Regular and Chaotic Dynamic Analysis for a Vibratically Vibrating and Rotating Elliptic Tube Containing a Particle*

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The paper is to present the detailed dynamic analysis of a vertically vibrating and rotating elliptic tube containing a particle. By subjecting to an external periodic excitation, it has shown that the system exhibits both regular and chaotic motions. By using the Lyapunov direct method and Chetaev's theorem, the stability and instability of the relative equilibrium position of the particle in the tube can be determined. The center manifold theorem is applied to verify the conditions of stability when system is under the critical case. The effects of the changes of parameters in the system can be found in the bifurcation and parametric diagrams. By applying various numerical results such as phase plane, Poincaré map and power spectrum analysis, a variety of the periodic solutions and the phenomena of the chaotic motion can be presented. Further, chaotic behavior can be verified by using Lyapunov exponents and Lyapunov dimensions.

Key Words: Bifurcation, Chaos

1. Introduction

The dynamic analysis of a particle moving in a rotating elliptic tube is a widely known problem. In the past, the study was focused on the stability of the motion(1), whereas the regular and chaotic dynamic analysis for a vertically vibrating and rotating elliptic tube containing a particle has not been studied, this paper will give a detailed analysis on these topics.

A lot of mechanical dynamical systems are nonlinear in nature. It has been realized that responses of many nonlinear dynamical systems do not follow simple, regular and predictable trajectories(2)-(6). Many numerical techniques are developed for analyzing the nonlinear system behavior. Both analytical and computational methods are employed to obtain the characteristics of the systems. By subjecting to an external periodic excitation, it has shown that the system exhibits both regular and chaotic motions. By using the Lyapunov direct method and Chetaev's theorem, the stability and instability of the relative equilibrium position of the particle in the tube can be determined. The center manifold theorem is applied to verify the conditions of stability when system is under the critical case. The effects of the changes of parameters in the system can be found in the bifurcation and parametric diagrams. By applying various numerical results such as phase plane, Poincaré map and power spectrum analysis, a variety of the periodic solutions and the phenomena of the chaotic motion can be presented. Further, chaotic behavior can be verified by using Lyapunov exponents and Lyapunov dimensions.

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2. Equations of Motion

The system considered here is depicted in Fig. 1. It is a vertically vibrating and rotating elliptic tube containing a particle with viscous damping where the damping coefficient is k. Because of the vertical vibration of the bearings, when the dynamic equations are established for the reference frame fixed with the bearings, the gravity acceleration appears as a constant term plus a harmonic term $g\left(1+\frac{A\omega^2}{g}\sin\omega t\right)$ in this noninertial frame. Since there is a non-conservative generalized force that is induced by damping, there exists energy dissipation. Then the equations of motion can be expressed as following:

$me^4d^2\ddot{\theta}(1+e\cos\theta)^{-4}\sin^2\theta$

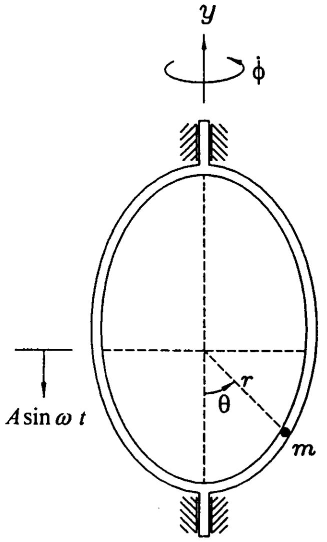


Fig. 1 A schematic diagram of the vertically vibrating and rotating elliptic tube containing a particle

$$
\begin{aligned}
&+me^2d^2\ddot{\theta}(1+e\cos\theta)^{-2}\\
&+2me^5d^2\dot{\theta}^2(1+e\cos\theta)^{-5}\sin^3\theta\\
&+me^4d^2\dot{\theta}^2(1+e\cos\theta)^{-4}\sin\theta\cos\theta\\
&+me^3d^2\dot{\theta}^2(1+e\cos\theta)^{-3}\sin\theta\\
&-me^3d^2\dot{\phi}^2(1+e\cos\theta)^{-3}\sin^3\theta\\
&-me^2d^2\dot{\phi}^2(1+e\cos\theta)^{-2}\sin\theta\cos\theta\\
&-mge^2d\left(1+\frac{A\omega^2}{g}\sin\omega t\right)\\
&\times(1+e\cos\theta)^{-2}\sin\theta\cos\theta\\
&+mged\left(1+\frac{A\omega^2}{g}\sin\omega t\right)(1+e\cos\theta)^{-1}\sin\theta\\
&+k\dot{\theta}=0 \qquad (1)
\end{aligned}
$$

