

Beam-propagation-dominant instability in an axially pumped solid-state laser near degenerate resonator configurations

Ching-Hsu Chen, Ming-Dar Wei, and Wen-Feng Hsieh

Institute of Electro-Optical Engineering, National Chiao Tung University, 1001 Ta-Hsueh Road, Hsinchu 30050, Taiwan

Received December 5, 2000

Propagation-dominant instabilities and chaos were found under so-called good-cavity conditions in an axially pumped solid-state laser operated near the 1/3-degenerate cavity configuration that had not previously been studied numerically. By using the generalized Huygens integral together with rate equations, we obtained a V-shaped configuration that depends on a quasi-periodic threshold. We call the propagation dominant because the laser behaves as a conservative system governed by beam propagation. Although it had previously been predicted that chaos would be impossible under nearly degenerate conditions, we have recognized that the laser is transformed into chaos as a result of the interplay of beam propagation and gain dynamics as the cavity is tuned close to degeneracy. © 2001 Optical Society of America

OCIS codes: 140.0140, 140.1540, 140.3580, 140.4780, 140.5560.

1. INTRODUCTION

On the basis of beam propagation in a laser cavity that contains a homogeneously broadened gain medium, Melnikov *et al.*¹ constructed and analyzed a three-dimensional point map that presented the evolution of the beam parameters (spot size and wave-front curvature) together with the field intensity with which to model the nonlinear laser dynamics. They found that in a uniformly pumped laser with a high-loss cavity the laser has a continuously smooth quasi-periodic threshold throughout the geometrically stable region, except that some singular points that correspond to transverse mode degeneration² may become chaotic at high-power pumping.

However, by using a diffraction integral and a rate equation, Hollinger and co-workers^{3–5} studied the instability of single-longitudinal but multitransverse modes. They found that, in a laser with a high-loss cavity and uniform high-power pumping, the laser's output appears to be chaotic at the configurations that have a g_1g_2 parameter equal to 0.4 but to be only quasi-periodic at $g_1g_2 = 0.5$, which corresponds to the 1/4-degenerate configuration. In the former case, chaotic behavior was considered a condition in which the phase shift between adjacent transverse modes per round trip is an irrational multiple of π and does not lie close to any rational number with a small denominator. As in the configuration at g_1g_2 , whose phase shift is a rational multiple of π , the laser output behaves quasi-periodically, even with much higher pumping. This result contradicts the conclusion described in Ref. 1 that the laser behaves chaotically as the transverse mode degenerate configurations.

Lugiato *et al.*^{6,7} expressed the Maxwell–Bloch equations in terms of modal amplitudes by using a suitably cylindrically symmetric empty-cavity-mode expansion. They presented a variety of spatiotemporal instabilities, including chaos and cooperative frequency locking, that

occur under uniform and low-power pumping, by tuning the mode spacing. They were able to do this because the Laguerre–Gauss modes are a set of good bases only when the uniform-field limit is applied for a so-called good cavity with small gain. Thus their results are valid only for a laser in which the pump size is larger than the minimum cavity beam waist.⁸

By applying Greene's residue theorem⁹ to analyze the iterative map of a Gaussian beam q parameter of a general optical resonator,¹⁰ we found that, even when only fundamental mode propagation is considered, some specific configurations that have so-called low-order resonance⁹ may become unstable under the influence of persistent nonlinear effects.^{10–12} These configurations with g_1g_2 parameters equal to 1/2, 1/4, and 3/4 correspond to 1/4, 1/3, and 1/6 transverse-mode degenerate configurations, respectively. When only the optical Kerr effect is considered as the nonlinear dynamic parameter and the gain saturation effect is excluded, optical bistability¹¹ and multiple-period bifurcation¹² are possible for Kerr-lens mode-locked lasers. However, gain saturation provides an inherently nonlinear effect. When the pump size was smaller than the waist of the cold cavity, peculiar lasing behavior^{13,14} was observed in an end-pumped Nd:YVO₄ laser near these degenerate configurations. Low lasing threshold and beam waist shrinkage accompanied by multiple-pass transverse modes were exhibited.¹⁴

In this paper we focus on the configuration-dependent instabilities near 1/4, 1/3, and 1/6 transverse-mode degenerate configurations and consider only gain saturation as the nonlinear effect that occurs when the pump size is larger than the waist of the cold cavity, as described by Lugiato *et al.*^{6,7} The generalized Huygens integral and the rate equations were used to model dynamics of the Gaussian end-pumped solid-state laser. We found that a good-cavity laser exhibits a V-shaped quasi-periodic insta-

bility threshold rather than a smooth variation about these transverse-mode degenerate configurations, as the pump size is one to two times that of the waist of the cold cavity. Here a “good cavity” means that the cavity loss per round trip is $<10\%$ and also that a class-B laser condition¹⁵ rather than a bad-cavity condition produces Lorentz–Haken instability.¹⁵ With higher Gaussian pumping at the point of degeneration, the laser output cannot lead to chaotic behavior, unlike the result described by Melnikov *et al.*¹ but the same as that of Hollinger and Jung. However, the chaotic region is close to degeneration, a result that is different from the results of Ref. 3 because the phase shift between adjacent transverse modes in a round trip is close to $2\pi/3$ for the 1/3-degenerate configuration. Moreover, a variety of instabilities were found near the 1/3-degenerate configuration, and a frequency characterized by precession in phase space has been used to recognize them. We have organized the remainder of this paper as follows: In Section 2 we formulate the equations of motion. The numerical results are discussed and compared with previous research in Sections 3–6. In Section 7 we summarize our conclusions.

