Asymptotic Rejection of Periodic Disturbances With Fixed or Varying Period

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A constructive derivation of repetitive control is obtained, through attempting to derive a control law for asymptotic rejection of periodic disturbances. This derivation not only reveals a close relationship between iterative operator inversion and repetitive control, but also suggests a unified design method for a learning control algorithm. Also, based on the observation, digital repetitive control can be generalized to reject periodic disturbance whose period is not exactly an integer multiple of the sampling interval. This study introduces a delay filter in the digital repetitive control law, which optimally interpolates the signal between samples, thus effectively reconstructing the signal of the previous period and making the learning process of repetitive control successful. The proposed optimal delay filter can be updated easily according to different signal periods. Thus it is specifically suitable for on-line tuning when the signal period is changing. Compared with the available tuning methods, the proposed tuning method has excellent steady-state performance while maintaining fast transient and system robustness. The simulations on active noise cancellation within a duct confirm the superiority of this tuning method.
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1 Introduction

Repetitive control is effective in asymptotic tracking or rejection of periodic signals [1,2]. It consists of a wide variety of applications, such as noncircular cutting [3], disk drive tracking [4], and active noise cancellation [5]. Repetitive control was explained and proved by the internal model principle ([1,2]). This work, however, proposes a constructive derivation based on the operator theory. First, the disturbance rejection problem is formulated as a question for solving an operator equation. The derivation begins with the iterative inversion of an operator by the Neumann series, which results in a Neumann series solution as well as a sufficient existence condition for an operator inversion problem. Additionally, this solution can be alternatively represented by successive iteration of an equation, which provides a good insight into deriving the learning algorithm for an operator equation. Consequently, a slight modification of this iteration equation leads to the well-known repetitive control law. Finally, in a mathematically rigorous manner, the fixed-point theorem is used to prove the performance and stability of repetitive control.

In the derivation of repetitive control, two operators are involved. One is an operator that is a rough inversion of the plant. This operator is closely related to the stability and transient behavior of repetitive control—the better it well inverses the plant, the more stable the overall system is and more quickly the system will reach steady state. The other is a delay operator, of which the delay time is equal to the signal period in order to achieve asymptotic signal tracking or disturbance rejection. Based on these observations, the repetitive control system with adjustable delay operator can reasonably track or reject signals with varying period. This idea naturally leads to an alternative repetitive control algorithm for asymptotic tacking or rejecting periodic signals with varying period.

Repetitive control repeatedly generates the present control force $u(t)$ by learning from the previous period of the control force $u(t-T)$ and the tracking or disturbance rejection error $\varepsilon(t-T)$. However, in a discrete-time system, where only the sampled signals are given, $u(t-T)$ and $\varepsilon(t-T)$ are unavailable unless that the signal period T is exactly an integer multiple of the sampling interval. When the signal period is precisely known and fixed, integer multiple condition can be easily achieved. However, it becomes a difficult task when the signal period varies. There are two types of methods for solving this problem. The first method ([6,7]) uses $u(t-NT_s)$ and $\varepsilon(t-NT_s)$ to approximate $u(t-T)$ and $\varepsilon(t-T)$, respectively, where NT_s is the nearest integer multiple of sampling interval to the signal period T. The second method ([7,8]) alters the sampling rate on-line while maintaining a fixed controller. The disadvantage of the first tuning scheme is the inevitable period mismatch due to the roundoff of the actual signal period. This mismatch may result in undesirable remaining oscillating errors, thus deteriorating the steady-state performance. As for the second tuning scheme, the most serious problem is that changing the sampling rate without changing the controller can affect system robustness, and even cause instability.

Based on the idea of the adjustable delay operator, this study attempts to provide an alternative method for a discrete-time repetitive controller. With the fixed sampling interval, a delay filter, which aims to optimally interpolate $u(t-T)$ and $\varepsilon(t-T)$ between samples, is introduced to enhance the steady-state performance. According to distinct signal periods, this optimal filter can be updated via only a small amount of computations. Thus it can be used as an on-line tuning scheme for the changeable signal period. This delay filter tuning method is applied to active noise cancellation within a duct. Simulations illustrate the effective enhancement of the steady-state performance.

2 A Pathway to Repetitive Control

This section provides a constructive derivation of repetitive control. The derivation begins with the formulation of a disturbance rejection problem, as shown in Fig. 1. The plant P is assumed to be linear time invariant (LTI) and stable. All signals are functions defined on $[0,\infty)$. The objective is to find control signal u, such that the periodic disturbance d can be asymptotically cancelled, i.e., the error signal $\varepsilon\rightarrow\infty$ as $t\rightarrow\infty$.