$$
\begin{aligned}
&me^2d^2\ddot{\phi}(1+e\cos\theta)^{-2}\sin^2\theta\\
&+2me^3d^2\dot{\theta}\dot{\phi}(1+e\cos\theta)^{-3}\sin^3\theta\\
&+2me^2d^2\dot{\theta}\dot{\phi}(1+e\cos\theta)^{-2}\sin\theta\cos\theta\\
&+I\ddot{\phi}=0 \qquad (2)
\end{aligned}
$$

where

m : the mass of the particle,
e : the eccentricity of the elliptic tube,
d : the length from the focus to the directrix,
θ : the angle between y-axis and the radius vector r through the particle,
$\dot{\phi}$: the angular velocity of the rotation of the elliptic tube about the vertical semiaxis y-axis,
I : the moment of inertia of elliptic tube with respect to y-axis.

Set state variables $x_1=\theta$, $x_2=\dot{\theta}$, $x_3=\dot{\phi}$ and change to dimensionless time $\tau=\omega t$, the state equations can be written as:

$$
\begin{aligned}
&\dot{x}_1=x_2\\
&\dot{x}_2=[-2me^5d^2\omega^2x_2^2\sin^3 x_1\\
&\quad-me^4d^2\omega^2x_2^2(1+e\cos x_1)\sin x_1\cos x_1\\
&\quad-me^3d^2\omega^2x_2^2(1+e\cos x_1)^2\sin x_1\\
&\quad+me^3d^2\omega^2x_3^2(1+e\cos x_1)^2\sin^3 x_1\\
&\quad+me^2d^2\omega^2x_3^2(1+e\cos x_1)^3\sin x_1\cos x_1\\
&\quad+mge^2d\left(1+\frac{A\omega^2}{g}\sin\tau\right)\\
&\quad\times(1+e\cos x_1)^3\sin x_1\cos x_1\\
&\quad-mged\left(1+\frac{A\omega^2}{g}\sin\tau\right)(1+e\cos x_1)^4\sin x_1\\
&\quad-k\omega x_2(1+e\cos x_1)^5]\\
&\quad/[me^4d^2\omega^2(1+e\cos x_1)\sin^2 x_1\\
&\quad+me^2d^2\omega^2(1+e\cos x_1)^3]\\
&\dot{x}_3=[-2me^3d^2\omega^2x_2x_3\sin^3 x_1\\
&\quad-2me^2d^2\omega^2x_2x_3(1+e\cos x_1)\sin x_1\cos x_1]\\
&\quad/[me^2d^2\omega^2(1+e\cos x_1)\sin^2 x_1\\
&\quad+I\omega^2(1+e\cos x_1)^3] \qquad (3)
\end{aligned}
$$

3. Stability Analysis by Lyapunov Direct Method

The stability of the system will be investigated by Lyapunov direct method in this section. The equilibrium points (x_1, x_2, x_3) in which we are interested, of the system can be found from Eq. (3) with $\sin\tau=0$ as $(0, 0, c)$ and $(\pi, 0, c)$, c is arbitrary constant. The generalized energy $E^*=T_2-T_0+\Pi$ does not contain τ explicity, therefore[1]:

$$\frac{dE^*}{dt}=-2R \qquad (4)$$

(1) For the equilibrium point $(0, 0, c)$, with no disturbance for $\dot{\phi}=c$, take the generalized energy dropping the constant terms as the Lyapunov function

$$
\begin{aligned}
V=&\frac{1}{2}me^2d^2(1+e)^{-2}\dot{\theta}^2\\
&-me^2d^2(1+e)^{-2}c^2\theta^2+\frac{1}{2}Ic^2\\
&+mged(1+e)^{-2}\theta^2+\cdots\\
=&\frac{1}{2}me^2d^2(1+e)^{-2}\dot{\theta}^2\\
&+med(g-edc^2)(1+e)^{-2}\theta^2+\cdots \qquad (5)
\end{aligned}
$$

where the higher order terms are not presented but not neglected.