2. MODELING

Consider the plano–concave axially pumped solid-state laser shown in Fig. 1. It consists of a laser crystal with one of its end faces (*I*) high-reflection coated as the flat mirror and with a curved mirror with radius of curvature R separated from it by a distance L . Let the reference plane be the place where the light beam just leaves the laser crystal in the direction of the curved mirror. Under cylindrical symmetry, propagation of the light field toward the curved mirror and back to the flat mirror (end face *I* of the crystal) according to the generalized Huygens diffraction integral is

$$E_{m+1}^-(r) = \frac{2\pi j}{B\lambda} \int \exp(jk2L)E_m^+(r')\exp[-(j\pi/B\lambda) \times (Ar'^2 + Dr^2)]J_0(2\pi rr'/B\lambda)r'dr', \quad (1)$$

with round-trip transmission matrix

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}.$$

Here $E_m^+(r')$ and $E_{m+1}^-(r)$ are the electric fields of the m th and the $(m+1)$ st round trips at the planes immediately after and before the gain medium (denoted by the

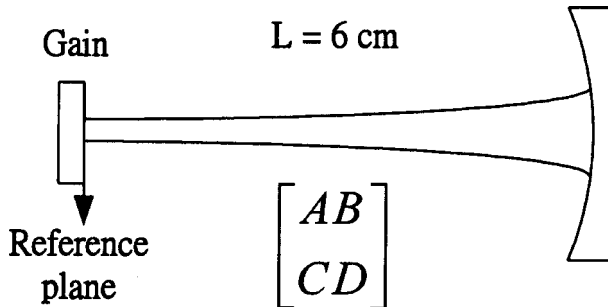


Fig. 1. Configuration of the laser system.

superscripts + and –), where r' and r are the corresponding radial coordinates, λ is the wavelength of laser, and J_0 is the Bessel function of zero order. In a thin-slab approximation, we can relate the electric fields E_{m+1}^+ to E_{m+1}^- (after and before the gain medium) in the same round trip as

$$E_{m+1}^+(r) = \rho E_{m+1}^-(r)\exp(\sigma\Delta Nd)\Pi(r/a), \quad (2)$$

where $1 - \rho^2$ is the round-trip energy loss, σ is the stimulated-emission cross section, ΔN is the population inversion per unit volume, d is the length of the active medium, and $\Pi(r/a)$ is an aperture function that equals 1 for r less than aperture radius a and equals 0 otherwise. Furthermore, assuming that the evolution of the population inversion follows the rate equation of a four-level system, we can write the rate equation as

$$\Delta N_{m+1} = \Delta N_m + R_{\text{pm}}(N_0 - \Delta N_m)\Delta t - \gamma\Delta N_m\Delta t - (|E_m|^2/E_s^2)\Delta N_m\Delta t, \quad (3)$$

where R_{pm} is the pumping rate, Δt is the travel time through the gain medium, E_s is the saturation parameter, γ is the spontaneous decay rate, and N_0 is the total density of the active medium. This method was used to model a single-longitudinal multitransversal high-power solid-state ring laser^{3–5} and to analyze the decay rate of standing-wave laser cavities in the linear regime.¹⁶ It was found that a standing-wave resonator can be approximated by a ring resonator if a thin gain medium is placed close to one of the end mirrors.¹⁷ For a continuous Gaussian pump profile $R_{\text{pm}} = R_{p0} \exp(-r^2/2w_p^2)$ with constant pumping beam radius w_p throughout the active medium (thin slab), the total pumping rate over the entire active medium is

$$\int R_{\text{pm}}dV = P_p/h\nu_p, \quad (4)$$

where P_p is the effective pumping power and $h\nu_p$ is the photon energy of the pumping laser. Because we considered only single-longitudinal-mode dynamics, we have omitted the dispersion of the active medium, so the gain is assumed to be real. Therefore we have four control parameters: ρ , R , w_p , and P_p , which play important roles in the laser system and are investigated in detail as follows.

In an ordinary axially pumped solid-state laser, the round-trip propagation time is many orders of magnitude shorter than the spontaneous decay time, especially in a short cavity. As a result, it would take a large number of iterations to arrive at the final state (which may be stable or unstable). To reduce computation time and because the quasi-periodic bifurcation point is just above the stable continuous-wave solution, we used the scaling method¹⁶ to magnify γ by 10^4 times to determine the bifurcation points. We also checked some important points without scaling that showed no promising change in the quasi-periodic threshold. To reduce the influence of the diffraction loss, we slightly varied R of the curved mirror rather than changing cavity length L to simulate tuning the laser cavity across the point of degeneration.

3. BEAM-PROPAGATION-DOMINANT DYNAMICS

As mentioned above, in the low-order-resonance configurations within the geometrically stable region the laser may become unstable under the influence of persistent nonlinear effects.¹⁰⁻¹² Thus we concentrate our simulation on these configurations, using the parameters of an axially pumped Nd:YVO₄ laser, a class B laser with $\lambda = 1.064 \mu\text{m}$, $1/\gamma = 50 \mu\text{s}$, $\sigma = 25 \times 10^{-19} \text{cm}^2$, $N_0 = 1.7 \times 10^{20} \text{cm}^{-3}$, $d = 1 \text{mm}$, a refractive index of 1.96, and $L = 6 \text{cm}$ at 808-nm pumping and setting aperture radius $a = 1 \text{mm}$, which is large enough for $g_1g_2 = 1/4$. In as much as our results are ascribed mainly to the dependence of laser dynamics on configuration as discussed below, we chose the cavity parameter product $g_1g_2 = 1 - L/R$ to be 1/2, 1/4, and 3/4 for studying nonlinear dynamics. These configurations correspond to $R = 8, 12, 24 \text{cm}$, respectively.

In the numerical simulations we set the initial values of E and ΔN to zero and added to Eq. (2) a term that simulates the spontaneous emission whose amplitude is given by the spontaneous decay term in Eq. (3) and a phase obtained from a random generator. To implement the generalized Huygens integral by the Romberg method we divided a 1-mm aperture into 1024 segments.