In a linear system, the candidate of control signal for asymptotically rejecting a periodic disturbance, must be some linear combination of a transient signal, which vanishes as time tends to infinity, and a periodic signal whose period is the same as that of

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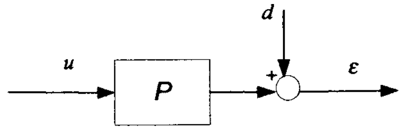

Fig. 1 A classical disturbance rejection problem

the disturbance. To analyze and synthesize such control signals, an appropriate linear vector space that includes both transient and periodic signals must be formed. Notably, a collection of periodic signals could not make a linear vector space because linear combinations of two or more periodic signals of arbitrary periods will generally not be periodic, for example, $f(t)=\sin(2\pi t)+\sin(2\sqrt{2}\pi t)$ is not periodic since there exists no $T\in\mathbf{R}$ such that $f(t+T)=f(t)$. Such signals with frequencies of the components not related by rational numbers have some ''almost-periodic'' characters. Let $\mathbf{H}$ be a linear vector space of complex-valued functions $x(t), 0\leqslant t<\infty$, spanned by $\{e^{j\omega t}, \omega\in\mathbf{R}\}$, with the norm

$$\|x\|=\left(\lim_{T\to\infty}\frac{1}{T}\int_0^T|x(t)|^2dt\right)^{1/2}<\infty \tag{1}$$

and the inner product

$$\langle x,y\rangle=\lim_{T\to\infty}\frac{1}{T}\int_0^T x(t)\overline{y(t)}dt, \quad \forall x,y\in\mathbf{H} \tag{2}$$

where $\overline{y(t)}$ represents the complex conjugate of $y(t)$. Notably, $\mathbf{H}$ is a Hilbert space of almost periodic functions ([9], sections 13 and 57); it contains either finite linear combinations $\Sigma_{n=1}^{N}c_n e^{j\omega_n t}$, or their limits in the norm defined in (1). The space $\mathbf{H}$ is not separable since its orthonormal basis $\{e^{j\omega t}, \omega\in\mathbf{R}\}$ is not countable. The norm defined in (1), known as the root-mean-square (RMS) norm in many areas of engineering, is a steady-state measure, which is unaffected by the transient of a signal. Accordingly, the transient signals can be easily defined as signals with zero RMS norm; let $\mathbf{M}$ be a collection of all such signals:

$$\mathbf{M}=\left\{x: \lim_{T\to\infty}\frac{1}{T}\int_0^T|x(t)|^2dt=0\right\} \tag{3}$$

$\mathbf{M}$ is a closed set. Thus, the union of $\mathbf{H}$ and $\mathbf{M}$, denoted by $\mathbf{H}\cup\mathbf{M}$ makes a new complete linear vector space, which includes both transient and almost periodic functions. In this new space, however, the RMS norm becomes only a seminorm. That is, $\|x\|=0$ does not imply $x=\theta$, rather in this case it only implies $x\to 0$ as $t\to\infty$. Consequently, two signals, x and y in $\mathbf{H}\cup\mathbf{M}$, that differ only by a transient (i.e., $\|x-y\|=0$) are equivalent in the RMS sense. Suppose that two functions $x,y\in\mathbf{H}\cup\mathbf{M}$ are said to belong to the same class if the difference $x-y$ belongs to $\mathbf{M}$, then the set of all such classes is called the quotient space of $\mathbf{H}\cup\mathbf{M}$ relative to $\mathbf{M}$, represented by $\mathbf{E}$. This quotient space $\mathbf{E}$ is a Hilbert space with the inner product defined in the following way: Given two elements of $\mathbf{E}$, i.e., two classes ξ and η, we choose a representative from each class, say x from ξ and y from η, then

$$\langle\xi,\eta\rangle=\lim_{T\to\infty}\frac{1}{T}\int_0^T x(t)\overline{y(t)}dt \tag{4}$$

In a completely analogous manner, the norm can be defined as follows

$$\|\xi\|=\left(\lim_{T\to\infty}\frac{1}{T}\int_0^T|x(t)|^2dt\right)^{1/2} \tag{5}$$

where x is any representative from the class ξ in $\mathbf{E}$. Let e_ω be the class containing the function $e^{j\omega t}$ in $\mathbf{E}$, then the set $\{e_\omega, \omega\in\mathbf{R}\}$ forms an orthonormal basis in $\mathbf{E}$.

