Output Feedback Sliding-Mode Controller Design for Minimum Phase Linear Systems*

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This paper proposes an output sliding-mode control under three assumptions given to the system matrices and the matching uncertainty. Based on these assumptions, first one important transformation matrix is introduced and then a state-estimator is constructed. The output sliding-mode control is designed when the system state is well estimated. Finally, a numeric example is included to demonstrate the developed controller.

Key Words: Output Feedback, Sliding-Mode Control, State-Estimator, Matching Disturbance

1. Introduction

In general, the sliding-mode controller design is based on the assumption that all the system states are available[(1)]. In most physical systems, this is not the case. Often, controller designers face a system that only output information is obtainable. To deal with such stringent situation, recently several output sliding-mode control algorithms have been proposed for systems restricted to some assumptions on system matrices and disturbance[(2)-(9)]. The work presented here is still limited to the assumptions mentioned above; however, a novel transformation matrix is introduced and then a state-estimator. Based on this state-estimator, an output sliding-mode control is proposed to suppress the matching disturbance depending on all the system state variables.

In output sliding-mode controller design, a suitable transformation matrix is important to appropriately transform the original system into a new model with two sub-systems, one is related to the control input and the other is not[(4),(9),(10)]. Based on this new model, estimator and controller are designed to achieve the control goal. The main contribution of this paper is to propose a simple and straightforward procedure to find such kind of transformation matrix.

In the next section, a class of system to be controlled is first introduced with three important assumptions related to the system matrices and the matching disturbance. Section 3 presents the state-estimator in accordance with a novel transformation matrix. Once the system's state is well estimated, the output sliding-mode control algorithm is proposed in Section 4. To demonstrate the developed controller, a numeric example is shown in Section 5. Finally, Section 6 gives the concluding remarks.

2. System Description

This paper considers the problem of designing the sliding-mode control in such a way that only output feedback is available. The system discussed is expressed as

$$\dot{\boldsymbol{x}}=\boldsymbol{A}\boldsymbol{x}+\boldsymbol{B}\boldsymbol{u}+\boldsymbol{B}\boldsymbol{\xi}(t, \boldsymbol{x}) \quad (1)$$

$$\boldsymbol{y}=\boldsymbol{C}\boldsymbol{x} \quad (2)$$

where $\boldsymbol{x}\in\Re^n$, $\boldsymbol{u}\in\Re^m$, and $\boldsymbol{y}\in\Re^m$ with $m<n$. Only the output information $\boldsymbol{y}$ is obtainable. Besides, there are three assumptions required for the state-estimator and sliding-mode controller design, listed as below:

A1. The input and output matrices $\boldsymbol{B}$ and $\boldsymbol{C}$ are both of full rank and rank $(\boldsymbol{CB})=m$.

A2. The system (1) and (2) is minimum phase. It follows that $(\boldsymbol{I}_n-\boldsymbol{B}(\boldsymbol{CB})^{-1}\boldsymbol{C})\boldsymbol{A}$ has $n-m$ non-zero eigenvalues λ_i, $i=1, 2, \cdots, n-m$, satisfying $Re\{\lambda_i\}<0$.

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A3. $\zeta(t, x)$ is a matching uncertainty satisfying

$$\|\zeta(t, x)\| < \alpha + \beta\|x\|,\ \alpha>0,\ \beta>0. \quad (3)$$

where $\|\cdot\|$ indicates the vector's Euclidean norm. The assumption A1 can be found in most of the researches related to output feedback sliding-mode control[2],[4],[5]. The assumption A2 is important for the state-estimator design. As for the assumption A3, it is different from many other researches in that the matching uncertainty $\zeta(t, x)$ depends on the system state x, not the output y[3],[9]. In order to suppress the uncertainty $\zeta(t, x)$, the system state x must be effectively estimated.