As expected, when the laser is continuously pumped slightly above the lasing threshold it starts with relaxation oscillation and eventually converges to the continuous-wave steady state. Because Δt is 1/30 of the round-trip time T_{cav} under magnification of γ by p times, the actual relaxation oscillation frequency f_r is equal to the numerical frequency multiplied by $[(T_{\text{cav}}/\Delta t)/p]^{1/2}$. When one increases the pump power beyond a certain level, bifurcation, or instability threshold, the laser output is no longer stable but becomes multiperiodic. Figure 2(a) shows the evolution of the laser pumped with Gaussian pump radius $w_p = 330 \mu\text{m}$ at $P_p = 313 \text{mW}$ above the bifurcation and $w_p = 309 \text{mW}$ at the configuration $g_1g_2 = 1/2$ with $\rho = 0.95$. Note that cavity beam radius w_0 is $\sim 142 \mu\text{m}$ and that Ψ , defined as w_p/w_0 , equals 2.32. The laser begins with the relaxation oscillation ($\sim 6.08 \text{MHz}$, corresponding to $f_r = 333 \text{kHz}$) followed by a short period of metastable output and finally develops into a flip-flopping steady-state period-2 solution. The corresponding field intensity profile, like the spot size on the plane-mirror end shown in Fig. 2(b), also flip-flops to repeat itself after two round trips, in contradiction to the regular situation of self-consistency after only one round trip. This result is equivalent to what is obtained from a stability analysis of a conservative map involving only Gaussian beam propagation, as in Fig. 4(a) of Ref. 10, where the rotation angle in phase space (spanned by spot size w and the curvature $1/R_g$) per round trip equals π for $g_1g_2 = 1/2$.

Similarly, both of the transverse-mode-degenerate configurations for the 1/3-degenerate configuration at $g_1g_2 = 1/4$ and the 1/6-degenerate configuration at $g_1g_2 = 3/4$ belong to the third-order resonance and need three round trips to repeat themselves (or period-3 solutions) in phase space.¹⁰ For the configuration slightly tuned away from its corresponding point of degeneration, e.g., at

$g_1g_2 = 0.25466$ or $R = 8.05 \text{cm}$, the laser shows nondecaying quasi-periodic oscillation [Fig. 3(a) and its inset]. We can see that the laser emission successively circulates in the resonator to form three branches of oscillation with a period of roughly 293 iterations. This is similar to the evolution of spot size in Fig. 3(b) of Ref. 10.

This period can be determined from $2\pi/|\theta - 2\pi/n|$, where θ is the rotation angle in phase space per round trip,¹⁰ $2\pi/n$ is the closest rational fraction rotation angle in phase space, and $n = 3$ in this case. In Fig. 3(b) we plot the evolution of the three consecutive states, 1, 2, and 3, in $(w, 1/R_g)$ space for the quasi-periodic case. Assume that the initial state is state 1 and that it will evolve in sequence and rotate at an angle θ per iteration (or per round trip). If $\theta/2\pi$ is a rational number, the dynamics is periodic. Contrarily, if $\theta/2\pi$ is irrational, the initial state will never repeat itself but will precess an angle $n(\theta - 2\pi/n)$ (or recede for a negative angle) in phase space after n iterations. As a consequence, an arbitrary initial state will nearly return to itself but will precess (or recede) a minimal angle after $T_p = 2\pi/(n|\theta - 2\pi/n|)$ iterations. We therefore define the precession frequency f_p as $c/2LT_p$.

The power spectrum [see Fig. 3(c)] of Fig. 3(a) shows that a low-frequency peak at 25.6 MHz is f_p , which equals the beat frequency (f_b) of the two nearly degener-

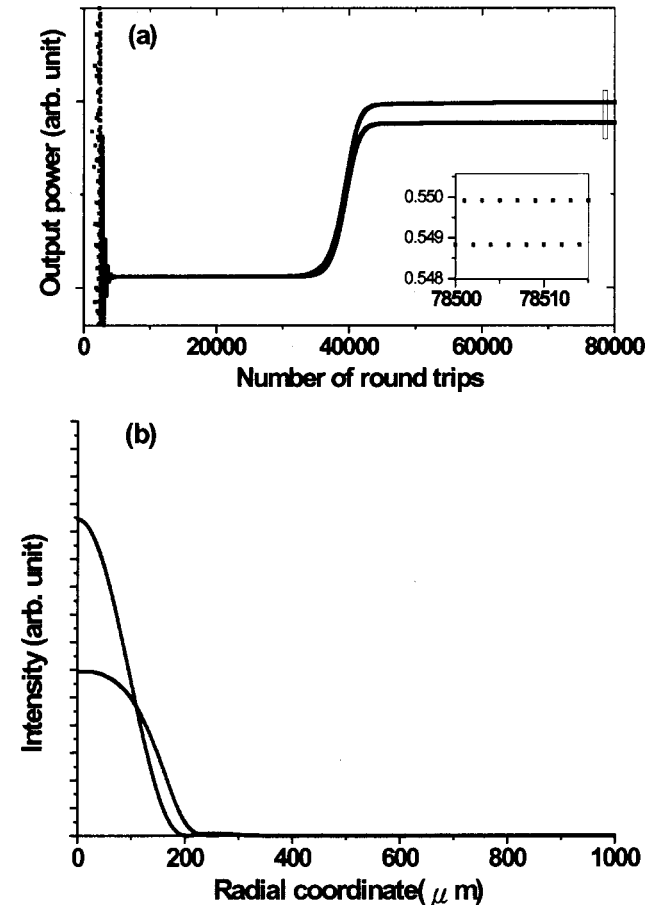


Fig. 2. (a) Output power evolution and (b) beam profile of the period-2 steady state for $g_1g_2 = 1/2$ with $\rho = 0.95$ above the instability threshold.