Similarly, linear transformations on $\mathbf{E}$ can also form a normed space. Let $\mathbf{L}(\mathbf{E})$ be a space of all LTI stable systems P on $\mathbf{E}$, with the operator norm

$$\|P\|=\sup_{\|\xi\|=1,\xi\in\mathbf{E}}\|P\xi\| \tag{6}$$

The frequency-response function of an operator $P\in\mathbf{L}(\mathbf{E})$ is defined by

$$P(\omega)=\langle Pe_\omega, e_\omega\rangle, \quad \omega\in\mathbf{R} \tag{7}$$

The operator norm defined in (6) turns out to be the least upper bound of the associated frequency-response function $P(\omega)$ ([10], pp. 19–25).

$$\|P\|=\sup_\omega|P(\omega)| \tag{8}$$

The normed space $\mathbf{L}(\mathbf{E})$ is a Banach space due to the completeness of $\mathbf{E}$.

Accordingly, in this RMS norm setting, the problem of asymptotic rejection of periodic disturbances can be restated: Given $P\in\mathbf{L}(\mathbf{E})$ and a periodic disturbance d with finite RMS norm, find a class of control signals $\xi\in\mathbf{E}$ such that the response class $P\xi$ contains the signal $-d$. Our initial concern is to determine if a solution exists and also find a solution if it exists.

Proposition 1. (Solution to Asymptotic Rejection of Periodic Disturbances) For an operator $P\in\mathbf{L}(\mathbf{E})$, if another operator $C\in\mathbf{L}(\mathbf{E})$ exists, such that $\|I-CP\|<1$, asymptotic rejection of periodic disturbances can be achieved for any periodic d with finite RMS norm, and the corresponding control signal class can be expressed in the following

$$\xi=\left\{u: \ u=-\sum_{n=0}^{\infty}(I-CP)^n Cd+x, \quad \forall x\in\mathbf{M}\right\}\in\mathbf{E} \tag{9}$$

Proof: Assume that the periodic disturbance d belongs to a class η in $\mathbf{E}$. Since $\mathbf{L}(\mathbf{E})$ is a Banach space, if an operator $C\in\mathbf{L}(\mathbf{E})$ exists, such that $\|I-CP\|<1$, the solution to the following operator equation

$$CP\xi=-C\eta \tag{10}$$

is unique and can be obtained by inverting the operator CP using the Neumann series ([11], p. 70).

$$\xi=-(CP)^{-1}C\eta=-\sum_{n=0}^{\infty}(I-CP)^n C\eta \tag{11}$$

Rewriting (10) gives

$$C(P\xi+\eta)=0 \tag{12}$$

The frequency-response function $C(\omega)$ of C cannot vanish on any frequency ω, since $\|I-CP\|<1$. This implies that the operator C cannot annihilate any almost periodic signals, i.e., it has a trivial nullspace. Consequently, (12) implies $\|P\xi+\eta\|=0$. The signal $-d$ belongs to the class $-\eta$, thus also belonging to $P\xi$. That is, the class ξ causes asymptotic disturbance rejection.

Proposition 1 provides a sufficient condition for the existence of a solution in $\mathbf{E}$. Clearly, if the control signal for achieving asymptotic disturbance rejection exists, then it is not unique. Among all solutions, solution $u=-\Sigma_{n=0}^{\infty}(I-CP)^n Cd$ has a very interesting alternative expression. Solution u can be obtained by the following successive iteration

$$u_{n+1}=-Cd+(I-CP)u_n \tag{13}$$

or equivalently

$$u_{n+1}=u_n-C\varepsilon_n \tag{14}$$

where ε_n denotes the error vector Pu_n+d. That the sequence $\{u_n\}$ converges to u from any initial vector u_0 can be easily verified. The convergence rate is estimated as follows:

$$\|\varepsilon_n\| \le \|I-CP\|^n \|\varepsilon_0\| \tag{15}$$

In view of the iteration (14), it is just like a ''learning'' mechanism, correcting the guess via the error at each iteration. Roughly speaking, the more closely *CP* approximates an identity operator, the quicker $\{u_n\}$ could ''learn'' from the errors. Now, assume that the disturbance signal is periodic with the period *T*. In an analogous manner, the present control force $u(t)$ can be generated by learning from the previous period of signals $u(t-T)$ and $\varepsilon(t-T)$ as follows:

$$u(t)=u(t-T)-[C\varepsilon](t-T) \tag{16}$$

This is known as the repetitive control law, which automatically generates *u* belonging to the solution class presented in Proposition 1. This solution-generating algorithm can be proved in a straightforward manner by applying the fixed-point theorem.