3. State-Estimator Design

Based on the assumptions A1 and A2, a new state space model for the system (1) and (2) can be obtained. Define a new system state $s \in \Re^m$ of the form

$$s = (CB)^{-1}y = (CB)^{-1}Cx \quad (4)$$

Now, by direct calculation, we obtain

$$C(I_n - B(CB)^{-1}C) = C - C = 0 \quad (5)$$

$$(I_n - B(CB)^{-1}C)B = B - B = 0 \quad (6)$$

According to the assumption A2, there exists a full rank matrix $W \in C^{n\times(n-m)}$ such that

$$(I_n - B(CB)^{-1}C)AW = W\Lambda \quad (7)$$

where Λ is the Jordan form of eigenvalues $\{\lambda_1, \lambda_2, \cdots, \lambda_{n-m}\}$ and W contains the right eigenvectors corresponding to Λ. As for the other m eigenvalues of $(I_n - B(CB)^{-1}C)A$, they are all zeros. This can be easily seen from Eq.(5) that $C(I_n - B(CB)^{-1}C)A = 0$ and $\mathrm{rank}(C) = m$. Also, pre-multiplying C into Eq.(7) yields

$$C(I_n - B(CB)^{-1}C)AW = CW\Lambda = 0 \quad (8)$$

Because Λ is of full rank, we have

$$CW = 0 \quad (9)$$

Now, let's focus on $W \in C^{n\times(n-m)}$, a complex matrix possessing $n-m$ complex eigenvectors in accordance with λ_i, $i = 1, 2, \cdots, n-m$, satisfying $Re\{\lambda_i\} < 0$. It is known if a complex eigenvalue λ belongs to $\{\lambda_1, \lambda_2, \cdots, \lambda_{n-m}\}$, so is its conjugate $\bar{\lambda}$; therefore, $WW^H \in \Re^{n\times n}$ where $W^H = \bar{W}^T$, the conjugate and transpose matrix of W[11]. Therefore, applying the singular-value decomposition technique to W leads to

$$W = U\begin{bmatrix} Q \\ 0 \end{bmatrix} V^H \quad (10)$$

where $U \in \Re^{n\times n}$ is orthogonal, i.e. $U^TU = I_n$, $V \in C^{(n-m)\times(n-m)}$ is unitary, i.e. $V^HV = I_{n-m}$, and $Q \in \Re^{(n-m)\times(n-m)}$ is invertible. Let $U = [U_1\ \ U_2]$ where $U_1 \in \Re^{n\times(n-m)}$, then from Eq.(10) we obtain

$$W = U_1QV^H \quad (11)$$

Note that $U_1^TU_1 = I_{n-m}$ since $U^TU = I_n$. Substituting Eq.(11) into Eq.(9) leads to $CU_1QV^H = 0$. Obviously,

$$CU_1 = 0 \quad (12)$$

since $V^HV = I_{n-m}$ and Q is invertible. Based on Eqs.(6), (12) and $U_1^TU_1 = I_{n-m}$, we have

$$\begin{bmatrix} (CB)^{-1}C \\ U_1^T(I_n - B(CB)^{-1}C) \end{bmatrix}[B\ \ U_1] = I_n \quad (13)$$

Define

$$M = \begin{bmatrix} (CB)^{-1}C \\ U_1^T(I_n - B(CB)^{-1}C) \end{bmatrix} \text{ and } N = [B\ \ U_1] \quad (14)$$

then both $n \times n$ square matrices M and N are real and $M = N^{-1}$. Most significantly, M can be used as a transformation matrix. Let $z = U_1^T(I_n - B(CB)^{-1}C)x$, which is not measurable, then

$$Mx = \begin{bmatrix} (CB)^{-1}C \\ U_1^T(I_n - B(CB)^{-1}C) \end{bmatrix}x = \begin{bmatrix} s \\ z \end{bmatrix} \quad (15)$$

or

$$x = M^{-1}\begin{bmatrix} s \\ z \end{bmatrix} = N\begin{bmatrix} s \\ z \end{bmatrix} = Bs + U_1z \quad (16)$$

Pre-multiplying M into (1) becomes

$$\begin{bmatrix} \dot{s} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} (CB)^{-1}C \\ U_1^T(I_n - B(CB)^{-1}C) \end{bmatrix}(A(Bs + U_1z) + Bu + B\zeta(t, x)) \quad (17)$$

Obviously,

$$\dot{s} = (CB)^{-1}CABs + (CB)^{-1}CAU_1z + u + \zeta(t, x) \quad (18)$$

$$\dot{z} = U_1^T(I_n - B(CB)^{-1}C)AU_1z + U_1^T(I_n - B(CB)^{-1}C)ABs \quad (19)$$