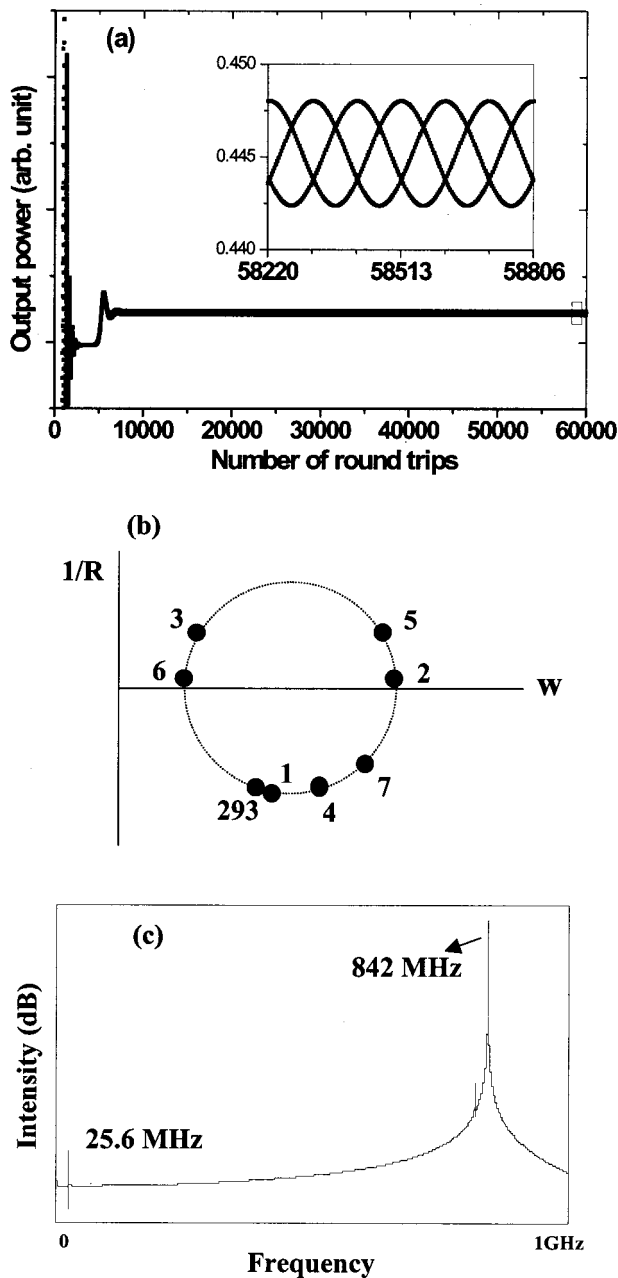


Fig. 3. (a) Evolution of the output power of the quasi-periodic oscillation at $g_1g_2 = 0.25466$, $\rho = 0.95$, $\Psi = 2.32$, and $P_p = 210$ mW. The inset is the magnification of six precession periods. (b) Phase space of $(w, 1/R)$, as in Ref. 10, provides an explanation. The numbered filled circles stand for the number of iterations. (c) The corresponding spectrum of (a).

ate Laguerre–Gauss modes, $LG_{n,0,0}$ and $LG_{n-2,3,0}$. Although there is a lower-order $LG_{n-1,1,1}$ mode that degenerates with the fundamental mode, it will not be excited under cylindrical symmetry. The highest peak at 842 MHz results from circulating among those three oscillating branches and returning to the initial branch every three round trips (as the longitudinal mode spacing is 2.5 GHz). This peak is accompanied by a sideband owing to beating with a 25.6-MHz peak. We say that the laser is beam-propagation dominant because it behaves as a conservative system governed by Gaussian beam propaga-

tion. Therefore, when the laser is axially pumped a bit above the quasi-periodic threshold about the point of degeneration, the laser behaves as if the beam propagation were dominant.

It is interesting to note that the laser output behaves the same as in a lossless optical resonator described by a conservative map. It seems that the laser will become a conservative system, although it does include dissipative elements such as gain and Gaussian (pump) aperture. For low pumping, the Gaussian gain profile is a weak Gaussian aperture and provides the damping mechanism; however, because there is already saturated gain above the instability threshold, the effective radius of the aperture increases. As a result, cavity field propagation dominates the laser dynamics and behaves as conservative propagation as illustrated in Ref. 10.

4. THRESHOLD OF A QUASI PERIOD

A. Good Cavity

Figure 4(a) is a three-dimensional bifurcation diagram that shows quasi-periodic instability threshold P_2 at various values of Ψ near $g_1g_2 = 1/4$ for $\rho = 0.95$. It is obvious that the system has a V-shaped quasi-periodic threshold with a local minimum at the point of degeneration over $1 < \Psi \leq 2$. The farther the cavity is tuned away from degeneration, the higher the quasi-periodic threshold is. This result confirms our previous prediction¹⁰ that the degenerate configuration is unstable under the nonlinear effect. Moreover, the V-shaped threshold is deeper as Ψ is close to 1 and becomes flat for large Ψ . This shows that the quasi-periodic threshold is independent of cavity configuration if a uniform pump is used, as reported in Ref. 6. Similar results can be obtained with other degeneration configurations.

Fixing $g_1g_2 = 1/4$, we plotted the ratio (P_2/P_1) of quasi-periodic threshold P_2 to lasing threshold P_1 versus Ψ [Fig. 4(b)]. The ratio approaches 1 with uniform pumping for Ψ approaching infinity, and it increases sharply as Ψ becomes close to 1. The result is the same as that derived by Lugiato *et al.*,⁶ namely, that instability in terms of the threshold ratio favors a cavity operated with large Ψ , where it is easier to excite multitransverse modes to develop spatiotemporal instabilities. However, as the lasing threshold increases monotonically as a function of Ψ , a minimal quasi-periodic threshold power of 175 mW occurs at $\Psi \approx 2.3$, where $P_2/P_1 \approx 1.8$. Furthermore, the lasing threshold is almost independent of g_1g_2 about the point of degeneration as $\Psi > 1$; thus the lowest quasi-periodic threshold at degeneration that is due to sensitivity to nonlinear effects is demonstrated in Ref. 10.