Proposition 2. (Repetitive Control Law) For an operator $P \in \mathbf{L}(\mathbf{E})$, if another operator $C \in \mathbf{L}(\mathbf{E})$ exists, such that $\|I-CP\| < 1$, asymptotic rejection of any periodic disturbance *d* with finite RMS norm, can be achieved by the control signal *u* generated by the following control law

$$u(t)=[Du](t)-[CD\varepsilon](t), \quad \text{for } t\in[0,\infty) \tag{17}$$

where *D* denotes an delay operator, which performs a *T*-second delay with zero initial states, i.e., $(Du)(t)=u(t-T)$, and $u(t-T)=0$ for $t-T<0$, and ε denotes the error vector $Pu+d$. Additionally, *u* is bounded in the RMS sense.

Proof: The proof consists of three parts. First, let the mapping $F:\mathbf{E}\to\mathbf{E}$ be defined by $F(\psi)=D\psi-CD(P\psi+d)$. Additionally, the mapping *F* is a contraction since

$$\|F(\psi)-F(\zeta)\| \le \|D(I-CP)\|\|\psi-\zeta\| < \|\psi-\zeta\| \quad \forall \psi,\zeta\in\mathbf{E} \tag{18}$$

Second, we show that the function *u* generated by (17) belongs to some class in **E**, thus having a finite RMS value. Rewrite (17) in the following

$$(I-D+CDP)u=-CDd \Rightarrow u=-(I-D+CDP)^{-1}CDd$$

where $(I-D+CDP)^{-1}\in\mathbf{L}(\mathbf{E})$ can be proved directly using Neumann series, since $I-D+CDP$ is an identity plus a small operator with the norm $\|D(I-CP)\|<1$. Clearly, the disturbance is an element from some class in **E**, since it is periodic with finite RMS value. Therefore, *u* also belongs to some class in **E**. Assume the class contains *u* is ξ'. It follows from (17) that ξ' is a fixed point of *F*.

Third, according to Proposition 1, the condition $\|I-CP\|<1$ guarantees the existence of a class of control signals $\xi\in\mathbf{E}$ for achieving asymptotic disturbance rejection, i.e., $P\xi=-\eta$, where η denotes the class containing the disturbance *d*. Also, the periodicity of *d* yields the relation $\eta=D\eta$, which in turn implies $\xi=D\xi$. Consequently, we obtain $F(\xi)=D\xi-CD(P\xi+\eta)=\xi$, i.e., ξ is also a fixed point of *F*.

Finally, since the contraction mapping *F* could only have one fixed point in **E** ([11], p. 256), we have $\xi'=\xi$. The signal *u* thus belongs to the equivalence class of the control signals that cause asymptotic disturbance rejection.

■

The proof of Proposition 2 provides a promising method for designing a learning control law or an adaptive algorithm if the desired solution can be represented in an operator equation like $Pu=d$, which is assumed to have a solution *u*. If we can find a contraction mapping *F* of which the solution *x* is a fixed point, then the control law $\hat{u}=F(\hat{u})$ can generate $\hat{u}$ that approaches to *u* in steady state. In practice, the contraction mapping *F* must be causal so that the associated control law can be physically realizable. In the repetitive control law, *F* is causal if the operator *D* and *CD* are designed to be causal; thereby, the control signal can be generated in real-time by processing current or past values of the control signal and cancellation error.

Compared with direct operator inversion, repetitive control law (17) has the advantage of precise knowledge of the plant is not necessary to obtain asymptotic disturbance rejection. Instead, only a rough inversion *C* and the cancellation error ε are required. Practically speaking, this is a very appealing feature, since a mathematical model can hardly represent a physical system precisely, and the disturbance signal is generally not measurable in many applications.

Inspection of the repetitive control law (17) shows that two operators *D* and *C* are involved. Operator *D* is a delay operator, and *C* is a rough inversion of the plant *P*. These two operators play vital roles in the steady state and transient behavior of repetitive control. In the following section, the design of digital repetitive controller is considered, and the emphasis is placed on the design of the delay filter *D*.

3 Digital Repetitive Controller With Lowpass Delay Filter

Consider a discrete-time repetitive control system, as illustrated in Fig. 2, the plant $P(z)$ is stable and the disturbance *d* is periodic with the period *T*, which may not be an integer multiple of the sampling interval. Rather than using the integer-delay operator as z^{-N} [2], a general filter $D(z)$ is used. The transfer function from the disturbance *d* to the residual error ε is

$$\frac{\varepsilon(z)}{d(z)}=\frac{1-D(z)}{1-D(z)(1-P(z)C(z))} \tag{19}$$

Assume that both $C(z)$ and $D(z)$ are stable, then, according to the small gain theorem, the sufficient condition for closed-loop stability is

$$\|D(z)(1-P(z)C(z))\|_\infty < 1 \tag{20}$$

This sufficient condition is reduced to the condition given in proposition 2 when the transfer function *D* is an ideal delay filter. Thus, if *D* is an ideal delay filter and *C* is well designed such that the overall system is stable, then perfect asymptotic disturbance rejection can be obtained. Therefore, the design of a repetitive controller can be separated into two model-matching problems. The first is to design a causal stable filter $D(z)$, which closely approximates a delay operator with the delay time *T*. The second is to design a stable compensator $C(z)$, such that $P(z)C(z)$ is approximately an identity operator. Note that the multiplication $C(z)D(z)$ must be causal. The design of $C(z)$ has been extensively studied, for example, Tomizuka et al. [2]; Hu et al. [12]. This study however, focuses on the design of $D(z)$.