V is the positive definite function for θ, $\dot{\theta}$ if $c^2<\frac{g}{ed}$, and the time derivative of V is

$$\dot{V}=\frac{dE^*}{dt}=-2R=-k\dot{\theta}^2 \qquad (6)$$

$\dot{V}$ is negative semidefinite for θ, $\dot{\theta}$. By Lyapunov theorem of stability[7], the conditional stability condition of θ and $\dot{\theta}$ is $c^2<\frac{g}{ed}$.

(2) For the equilibrium point $(\pi, 0, c)$, let $\theta=\pi+\zeta$, $\dot{\theta}=\dot{\zeta}$, $\dot{\phi}=c+\xi$, take the generalized energy dropping the constant terms as the Lyapunov function:

$$
\begin{aligned}
V=&\frac{1}{2}me^2d^2(1+e)^{-2}\dot{\zeta}^2\\
&+me^2d^2(1+e)^{-2}(\pi+\zeta)^2(c+\xi)^2\\
&+\frac{1}{2}I(c+\xi)^2+mged(1+e)^{-2}(\pi+\zeta)^2+\cdots\\
=&2\pi c^2me^2d^2(1+e)^{-2}\zeta+2\pi^2cme^2d^2(1+e)^{-2}\xi\\
&+Ic\xi+2\pi mged(1+e)^{-2}\zeta+\cdots \qquad (7)
\end{aligned}
$$

where the higher order terms are not presented but not neglected.

The time derivative of V is

$$\dot{V}=\frac{dE^*}{dt}=-2R=-k\dot{\zeta}^2 \quad (8)$$

In this case, the Chetaev theorem fails. We take another method to solve this problem. It is successfully to deal with the stability problem by using the second Ge-Liu theorem of instabililty in Ref.(8). After a rearrangement, the derivative $\ddot{V}$ and $\dddot{V}$ are shown bellow :

$$\ddot{V}=-2k\dot{\zeta}\ddot{\zeta} =-2k\left(\frac{ed\omega^2c^2+g+A\omega^2\sin\tau}{ed\omega}\right)\zeta\dot{\zeta}+\cdots \quad (9)$$

$$\begin{aligned}\dddot{V}&=-2k\left(\frac{ed\omega^2c^2+g+A\omega^2\sin\tau}{ed\omega}\right)\dot{\zeta}^2\\&-2k\left(\frac{ed\omega^2c^2+g+A\omega^2\sin\tau}{ed\omega}\right)\zeta\ddot{\zeta}+\cdots\\&=-2k\left(\frac{ed\omega^2c^2+g+A\omega^2\sin\tau}{ed\omega}\right)\dot{\zeta}^2\\&-2k\left(\frac{ed\omega^2c^2+g+A\omega^2\sin\tau}{ed\omega}\right)^2\zeta^2+\cdots\end{aligned} \quad (10)$$

No matter what value of ζ and ξ may be, $\dddot{V}$ is always negative if

$$ed\omega^2c^2+g-A\omega^2>0. \quad (11)$$

The conditions of instability of the above theorem are all satisfied, therefore the equilibrium point $(\pi, 0, c)$ is unstable in this case.

4. Application of the Center Manifold Theorem

Using linearization to study stability of equilibrium points of a nonlinear dynamical system is a well-known method[7]. Linearization fails when the Jacobian matrix, evaluated at the equilibrium point, has some eigenvalues with zero real parts and no eigenvalues with positive real parts. Variation of system parameters may cause this critical case. Here we use the center manifold theorem to study the conditions of stability of the equilibrium points in the critical case when linearization fails.

We consider the state equations as

$$\begin{aligned}\dot{x}_1&=x_2\\ \dot{x}_2&=[-2me^5d^2x_2^2\sin^3 x_1\\&-me^4d^2x_2^2(1+e\cos x_1)\sin x_1\cos x_1\\&-me^3d^2x_2^2(1+e\cos x_1)^2\sin x_1\\&+me^3d^2x_3^2(1+e\cos x_1)^2\sin^3 x_1\\&+me^2d^2x_3^2(1+e\cos x_1)^3\sin x_1\cos x_1\\&+mge^2d(1+e\cos x_1)^3\sin x_1\cos x_1\\&-mged(1+e\cos x_1)^4\sin x_1]\\&/[me^4d^2(1+e\cos x_1)\sin^2 x_1\\&+me^2d^2(1+e\cos x_1)^3]\\ \dot{x}_3&=[-2me^3d^2x_2x_3\sin^3 x_1\\&-2me^2d^2x_2x_3(1+e\cos x_1)\sin x_1\cos x_1\\&-k_1x_3+M]\\&/[me^2d^2(1+e\cos x_1)\sin^2 x_1+I(1+e\cos x_1)^3]\end{aligned} \quad (12)$$

where k_1 is the viscous damping between the rotating elliptic tube and y-axis, M is the moment acting on the elliptic tube along the y-axis.