B. High-Loss Cavities

We have discussed cavity-configuration-dependent laser dynamics under the good-cavity condition with $\rho = 0.95$. To examine the influence of cavity loss on laser dynamics, we have chosen values of ρ of 0.95, 0.9, 0.8, and 0.7 for $\Psi = 1.3$. From Fig. 4(c) we found that the V-shaped threshold behavior disappears as ρ decreases to 0.7. It develops into a monotonically increasing smooth curve with respect to g_1g_2 , and the threshold at degeneration is no longer a local minimum. This smooth curve

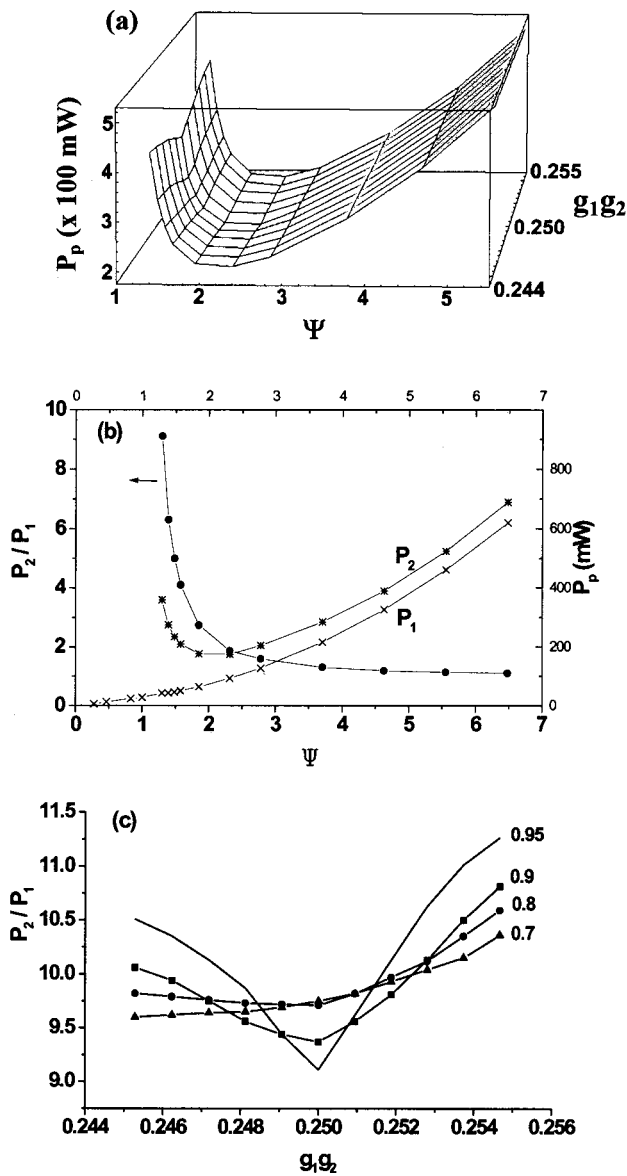


Fig. 4. (a) Three-dimensional quasi-periodic bifurcation diagram in terms of P_p , Ψ , and $g_1 g_2$ for $\rho = 0.95$. (b) Dependence of the ratio of the instability threshold (P_2) to the lasing threshold (P_1) on parameter Ψ for $g_1 g_2 = 1/4$ and $\rho = 0.95$. (c) Dependence of P_2/P_1 on $g_1 g_2$ for $\Psi = 1.3$ with different values of ρ as indicated.

is similar to that described in Ref. 1 for 50% mirror reflection or $\rho^2 = 0.5$. As mentioned above, the laser with high mirror reflectivity mimics a conservative system and becomes propagation dominant when it is operated in a quasi-periodic state. Thus the V-shaped quasi-periodic threshold will not be found in the research reported in Ref. 1, where a high-loss cavity was considered, nor in Ref. 7, with uniform pumping. Note that these curves are asymmetric. Normally, if the aperture radius and the cavity length are both constant, the larger the $g_1 g_2$ parameter, the larger the spot size is on the flat mirror, which is also a gain medium. Because diffraction loss is minimized by choice of a sufficiently large aperture, in our simulation the asymmetry is ascribed mainly to a change in the overlap integral of the cavity field with the pumping as $g_1 g_2$ varies.

5. BIFURCATION DIAGRAM

It is worth noting that, when the laser is pumped just above the bifurcation, the stripe denoted for quasi-periodic oscillation in Fig. 3(a) has less than a 1% varia-