The filter $D(z)$ is designed to approximate an ideal delay operator. In theory, an ideal delay filter is ([13], eq. (2))

$$D(z)=\sum_{n\in Z}\mathrm{sinc}(n-\eta)z^{-n} \tag{21}$$

where $\eta=T/T_s$, *T* is the desired delay time and T_s is the sampling period. The sinc function is defined as $\mathrm{sinc}(\omega)=\sin(\pi\omega)/(\pi\omega)$. When the desired delay is exactly an integer multiple *N* of the sampling time, the ideal delay filter reduces to z^{-N}. If $\eta=T/T_s$ is not an integer, the ideal delay filter of (21) is

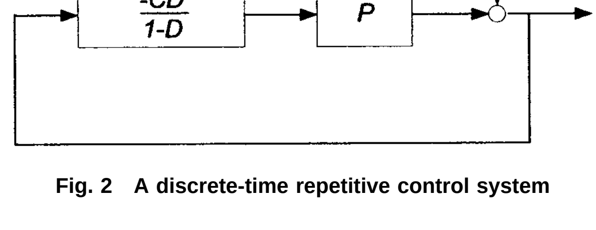

Fig. 2 A discrete-time repetitive control system

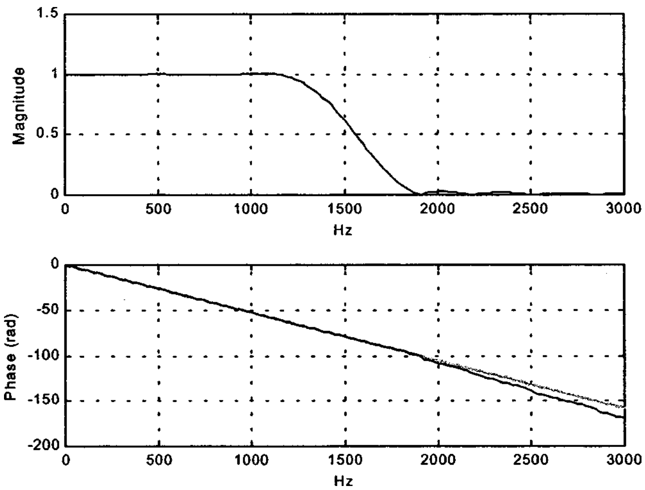

Fig. 3 Frequency response of the lowpass delay filter *D* (the gray line in the phase diagram is the desired phases)

noncausal with infinite taps. Thus, in practice, a causal and finite-length filter that well approximates an ideal delay filter must be designed. To simplify the problem, $D(z)$ is restricted to be an FIR filter. Also, to optimally realize a long delay with a short-length FIR filter $D(z)$, the desired delay is split into an integer multiple part and a fractional part of the sampling interval [14,15]. That is, if the desired delay is $T=(N+\tau)T_s$, where T_s is the sampling interval, N is an integer, and $0\leqslant\tau<1$, then $D(z)$ can be represented as $z^{-N}Q(z)$, where $Q(z)$ well approximates an ideal delay filter with delay τ. Moreover, as implied in (20), the magnitude of $D(e^{j\omega})$ can be made small in the high-frequency band, such that high-frequency model uncertainty can be tolerated. In this way, the system ''gives up'' the rejection of the high-frequency disturbance. Therefore, in frequency domain, the filter $Q(e^{-j\omega})$ should closely approximate an ideal delay response $e^{-j\omega\tau}$ within the control bandwidth, while the magnitude of $Q(e^{j\omega})$ is as small as possible over the high-frequency band for robustness concern.