System has an equilibrium point $\left(0, 0, \frac{M}{k_1}\right)$ and rewrite (12) in matrix form

$$\dot{X}=GX+F(X)+O(3), \quad X\in R^2 \quad (13)$$

where

$$X=[x_1, x_2, x_3]^T$$

$$G=\begin{bmatrix}0 & 1 & 0\\ \dfrac{\frac{M^2}{k_1^2}ed-g}{ed} & 0 & 0\\ 0 & 0 & \dfrac{-k_1}{I(1+e)^3}\end{bmatrix}$$

$$F=\frac{1}{2}\begin{bmatrix}0\\ 4\dfrac{M}{k_1}x_1x_3\\ -4\dfrac{me^2d^2M}{Ik_1(1+e)^2}x_1x_2\end{bmatrix} \quad (14)$$

Now the matrix has three eigenvalues $\bar{\lambda}_1$, $\bar{\lambda}_2$ and $\bar{\lambda}_3$

$$\bar{\lambda}_1, \bar{\lambda}_2=\pm\sqrt{\frac{\frac{M^2}{k_1^2}ed-g}{ed}}=\pm i\Omega \quad \bar{\lambda}_3=\frac{-k_1}{I(1+e)^3}$$

One of the eigenvalues has no positive real part and the others are pure imaginary pair. This is a critical case, the eigenvalues of matrix G fails to determine the stability of the equilibrium point and it becomes necessary to consider the higher order terms.

So one employs the center manifold theorem to reduce the dimension of the state space at the critical parameter[9]. A transformation matrix P is used to transform our state equation (13) :

$$P=\begin{bmatrix}1 & 1 & 0\\ \Omega & -\Omega & 0\\ 0 & 0 & 1\end{bmatrix} \quad (15)$$

which is formed by eigenvectors of G.

Let

$$\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=P\begin{bmatrix}q_1\\q_2\\q_3\end{bmatrix} \quad (16)$$

then Eq.(13) is transformed into

$$\begin{bmatrix}\dot{q}_1\\ \dot{q}_2\\ q_3\end{bmatrix}=\begin{bmatrix}0 & -\Omega & 0\\ \Omega & 0 & 0\\ 0 & 0 & -\dfrac{k_1}{I(1+e)^3}\end{bmatrix}\begin{bmatrix}q_1\\q_2\\q_3\end{bmatrix}+\begin{bmatrix}\dfrac{M}{k_1\Omega}(q_1q_3-q_2q_3)\\ \dfrac{-M}{k_1\Omega}(q_1q_3+q_2q_3)\\ -2\dfrac{me^2d^2\Omega M}{Ik_1(1+e)^2}(q_1^2-q_2^2)\end{bmatrix}+\text{H.O.T} \quad (17)$$

By the center manifold theorem, we can reduce the system to two dimensional system with the center manifold

$$q_3=h(q_1, q_2)=aq_1^2+bq_2^2+cq_1q_2 \tag{18}$$

where

$$a=\frac{2\Omega me^2d^2M(1+e)}{k_1^2+4I^2\Omega(1+e)^6} \quad b=\frac{2\Omega me^2d^2M(1+e)}{k_1^2+4I^2\Omega(1+e)^6}$$

$$c=-\frac{8I\Omega^2me^2d^2M(1+e)^4}{k_1^3+4I^2\Omega k_1(1+e)^6} \tag{19}$$

then the reduced system becomes

$$\begin{aligned}\begin{bmatrix}\dot{q}_1\\ \dot{q}_2\end{bmatrix}&=\begin{bmatrix}0 & -\Omega\\ \Omega & 0\end{bmatrix}\begin{bmatrix}q_1\\ q_2\end{bmatrix}+\begin{bmatrix}f_1\\ f_2\end{bmatrix}+\text{H.O.T}\\ &=\begin{bmatrix}0 & -\Omega\\ \Omega & 0\end{bmatrix}\begin{bmatrix}q_1\\ q_2\end{bmatrix}\\ &+\begin{bmatrix}\frac{M}{k_1\Omega}(aq_1^3+bq_2^3+(b+c)q_1q_2^2+(a+c)q_1^2q_2)\\ \frac{-M}{k_1\Omega}(aq_1^3+bq_2^3+(b+c)q_1q_2^2+(a+c)q_1^2q_2)\end{bmatrix}\\ &+\text{H.O.T}\end{aligned} \tag{20}$$