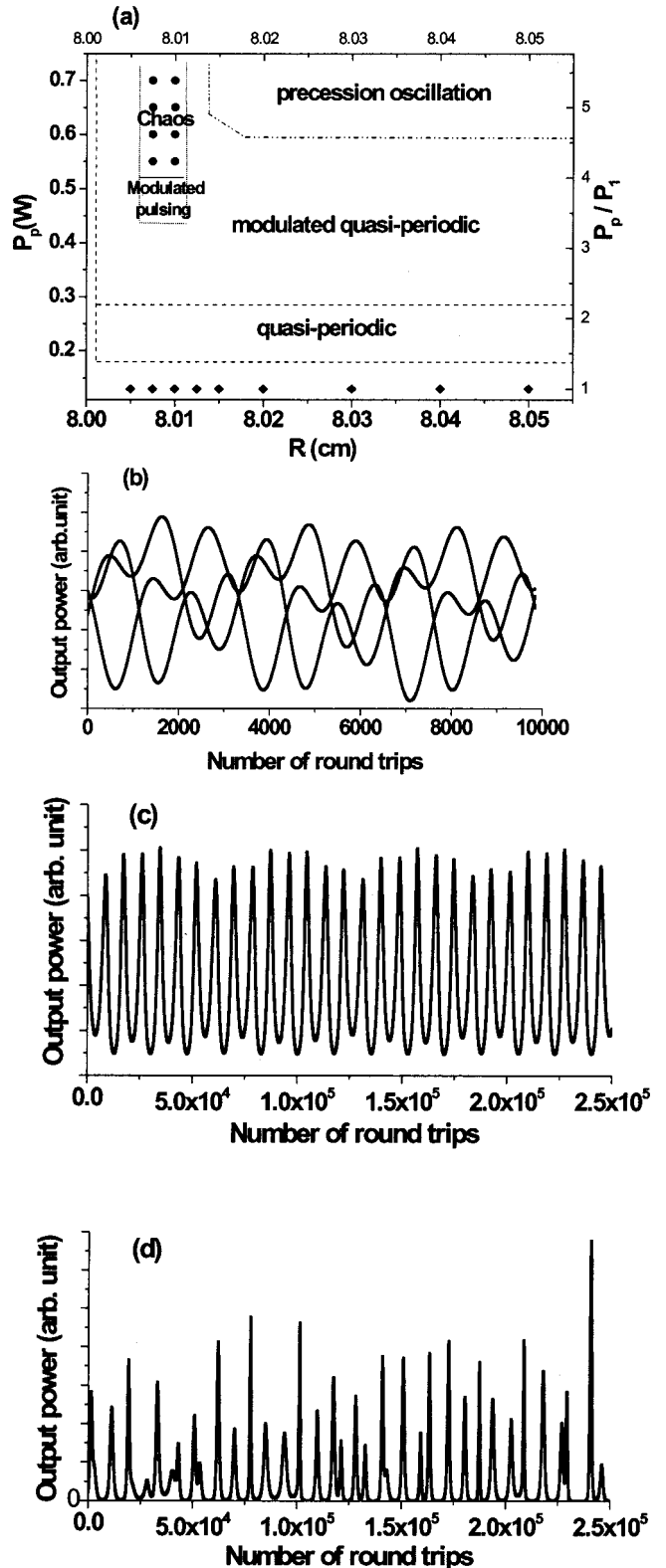


Fig. 5. (a) Bifurcation diagram for higher pumping with $\rho = 0.95$ and $\Psi = 2.78$ near $g_1 g_2 = 1/4$, (b) power evolution of a modulated quasi period, (c) modulated pulsing, (d) chaos.

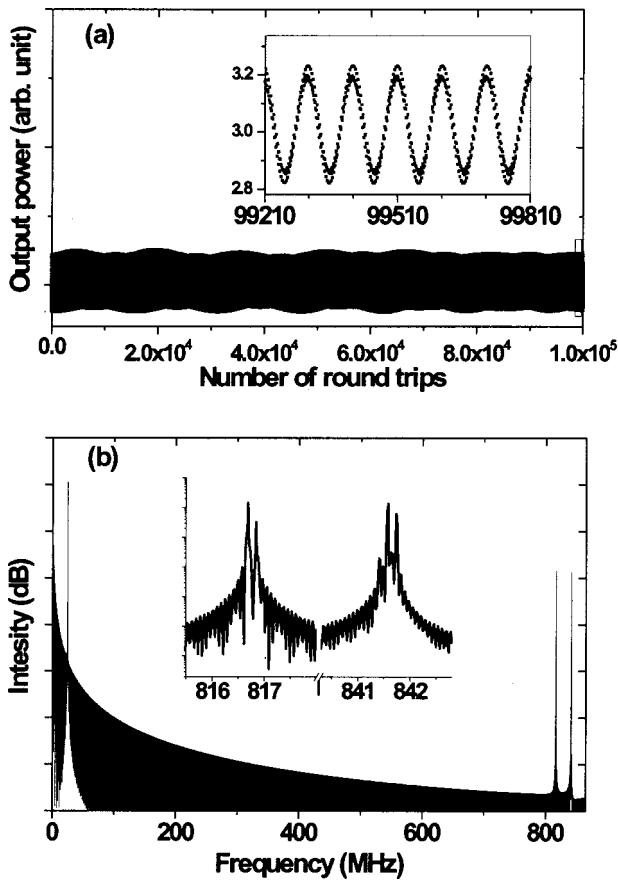


Fig. 6. (a) Precession oscillation and (b) power spectrum at $g_1g_2 = 0.25466$ with $\rho = 0.95$, $\Psi = 2.78$, and $P_p = 650$ mW.

tion. Raising the pump power makes the stripe wider. When the pump power is further increased, the laser is operated far away from the linear regime, and its highly saturated gain may cause serious instabilities. Because the instabilities induced by higher pumping may depend on the spontaneous decay rate, we magnify γ only 10 times to investigate the high-pumping condition. In fact, there are minor differences compared with scaling γ by 10 times and 100 times if we simply want to classify the types of instability.

With a spatially inhomogeneous pump, because of competition between two transverse modes a laser can produce chaotic emission.¹⁸ Thus we suspect that it might be easier to obtain chaos at a minimal quasi-periodic threshold where $\Psi = 2.32$. Indeed, we tried the pump power up to $7 P_1$ at $\Psi = 2.32$, but chaos was not found. By using $\Psi = 2.78$ we can classify the many kinds of instability shown in Fig. 5(a) for $\rho = 0.95$. For instance, we defined the so-called modulated quasi-periodic state shown in Fig. 5(b) for $R = 8.0075$ cm and $P_p = 400$ mW. Further increasing the pump power, we found so-called modulated pulsing and chaos when $R = 8.0075$ cm, as shown in Figs. 5(c) and 5(d) for $P_p = 500$ and $P_p = 600$ mW, respectively. In the region where $R > 8.015$ cm, the laser is in the so-called precession oscillation state, showing three overlapped sinusoidallike oscillations, as illustrated in Fig. 6(a) for $R = 8.05$ cm and $P_p = 650$ mW. Its corresponding power spectrum is shown in Fig. 6(b). We can see the power spectrum that

has a precession frequency of 24.75 MHz that is close to 25.6 MHz for lower pumping as in Fig. 3(c). The precession oscillation appears to be soft in amplitude and hard in frequency, even for P_p as much as 1 W. The inset in Fig. 6(b) is the expansion of a high-frequency spectrum. The main peak at 841.6 MHz again corresponds to one third of the longitudinal mode spacing, and the peak at 816.6 MHz, which has downshifted ~ 25 MHz, corresponds to f_p . The presence of small peak, located 150 ± 25 kHz beside the main peaks, is ascribed to beating with the subharmonics of the relaxation oscillation. Note that the numerical relaxation oscillation frequency is now ~ 350 kHz.