Assume that the normalized control bandwidth is $[0,\omega_0\pi]$ and the high-frequency band is set as $[\omega_1\pi,\pi]$, where $0<\omega_0<\omega_1<1$. Given the filter length $2n+1$, the FIR filter $Q(z)$ is fixed as the following form

$$Q(z)=q_{-n}z^n+q_{-n+1}z^{n-1}+\ldots+q_0+q_1z^{-1}+\ldots+q_nz^{-n} \tag{22}$$

To design the filter $Q(z)$, the following cost function is minimized.

$$J=\int_0^{\omega_0\pi}|e^{-j\omega\tau}-Q(e^{j\omega})|^2d\omega+\lambda\int_{\omega_1\pi}^{\pi}|Q(e^{j\omega})|^2d\omega \tag{23}$$

where λ is a non-negative real number. The larger the value λ selected, the more emphasis there is on the stopband minimization. The design criterion leaves the transition band as ''don't care.'' The cost function J can be written in the following matrix-vector form

$$J=\pi\mathbf{q}^T\{\omega_0\mathbf{R}(\omega_0)+\lambda[\mathbf{R}(\pi)-\omega_1\mathbf{R}(\omega_1)]\}\mathbf{q}-2\omega_0\pi\mathbf{h}^T\mathbf{q}+\omega_0\pi \tag{24}$$

where

$$\mathbf{q}=[q_{-n}\quad q_{-n+1}\quad\cdots\quad q_n]^T$$

$$\mathbf{h}=[\mathrm{sinc}[\omega_0(\tau+n)]\quad \mathrm{sinc}[\omega_0(\tau+n-1)]\quad\cdots\quad \mathrm{sinc}[\omega_0(\tau-n)]]^T$$

$$\mathbf{R}(\omega)=\begin{bmatrix} 1 & \mathrm{sinc}(\omega) & \mathrm{sinc}(2\omega) & \cdots & \mathrm{sinc}(2n\omega)\\ \mathrm{sinc}(\omega) & 1 & \mathrm{sinc}(\omega) & & \mathrm{sinc}[(2n-1)\omega]\\ \vdots & & & & \vdots\\ \mathrm{sinc}(2n\omega) & \cdots & & & 1\end{bmatrix}$$

Differentiating J with respect to $\mathbf{q}$ and setting it equal to zero, we obtain the optimal solution

$$\mathbf{q}=\left\{\mathbf{R}(\omega_0)+\frac{\lambda}{\omega_0}[\mathbf{R}(\pi)-\omega_1\mathbf{R}(\omega_1)]\right\}^{-1}\mathbf{h} \tag{25}$$

Note that the matrix inversion part of (25) is independent of the delay τ and thus can be computed beforehand. Given a fractional delay τ, the vector $\mathbf{h}$ can be computed. The coefficients of the optimal low-pass delay filter are then readily obtained by a single matrix multiplication. Therefore, This explicit solution is suitable for a real-time coefficient update. Given a new signal period, the repetitive controller can be easily updated to minimize the period mismatch on-line. Another alternative is to calculate a set of $2n+1$ filter coefficients of $Q(z)$, with respect to some prescribed delays between 0 to 1 sampling time and store all these filter coefficients in a digital memory. Thus, by the table lookup, the filter coefficients can be updated more quickly. Notably, the closed-loop stability for these optimal delay filters can be easily verified. The system stability associated with any one of these filters generally implies the stability associated with the entire set of filters since these lowpass delay filters have roughly the same magnitude response.

Consider a discrete repetitive control system as a design example, where the sampling frequency is 6 kHz and the period of the disturbance is 50.4. Assume the control bandwidth is 1200 kHz, then the filter $D(z)$ in the repetitive controller can be obtained via (25) as follows:

$$\begin{aligned}D(z)=z^{-50}(&0.0127z^8-0.0102z^7-0.025z^6+0.0269z^5+0.045z^4\\&-0.0609z^3-0.0877z^2+0.177z+0.4842+0.4274z^{-1}\\&+0.0951z^{-2}-0.0888z^{-3}-0.0269z^{-4}+0.0347z^{-5}\\&+0.0071z^{-6}-0.0169z^{-7}+0.0024z^{-8})\end{aligned}$$

Figure 3 displays the resulting frequency response of the optimal lowpass delay filter $D(z)$. This delay filter is obviously a good approximation of a desired delay operator within the bandwidth, with the magnitude of the error response $|e^{-j50.4\omega}-D(e^{j\omega})|$ ranging from -75 dB to -25 dB. Notably, since the multiplication $z^{-N}QC$ must be causal, the lowest realizable N is restricted, when the orders of Q and C are determined.

4 Application to Repetitive Acoustic Noise Cancellation

Periodic noises, mostly generated by engines, motors, and compressors, are very common in our living environment. Using the results obtained in Section 3, the repetitive control scheme is applied to active noise cancellation (ANC) in a duct (Fig. 4). Assume the noise source is a piece of machinery that constantly generates periodic noise with constant or varying periods. The error microphone feeds the residual noise back to the controller, and the controller calculates the control signal to drive the canceling loudspeaker, thus producing another noise with which to cancel the primary noise. Given the noise period, which can be directly obtained via a nonacoustic sensor [8], such as a tachometer or an accelerometer, then the asymptotic noise cancellation can be attained via learning from the past if the controller is properly designed.