From Eqs.(7), (11), $V^HV = I_{n-m}$, and $U_1^TU_1 = I_{n-m}$, we have

$$U_1^T(I_n - B(CB)^{-1}C)AU_1 = QV^{-1}\Lambda(QV^{-1})^{-1} \quad (20)$$

Evidently, $U_1^T(I_n - B(CB)^{-1}C)AU_1$ has eigenvalues related to Λ, i.e., $\{\lambda_1, \lambda_2, \cdots, \lambda_{n-m}\}$, all located in the left-half complex plane, as given in the assumption A2. Viewing from Eq.(19), an estimator for z can be built up as

$$\dot{\hat{z}} = U_1^T(I_n - B(CB)^{-1}C)AU_1\hat{z} + U_1^T(I_n - B(CB)^{-1}C)ABs \quad (21)$$

From Eqs.(19) and (21), we have

$$\dot{z} - \dot{\hat{z}} = U_1^T(I_n - B(CB)^{-1}C)AU_1(z - \hat{z}) \quad (22)$$

where all the eigenvalues of $U_1^T(I_n - B(CB)^{-1}C)AU_1$ possess negative real part, shown in Eq.(20). As a result, we can conclude

$$\hat{z} \to z \quad \text{for} \quad t \to \infty \quad (23)$$

This completes the design of state-estimator (21) for z. In addition, from Eqs.(3) and (16), as $t \to \infty$ the matching uncertainty is then bounded by

$$\|\zeta(t, x)\| < \alpha + \beta\|\hat{x}\|,\ \alpha>0,\ \beta>0. \quad (24)$$

This is the reason why the matching uncertainty can be considered depending on the whole system state x, not just the output y.

4. Output Sliding-Mode Control

With the use of the state-estimator (21), the total system is rewritten as

$$\dot{x} = Ax + Bu + B\zeta(t, x) \quad (25)$$

$$y = Cx \quad (26)$$

$$\dot{\hat{z}} = U_1^T(I_n - B(CB)^{-1}C)AU_1\hat{z} + U_1^T(I_n - B(CB)^{-1}C)ABs \quad (27)$$

where

$$\hat{z} \to z = U_1^T(I - B(CB)^{-1}C)x \qquad \text{for } t \to \infty \quad (28)$$

$$\|\zeta(t, x)\| < \alpha + \beta\|\hat{x}\|, \ \alpha > 0, \ \beta > 0, \qquad \text{for } t \to \infty \quad (29)$$

Now choose the sliding vector as $s = (CB)^{-1}Cx = (CB)^{-1}y$ in Eq. (4), then from Eq. (18) we obtain

$$\dot{s} = (CB)^{-1}CABs + (CB)^{-1}CAU_1z + u + \zeta(t, x) \quad (30)$$

Note that although z cannot be directly measured, it can be estimated by Eq. (27). Therefore, with the estimation of $\hat{z}$, the sliding-mode control law is established as

$$u = -(CB)^{-1}CA(Bs + U_1\hat{z}) - (\alpha + \beta\|Bs + U_1\hat{z}\| + \delta)sat(s, \varepsilon), \ \varepsilon > 0, \ \delta > 0 \quad (31)$$

$$sat(s, \varepsilon) = \begin{cases} \dfrac{s}{\|s\|} & \text{for } \|s\| > \varepsilon \\ \dfrac{s}{\varepsilon} & \text{for } \|s\| \le \varepsilon \end{cases} \quad (32)$$

where $\delta > 0$ is a constant decided by the designer and is used to guarantee the reaching and sliding condition[3]. For $\|s\| > \varepsilon$, substituting Eq. (32) into Eq. (31) results in

$$\dot{s} = (CB)^{-1}CAU_1(z - \hat{z}) - (\alpha + \beta\|Bs + U_1\hat{z}\| + \delta)\frac{s}{\|s\|} + \zeta(t, x(t)) \quad (33)$$