We have plotted in Fig. 7(a) the frequencies of the spectral peaks as P_p increases for $R = 8.05$ cm. At low pump, we had only two peaks, separated by f_p , until $P_p = 400$ mW, a sideband attributed to frequency beating with relaxation oscillation, appeared in the high-frequency region. To show how the spectrum develops as P_p increases, we used a filled circle bisected by a shortline to mark the highest peak in that group of spectral peaks. We found not only that the precession frequency is slightly redshifted but also that the subharmonic of the relaxation oscillation appears as increasing P_p ; for instance, the frequency spacing of the main peak and its sideband at $P_p = 600$ mW is half that at $P_p = 500$ mW. Figure 7(b) shows the bifurcation diagram for $R = 8.0075$ cm, which is closer to the 1/3-degenerate con-

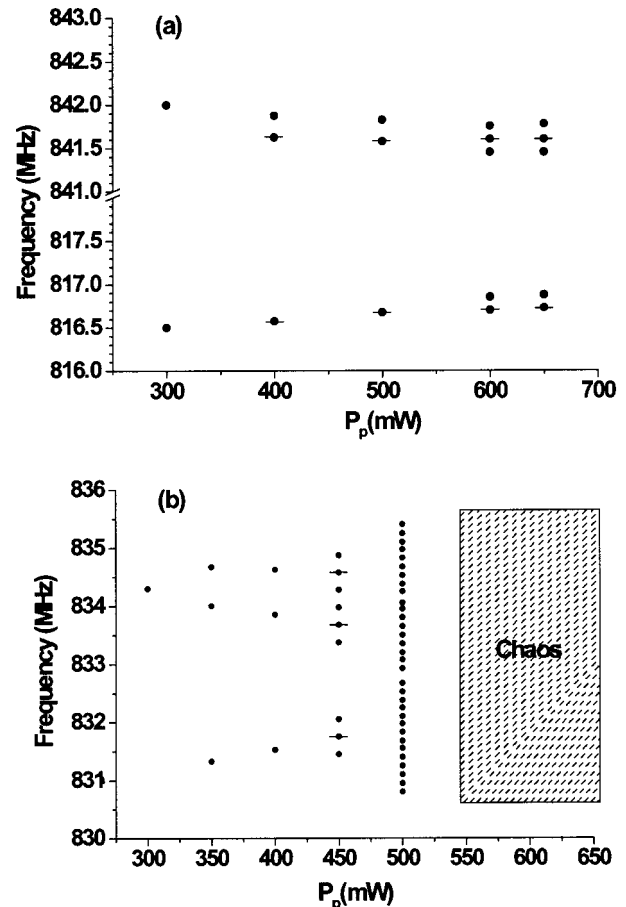


Fig. 7. Frequency bifurcation plot using P_p as the parameter for (a) $R = 8.05$ cm and (b) $R = 8.0075$ cm.

figuration. As $P_p = 350\text{--}400\text{ mW}$, the difference between the two peaks near 834 MHz approaches the third harmonic of the relaxation oscillation frequency. Each of the three main peaks shows sideband frequencies owing to beating with the relaxation oscillation at $P_p = 450\text{ mW}$. Further increasing the pumping, we observed increasingly more sidebands caused by beating with the subharmonics of the relaxation oscillation; finally, the laser became chaotic. We believe that either increasing the pumping or tuning the cavity configuration toward the point of degeneration will enhance the gain dynamic effect and will cause subharmonic bifurcation owing to nonlinear gain. A transition from a mechanism that is dominated by beam propagation to one dominated by gain dynamics will result. We can also suppress the diffraction effect by reducing the reflectance; for example, when $\rho = 0.8$, the region of chaos becomes wider and farther away from degeneration than for $\rho = 0.95$.

6. INTERPLAY OF BEAM PROPAGATION AND GAIN DYNAMIC BIFURCATION

If we maintain proper pump power and scan over the whole range of R in Fig. 5(a) we will obtain instabilities similar to those described in Ref. 7, in which the transverse mode spacing varies about $g_1g_2 = 1$. We can also achieve the results of Fig. 5(a) by using Fig. 6 of Ref. 19 where chaos exists within small ranges of phase difference (which corresponds to R in our case) and round trip loss (or $1 - \rho^2$).

Maintaining P_p at 650 mW, we show in Fig. 8 the transverse beat frequency (f_b) of the cold cavity and the precession frequency (f_p) relative to R for $\rho = 0.95$. We have found that the numerical precession frequency nearly equals the transverse beat frequency when the laser is propagation dominant as $R \geq 8.03\text{ cm}$. Another evidence that propagation-dominant instability is surely governed by the diffraction integral is that the precession frequency is independent of the spontaneous-emission rate or gain. As R is tuned toward the degenerate or the chaotic region, however, precession frequency f_p deviates from the transverse beat frequency because the gain dynamics, like the rate equations, begin to play a crucial role in change of the precession frequency. The gain aperture and saturation effects take control of the dynamics when the precession frequency declines to several times the relaxation oscillation frequency as $R \approx 8.01\text{ cm}$. In Fig. 7(b), for small P_p , two frequencies, ~ 834 and $\sim 831\text{ MHz}$, appear to be quasi-periodic, mainly because of beam propagation or diffraction, so their difference is understood as f_p . The laser has increasingly sideband frequencies as a result of period multiplication owing to the nonlinear gain through the rate equations for larger P_p . It seems that the route to chaos close to degeneration is the interplay (or the mixing effect) of the quasi-period and the period multiplied as shown in Fig. 7(b).

We believe that the cavity loss $1 - \rho^2$ is the key factor that differentiates the results of Melnikov *et al.*¹ and Hollinger *et al.*^{5,19} from ours. The V-shaped threshold becomes as smooth as Melnikov's result [Fig. 4(c)] for a high-loss cavity. Hollinger *et al.* obtained their results

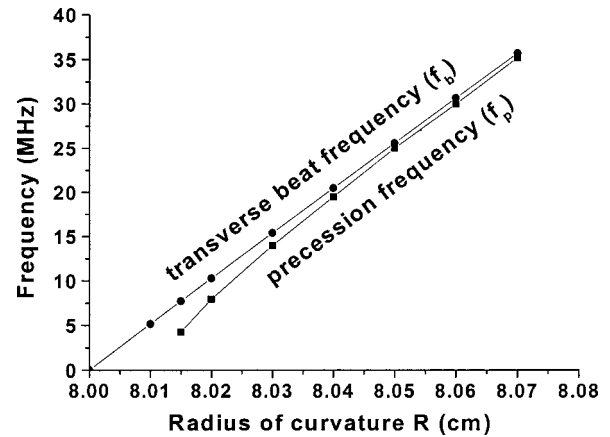


Fig. 8. Transverse beat frequency of the cold cavity and the precession frequency versus R for P_p fixed at 650 mW.

with a high-loss cavity, but they did not investigate how close g_1g_2 should be to 0.5 for the laser output to be quasi-periodic but not to become chaotic,³ however, in our good-cavity case the chaotic region becomes narrower and can be close to degeneracy.