The stability criterion for a general two dimensional system of the Eq. (20) is[10]

$$\begin{aligned}\alpha&=\frac{1}{16}\left(\frac{\partial^3 f_1}{\partial q_1^3}+\frac{\partial^3 f_1}{\partial q_1 \partial q_2^2}+\frac{\partial^3 f_2}{\partial q_1^2 \partial q_2}+\frac{\partial^3 f_2}{\partial q_2^3}\right)\\ &+\frac{1}{16\Omega}\left[\frac{\partial^2 f_1}{\partial q_1 \partial q_2}\left(\frac{\partial^2 f_1}{\partial q_1^2}+\frac{\partial^2 f_1}{\partial q_2^2}\right)\right.\\ &-\frac{\partial^2 f_2}{\partial q_1 \partial q_2}\left(\frac{\partial^2 f_2}{\partial q_1^2}+\frac{\partial^2 f_2}{\partial q_2^2}\right)\\ &\left.-\frac{\partial^2 f_1}{\partial q_1^2}\frac{\partial^2 f_2}{\partial q_1^2}+\frac{\partial^2 f_1}{\partial q_2^2}\frac{\partial^2 f_2}{\partial q_2^2}\right]\\ &=\frac{M}{2k_1\Omega}a<0\end{aligned} \tag{21}$$

For this case, parameters

$$\frac{M^2}{k_1^2}ed-g<0 \tag{22}$$

will cause $\alpha<0$, the equilibrium point $\left(0, 0, \frac{M}{k_1}\right)$ in the critical situation is stable.

5. Phase Portraits, Poincaré Map and Power Spectrum

In previous section, we have given the equations of motion. Now, we can set some parameters and use fourth order Runge-Kutta numerical integration method to simulate our system, then plot the results for three different amplitudes of external vertical excitation (displacement). In Eq. (3), we set the parameters $I=2$, $m=1$, $e=0.5$, $d=10$, $k=10$, $\omega=1$, and $g=9.8$. Phase diagrams and Poincaré map are plotted in Fig. 2(a)-(c) for $A=10.85$, 11.0 and 11.185. Clearly, the motion is periodic for $A=10.85$, 11.0, and the Fig. 2(c) for $A=11.185$ shows the chaotic state.

Any function $x(t)$ may be represented as a superposition of different periodic components. If it is periodic, the spectrum may be a linear combination of oscillations whose frequencies are integer multiples of basic frequency. The linear combination is called a Fourier series. If it is not periodic, the spectrum then must be in terms of oscillations with a continuum of frequencies. Such a representation of the spectrum is called Fourier integral of $x(t)$. The representation is useful for dynamical analysis. The nonautonomous system are observed by the portraits of power spectrum in Fig. 3(a)-(c) for $A=10.85$, 11.0 and 11.185. In Fig. 3(a)-(b), period-1 and period-2 oscillations present respectively. S_x is the amplitude of the component in Fourier series expansion for x_1. In Fig. 3(c), the noise-like shape appears and the peak is still presented at the fundamental frequencies, it is the characteristic of the chaotic dynamical system.

6. Bifurcation Diagram, Lyapunov Exponent, Lyapunov Dimension and Parametric Diagram

The information about the dynamics of the nonlinear system for specific values of the parameters is provided. The dynamics may be viewed more completely over a range of parameter value. Let us vary one of the parameters in the system, and record the data of Poincaré map corresponding to every different parameter values. Then the steady state behavior of the system versus the range of parameter will be plotted. This is a well-known technique to describe a