From Eqs. (28) and (29), it is obvious that

$$\dot{s} \approx -(\alpha + \beta\|Bs + U_1\hat{z}\| + \delta)\frac{s}{\|s\|} + \zeta(t, Bs + U_1\hat{z}) \qquad \text{for } t \to \infty \quad (34)$$

Pre-multiplying s^T yields

$$s^T\dot{s} \approx -\|s\|(\alpha + \beta\|Bs + U_1\hat{z}\| + \delta) + s^T\zeta(t, Bs + U_1\hat{z}) < -\delta\|s\| \qquad \text{for } t \to \infty \quad (35)$$

It shows that, as the time t is getting larger and larger or $t \to \infty$, the sliding and reaching condition $s^T\dot{s} < -\delta\|s\|$ in Eq. (35) is guaranteed. In other words, when the system is in the region $\|s\| > \varepsilon$, its trajectory will be controlled to reach and then stay in the layer $\|s\| \le \varepsilon$. Now, from Eq. (19) and the assumption A2, the state z will be bounded due to the fact $\|s\| \le \varepsilon$. It can be also concluded from Eq. (16) that the system state $\|x\| = \|Bs + U_1z\| < \|B\|\|s\| + \|U_1\|\|z\|$ is also bounded. Clearly, the system is successfully stabilized by the output sliding-mode control law (31) and (32).

5. Numerical Example

In order to demonstrate the developed output sliding-mode control, a numerical example of Eqs. (1) and (2) is given below:

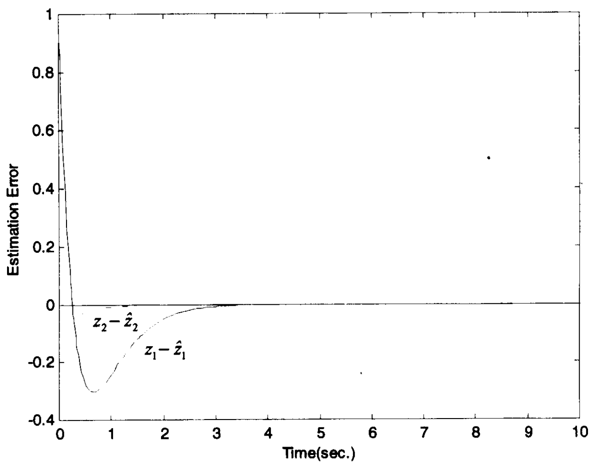

Fig. 1 Estimation error of z

$$A = \begin{bmatrix} -20 & 39 & -28 \\ -21 & 40 & -29 \\ -12 & 22 & -16 \end{bmatrix} \quad B = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} \quad C = [3 \ -1 \ -3] \quad (36)$$

Since $CB = 1$, the assumption A1 for rank $(CB) = 1$ is true. Besides, $(I_n - B(CB)^{-1}C)A$ has eigenvalues -2 and -3. The assumption A2 is guaranteed. For the matching disturbance, it is assumed that $\|\zeta(t, x(t))\| < 0.5 + 0.8 \cdot \|x(t)\|$, i.e., $\alpha = 0.5$ and $\beta = 0.8$. Now, the sliding vector in Eq. (4) is chosen as

$$s = y \quad (37)$$

The matrix W in Eq. (7) is obtained as

$$W = \begin{bmatrix} -0.6075 & 0.5976 \\ -0.7009 & 0.7171 \\ -0.3738 & 0.3586 \end{bmatrix} \quad (38)$$

The singular value decomposition of $W = U_1QV^H$ in Eq. (11) results in

$$U_1 = \begin{bmatrix} -0.6026 & 0.4040 \\ -0.7091 & -0.6668 \\ -0.3662 & 0.6263 \end{bmatrix}, \quad Q = \begin{bmatrix} 1.4141 & 0 \\ 0 & 0.0172 \end{bmatrix},$$

$$V^H = \begin{bmatrix} 0.7071 & -0.7071 \\ -0.7071 & -0.7071 \end{bmatrix} \quad (39)$$

Then we can use these matrices to construct the state estimator, described by Eq. (21). By using the sliding-mode control law (31) with $\varepsilon = 0.05$, $\delta = 0.5$, the numeric simulation results are given in Fig. 1 to Fig. 4. Besides, the system initial condition is $x(0) = [1 \ 1 \ 0.6]^T$, the initial condition of the state-estimator is $\hat{z}(0) = [0 \ \ 0]^T$ and the matching disturbance is $\zeta(t, x(t)) = 0.4 \sin(0.5t) + 0.8x_1$.