The plant $P(z)$, which represents the acoustic dynamics in a $0.5\times0.15\times0.15\ \mathrm{m}^3$ duct as well as the dynamics of the cancellation speaker, amplifiers, and the microphone, is identified using time-domain least square algorithms with frequency weighting. Figure 5 shows its frequency response. In the low-frequency band, the plant has transmission zeros around 0 Hz and 700 Hz. The control bandwidth is set as 600 Hz. To stabilize the overall system, the magnitude of $D(e^{j\omega})$ is designed to roll off before the nodal frequency 700 Hz. The parameters $n=10$, $\omega_0=0.1$, $\omega_1=0.1167$, $\lambda=0.01$ are selected. Thus, given the noise period, the filter $D(z)$ can be determined via the formula given in Section 3. Figure 6 displays the frequency response of the optimal $D(z)$ with the desired delay of 50.4 (in the unit of sampling interval). Also, the FIR compensator $C(z)$ with the tap length 20 is designed such that the multiplication $P(z)C(z)$ has zero phase and is close to 1 in the least-squares sense. Figure 7 illustrates the design result, where $|D(e^{j\omega})(1-C(e^{j\omega})P(e^{j\omega}))|<1$ for any ω, except when $\omega=0$. According to Nyquist stability criterion, the overall system is stable since the Nyquist locus does not encircle critical point $-1+0j$.

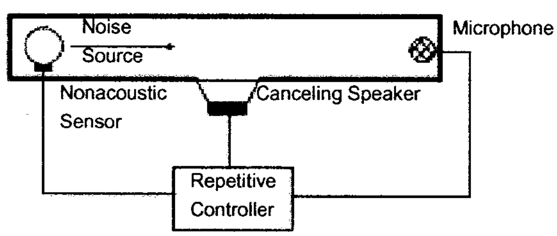

Fig. 4 An active noise cancellation system

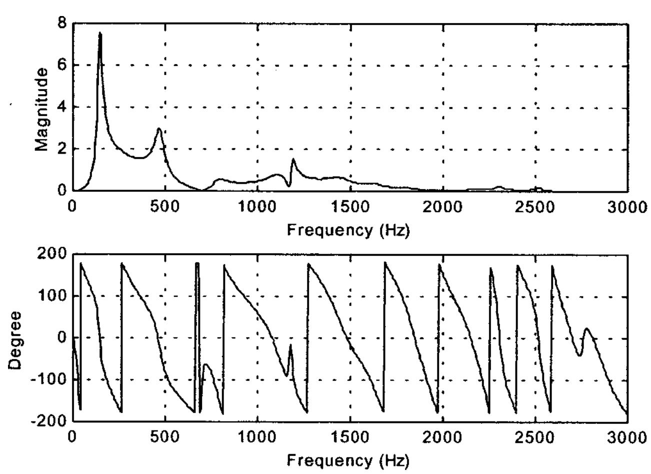

Fig. 5 Frequency response of the plant $P(z)$

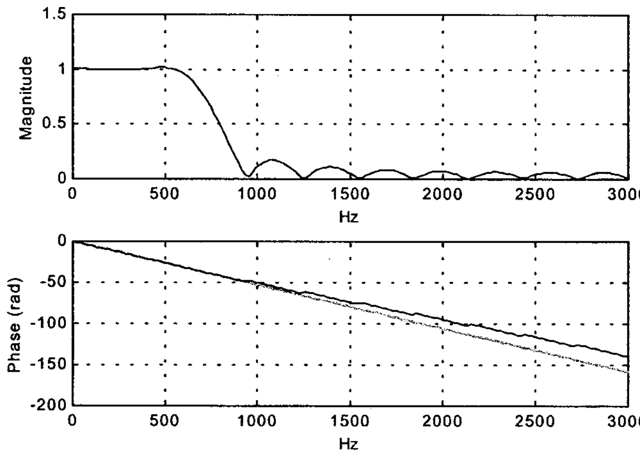

Fig. 6 Frequency response of the lowpass delay filter $D(z)$ (the gray line in the phase diagram is the desired phases)