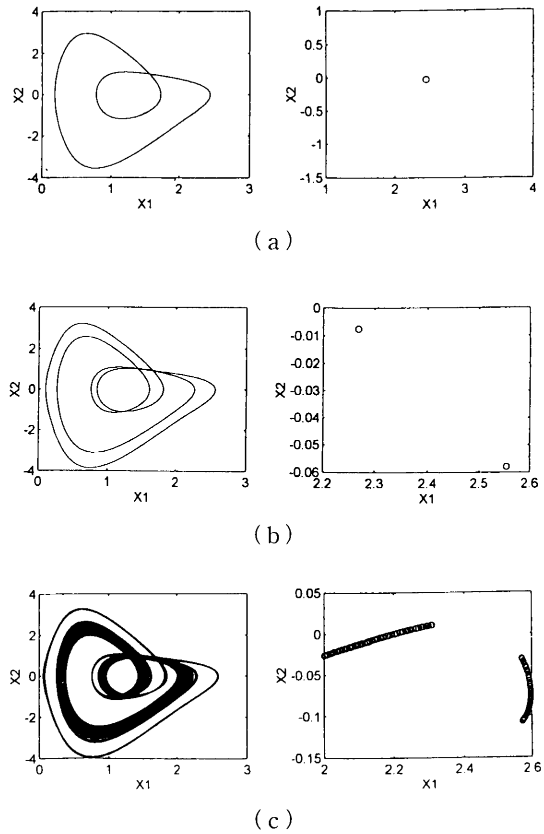


Fig. 2 Phase portraits and corresponding Poincaré maps of for (a) $A=10.85$ of period-1, (b) $A=11.0$ of period-2, (c) $A=11.185$ of chaos where $x_1=\theta$, $x_2=\dot{\theta}$

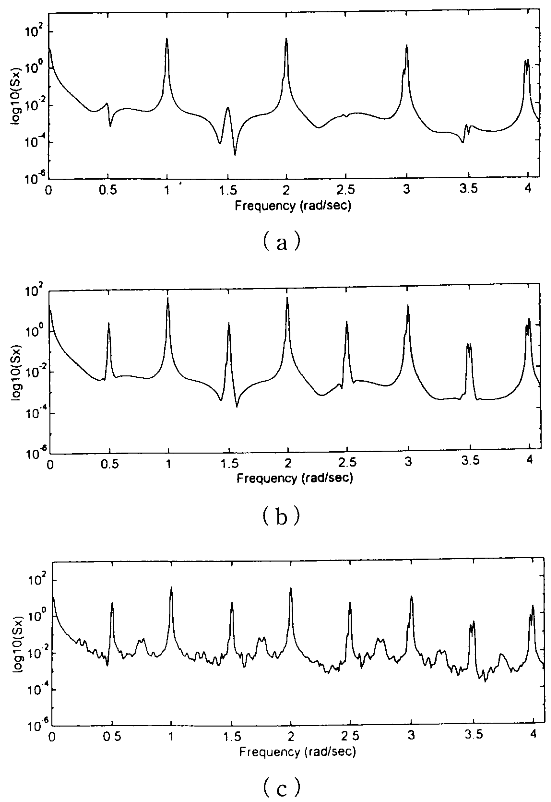


Fig. 3 The power spectra for (a) $A=10.85$ of period-1, (b) $A=11.0$ of period-2, (c) $A=11.185$ of chaos

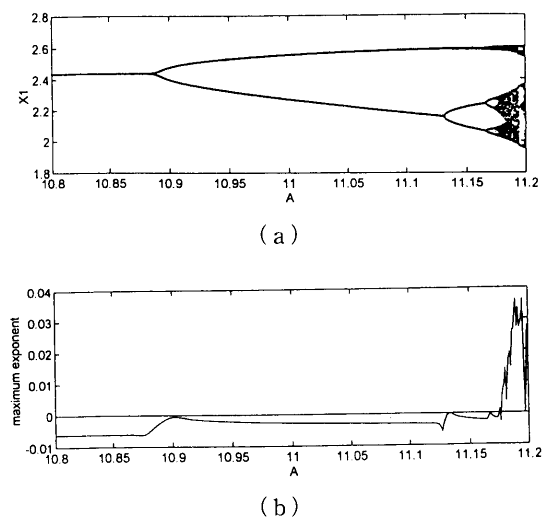


Fig. 4 (a) Bifurcation and, (b) the maximum Lyapunov exponent diagram

transition from periodic motion to chaotic motion for a dynamical system. This is called as a bifurcation diagram.