7. CONCLUSIONS

Numerically propagating a cavity field through the generalized Huygens integral and using the atomic rate equations for a homogeneously broadened gain medium with Gaussian pumping, we obtained propagation-dominant laser instabilities. We have investigated in detail the temporal behavior of the instabilities near the 1/3-degenerate configuration. We determined the quasi-periodic threshold as the cavity was tuned across the degenerate configurations. A laser with a good cavity including a saturated gain medium shows a V-shaped quasi-periodic threshold; however, a high-loss cavity has not a V shape but a smooth monotonic curve. Furthermore, the propagation-dominant V-shaped threshold depends not only on the resonator configuration but also on the pump size. There is a best value Ψ in a good cavity to produce the lowest-instability pump power. In addition to a quasi-periodic region, we obtained another region of propagation-dominant instability outside the chaotic region near the 1/3-degenerate configuration in the good-cavity conditions. We ascribed this type of instability to the special dependence of the geometrical configuration.

Furthermore, chaos was found in a good cavity close to the 1/3-degenerate configuration. Although the phase shift between adjacent transverse modes in one round trip is irrational multiples of π and lies close to $2\pi/3$, the laser output can become chaotic in a good cavity under Gaussian pumping. This result is different from the results of Hollinger *et al.*^{5,19} We believe that, as the cavity is tuned toward 1/3 degeneration, the beam-propagation-dominant laser dynamics is transformed into an interplay of beam propagation and gain dynamics. Thus the route to chaos close to the degenerate configuration involves the mixing effect of quasi-period- and period-multiplying bifurcation.

ACKNOWLEDGMENTS

This research was partially supported by the National Science Council (NSC) of the Republic of China under grant NSC89-2112-009-071 and by the Ministry of Education of the Republic of China under grant 89-E-FA-1-4. C. H. Chen gratefully acknowledges fellowship from the NSC.

W.-F. Hsieh's e-mail address is wfhsieh@cc.nctu.edu.tw.

REFERENCES

1. L. A. Melnikov, S. A. Tatarkova, and G. N. Tatarkov, "Non-linear dynamics of beam parameters and intensity in a unidirectional ring laser with a homogeneously broadened line," *J. Opt. Soc. Am. B* **7**, 1286–1292 (1990).
2. A. E. Siegman, *Lasers* (University Science, Mill Valley, California, 1986), p. 761.
3. F. Hollinger and Chr. Jung, "Single-longitudinal-mode laser as a discrete dynamical system," *J. Opt. Soc. Am. B* **2**, 218–225 (1985).
4. R. Hauck, F. Hollinger, and H. Weber, "Chaotic and periodic emission of high power solid state lasers," *Opt. Commun.* **47**, 141–145 (1983).
5. F. Hollinger, Chr. Jung, and H. Weber, "Quasiperiodicity versus chaos in high power solid state lasers in multitransversal mode operation," *Opt. Commun.* **75**, 84–92 (1990).
6. L. A. Lugiato, F. Prati, L. M. Naducci, P. Ru, J. R. Tredicce, and D. K. Bandy, "Role of transverse effects in laser instabilities," *Phys. Rev. A* **37**, 3847–3866 (1988).
7. L. A. Lugiato, G. L. Oppo, J. R. Tredicce, L. M. Narducci, and M. A. Pernigo, "Instabilities and spatial complexity in a laser," *J. Opt. Soc. Am. B* **7**, 1019–1033 (1990).
8. G. D'Alessandro and G. L. Oppo, "Gauss–Laguerre modes: a 'sensible' basis for laser dynamics," *Opt. Commun.* **88**, 130–136 (1992).
9. J. M. Greene, "A method for determining a stochastic transition," *J. Math. Phys.* **20**, 1183–1201 (1979).
10. M. D. Wei, W. F. Hsieh, and C. C. Sung, "Dynamics of an optical resonator determined by its iterative map of beam parameters," *Opt. Commun.* **146**, 201–207 (1998).
11. M. D. Wei and W. F. Hsieh, "Bifurcation of fundamental gaussian modes in Kerr-lens mode-locked lasers," *Opt. Commun.* **168**, 161–166 (1999).
12. M. D. Wei and W. F. Hsieh, "Cavity configuration dependent nonlinear dynamics in Kerr lens mode-locked lasers," *J. Opt. Soc. Am. B* **17**, 1335–1342 (2000).
13. H. H. Wu and W. F. Hsieh, "Observations of multipass transverse modes in an axially pumped solid state laser with different fractionally degenerate resonator configurations," *J. Opt. Soc. Am. B* **18**, 7–12 (2001).
14. H. H. Wu, C. C. Sheu, T. W. Chen, M. D. Wei, and W. F. Hsieh, "Observation of power drop and low threshold due to beam waist shrinkage around critical configurations in an end-pumped Nd:YVO₄ laser," *Opt. Commun.* **165**, 225–229 (1999).
15. C. O. Weiss and R. Vilaseca, *Dynamics of Lasers* (Weinheim, New York, 1991), pp. 14, 26.
16. Y. J. Cheng, P. L. Mussche, and A. E. Siegman, "Cavity decay rate and relaxation oscillation frequency in unconventional laser cavities," *IEEE J. Quantum Electron.* **31**, 391–398 (1995).
17. M. Moller, L. M. Hoffer, G. L. Lippi, T. Ackemann, A. Gahl, and W. Lange, "Fabry–Perot and ring cavity configurations and transverse optical patterns," *J. Mod. Opt.* **45**, 1913–1926 (1998).
18. M. L. Shih and P. W. Milonni, "Chaotic two-mode lasing," *Opt. Commun.* **49**, 155–160 (1984).
19. F. Hollinger, Chr. Jung, and H. Weber, "Simple mathematical model describing multitransversal solid-state lasers," *J. Opt. Soc. Am. B* **7**, 1013–1018 (1990).