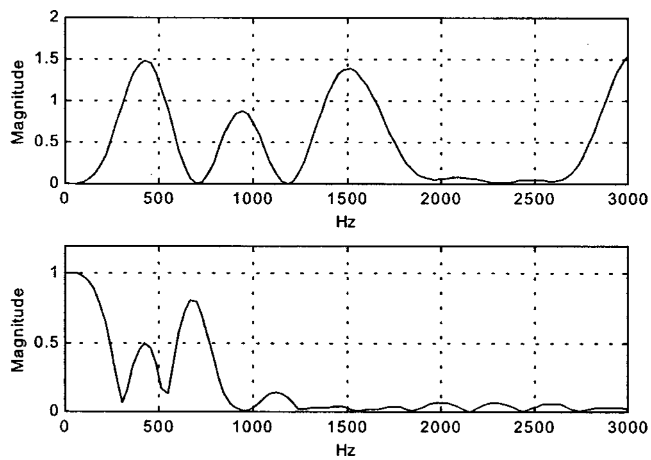

Fig. 7 Magnitude response of $C(e^{j\omega})P(e^{j\omega})$ (above) and $D(e^{j\omega})(1-C(e^{j\omega})P(e^{j\omega}))$ (below)

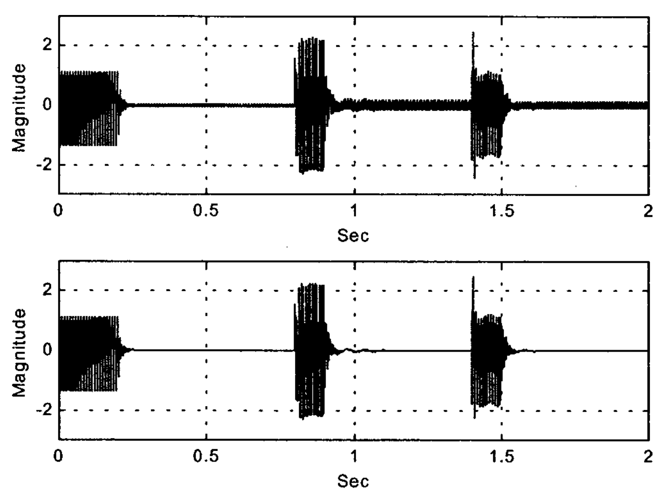

Fig. 8 The cancellation error signals (above: the integer delay tuning method; below: the fractional delay tuning method).

Table 1

Noise period	56.1	45.56	50.67
E/N for the integer delay tuning method	−33.63 dB	−10.20 dB	−22.64 dB
E/N for the fractional delay tuning method	−49.71 dB	−46.27 dB	−47.51 dB

The noise period may alter as the motor or compressor is operated at different speeds. Two controller-tuning schemes are considered in the simulations. One is the integer delay tuning method proposed by Tsao and Nemani [7] and Hu [6], that is, adjusting the order of the repetitive controller according to the roundoff of the given noise period. The other is the fractional delay tuning method proposed herein. That is, the coefficients of the optimal delay filter are updated according to the altered noise period. A periodic noise with five harmonic components is created. The noise period varied from 56.1 samples, to 45.56 samples, and finally to 50.67 samples. Figure 8 shows the simulation results for the assumption that the noise periods can be estimated, and that the repetitive controllers are updated for every 600 sampling intervals. As a steady-state noise-cancellation measure, cancellation-error to noise ratio (E/N) is defined by the RMS norm as follows

$$E/N = 20\log_{10}\left(\frac{\|\text{cancellation error}\|}{\|\text{uncanceled noise}\|}\right)\text{dB}$$

Table 1 indicates E/N values for the simulation results. Obviously, the fractional delay tuning method has superior performance over the integer delay tuning method. The remaining cancellation error for the integer delay tuning method is primarily attributed to the roundoff error of the noise period.

Although, the adjustable sampling rate method may achieve the similar steady-state performance to the fractional delay tuning method, adjusting the sampling rate without changing the controller on-line cannot maintain the best transient performance as the proposed tuning method does, as illustrated in Fig. 8. Worse still, it may affect the system robustness, and even cause instability, whereas the proposed tuning method still maintains good system robustness.

5 Conclusion

A constructive derivation of repetitive control is presented. The stability and performance of repetitive control is proved by the fixed-point theorem. This derivation not only leads to a better understanding of the learning mechanism of repetitive control, but also provides an alternative method with which to design a learning control law or an adaptive algorithm for a desired solution that can be represented in an operator equation.

Also, an optimal delay filter is introduced in the digital repetitive control law to enhance the steady-state performance of repetitive control, when the signal period is not an integer multiple of the sampling interval. The main purpose of this filter is to reconstruct the inter-sample signals of previous period and makes the learning process of digital repetitive control more accurately implemented. The proposed delay filter has the feature that, with respect to different signal periods, its coefficients can be updated quickly. Thus, it is suitable for on-line tuning. Additionally, once the nominal controller is well designed, the transient performance and system stability can be guaranteed for each update. Therefore, when the period of the signal varies, it is perfectly suitable for on-line tuning of the repetitive controller.

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