In order to determine chaos existing in nonlinear system, how to detect chaos becomes very important. Here, Lyapunov exponent is an index for chaotic behavior. It has proven to be the most useful dynamical diagnostic tool for examining chaotic motion. For

Table 1 The Lyapunov dimension and the Lyapunov exponent of the nonlinear system

A	10.85	11.0	11.15	11.17	11.185
λ_1	-0.0061122	-0.0026641	-0.001778	-0.0013303	0.0146802
λ_2	-0.0157942	-0.1001171	-0.1499486	-0.0477297	-0.0133249
λ_3	-0.3607489	-0.2840784	-0.2410326	-0.3439619	-0.3947034
λ_4	0	0	0	0	0
d_L	1	1	1	1	2.0034
	Period-1	Period-2	Period-4	Period-8	Chaos

a three-dimensional dynamical system, the calculation of Lyapunov exponent is described briefly[11]. Positive value of Lyapunov exponent indicates chaos, negative value of Lyapunov exponent indicates stable orbit. The criterion is

$\lambda>0$ (chaotic)

$\lambda\leq 0$ (regular motion)

The periodic and chaotic motions can be distinguished by the bifurcation diagram and Lyapunov exponent shown in Fig. 4.

The Lyapunov dimension is a measure of the complexity of the attractor. It has been developed by Kaplan and Yorke[12], that the Lyapunov dimension d_L is introduced as

$$d_L=j+\frac{\sum_{i=1}^{j}\lambda_i}{|\lambda_{j+1}|} \tag{23}$$

where j is defined by the condition that

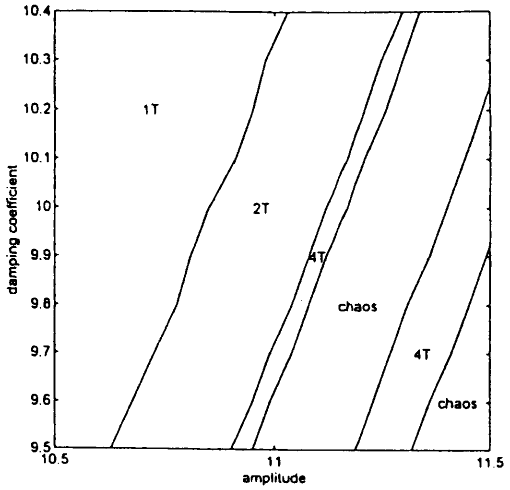
$$\sum_{i=1}^{j}\lambda_i>0 \quad \text{and} \quad \sum_{i=1}^{j+1}\lambda_i<0$$

The Lyapunov dimension for a strange attractor is a noninteger number. The Lyapunov dimension and the Lyapunov exponent of the nonlinear system are listed in Table 1.

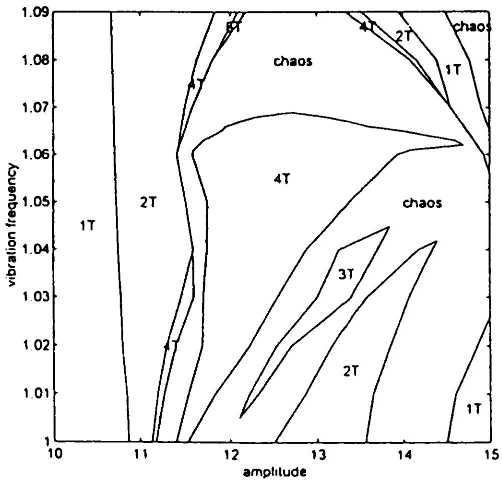
Further, the parameter values, such as damping coefficient k, vibrating frequency ω and vibrating amplitude A, may also be varied together to observe the behaviors of bifurcation of the system. By varying any two of these three parameters, the enriched information of the behaviors of the system can be described and shown as Fig. 5, in which a region of period-three response is found.

7. Conclusions

The dynamical system of the vertically vibrating and rotating elliptic tube containing a particle has a rich variety of nonlinear behaviors as the parameters varied. Due to the effect of nonlinearity, regular and chaotic motions appear. In this paper, analytical and computational methods have been employed to study the dynamical behaviors of the nonlinear system. In Section 3, the stability for the mechanical system with vertical vibration and viscous damping has been found by using the Lyapunov direct method. The condition of stability in critical case has also found by using center manifold theorem.



(a)



(b)

Fig. 5 (a) Parametric diagram of amplitude versus damping coefficient for $\omega=1$, (b) Parametric diagram of amplitude versus vibration frequency for $k=10$

The computational analyses have also been studied. The periodic and chaotic phenomena are described by phase portraits, Poincaré map, bifurcation diagram and power spectrum. The occurrence of chaotic attractor has been verified by evaluating Lyapunov exponents and Lyapunov dimension.

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