Figure 1 shows the estimation error $z - \hat{z}$. Clearly, the state z is well estimated after $t = 3$ sec. In Fig. 2, it can be seen that the reaching and sliding condition $s^T\dot{s} < -\delta\|s\|$ is satisfied after $t = 1$ sec. Most importantly, the system is restricted in the sliding layer $\|s\| \le 0.05$ after $t = 3$ sec, i.e., the state z is well estimated. The control law is given in Fig. 3 without any chattering. The success of this developed output sliding-mode control is demonstrated by the state variables in

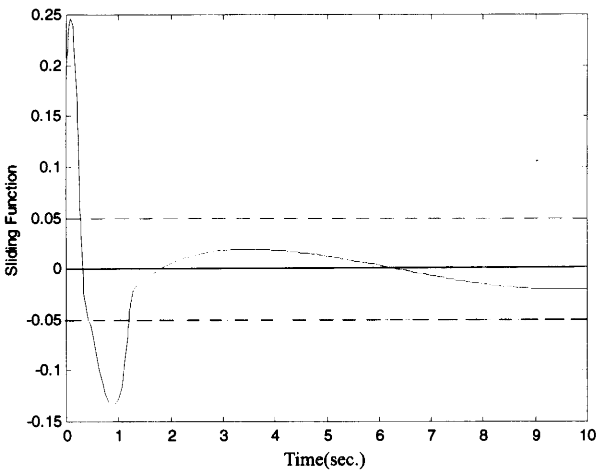

Fig. 2 Sliding function $\boldsymbol{s}$

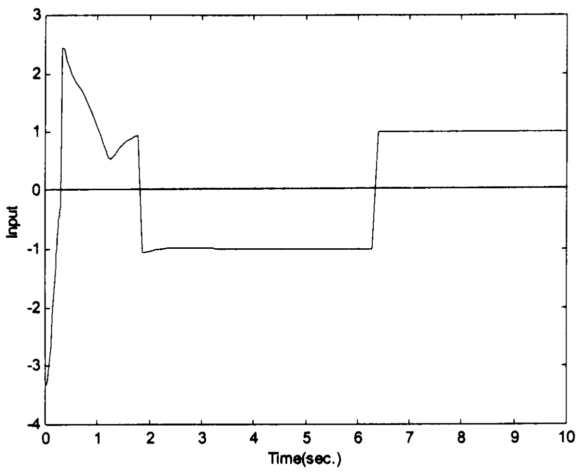

Fig. 3 Input $\boldsymbol{u}$

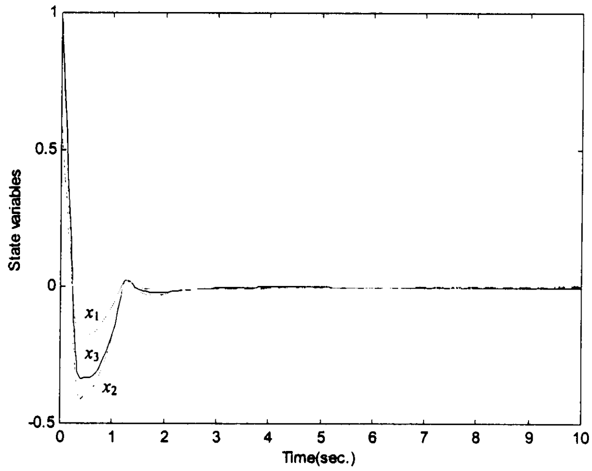

Fig. 4 State variables $\boldsymbol{x}$

Fig. 4, where $\boldsymbol{x} \to 0$.

6. Conclusions

An output sliding-mode control has been successfully developed to deal with the situation that only output information is available. It is still restricted to three assumptions concerning the system matrices and the matching uncertainty. However, the matching uncertainty depends on all the system state variables, not just the output. By adopting a state-estimator, the system state can be well estimated and then the sliding-mode control succeeds in suppressing the matching disturbance that fulfills the control goal.

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