況，而當產品的規格是非對稱的情況時常常被忽略。Boyles（1994）回顧一些指標並提出一些處理非對稱規格區間的製程能力指標，在本文中我們研究並比較這些處理非對稱規格區間的製程能力指標，同時提出一個更適合處理非對稱規格區間的製程能力新指標。本文以製程良率、製程偏離目標值的量（製程集中量）與製程期望損失爲基準，比較新指標與現有指標結果顯示新指標優於現有指標。

關鍵詞：製程能力指標，製程良率，製程集中量，目標值。