



SPATIAL DISORDER OF CNN — WITH ASYMMETRIC OUTPUT FUNCTION

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This investigation will describe the spatial disorder of one-dimensional Cellular Neural Networks (CNN). The steady state solutions of the one-dimensional CNN can be replaced as an iteration map which is one dimensional under certain parameters. Then, the maps are chaotic and the spatial entropy of the steady state solutions is a three-dimensional devil-staircase like function.

1. Introduction

Following their introduction by Chua and Yang [1988a, 1988b], Cellular Neural Networks have been extensively studied and applied mainly in image processing and pattern recognition [Thiran *et al.*, 1995; Chua & Roska, 1993]. An important class of solutions of one-dimensional CNN

$$\frac{dx_i}{dt} = -x_i + z + \alpha f(x_{i-1}) + a f(x_i) + \beta f(x_{i+1}), \quad (1)$$

is the steady state solutions, thus necessitating the study of the complexity of steady state solutions of (1). Juang and Lin [1998, 2000] and Hsu and Lin [1999a, 1999b, 2000] recently considered some mathematical results about the complexity of steady state solutions and multiplicity of traveling wave solutions. Hsu and Lin [1999a] considered the

output function of (1) with

$$f(x) = \begin{cases} rx + 1 - r & \text{if } x \geq 1, \\ x & \text{if } |x| \leq 1, \\ rx + r - 1 & \text{if } x \leq -1. \end{cases} \quad (2)$$

They described the spatial entropy of steady state solutions as a devil-staircase like function.

The investigation elucidates the complexity of a set of bounded steady state solutions of (1). Herein $f(x)$ is a piecewise-linear output function defined by

$$f(x) = \begin{cases} rx + m - r & \text{if } x \geq 1, \\ mx & \text{if } |x| \leq 1, \\ lx + l - m & \text{if } x \leq -1, \end{cases} \quad (3)$$

where $r, m, l \in \mathbb{R}^+ \setminus \{0\}$ are constants and the quantity z is called threshold, which is related to independent voltage sources in electric circuits. The

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coefficients of $f(x)$ are real constants and called the space invariant A -template denoted by

$$A \equiv [\alpha, a, \beta]. \tag{4}$$

For simplicity, let $m = 1$ in (3). That is,

$$f(x) = \begin{cases} rx + 1 - r & \text{if } x \geq 1, \\ x & \text{if } |x| \leq 1, \\ lx + l - 1 & \text{if } x \leq -1. \end{cases} \tag{5}$$

Let output $v = f(x)$ be taken as the unknown variable, i.e.

$$v_i = f(x_i), \tag{6}$$

and let F be the inverse function of f . When $\alpha = 0$, $\beta \neq 0$, the steady state solutions of (1) can be written as a one-dimensional iteration map:

$$T(v) = \frac{1}{\beta}(F(v) - z - av). \tag{7}$$

For this map, each bounded trajectory corresponds to the outputs of bounded steady state solutions. If the maps are chaotic, then the steady state solutions of (1) are of spatial disorder. However, only steady state solutions of (1) should be considered. Therefore, in addition to considering the set of all stable bounded orbits of T , the entropy h of T on the set must be computed as well. If the entropy is positive, then the steady state solutions of (1) are of spatial disorder. In fact, we have the following main theorem:

Main Theorem. Assume that $\alpha = 0$, $\beta > 0$, $z = 0$, $a > \beta + 1$ and $h(r, l)$ is the entropy function

of T with $F = f^{-1}$, $r, l > 0$. Denote

$$r_\infty = l_\infty = \frac{a - \beta - 1}{a(a - 1) + \beta(a - 2)}, \tag{8}$$

then there exists strictly decreasing sequences $\{r_p\}$, $\{l_q\}$, $p, q = 2, 3, \dots$, with

$$\lim_{p \rightarrow \infty} r_p = r_\infty, \tag{9}$$

$$\lim_{q \rightarrow \infty} l_q = l_\infty, \tag{10}$$

such that

- (i) If $0 < r \leq r_\infty$ and $0 < l \leq l_\infty$, then $h(r, l) = \ln 2$.
- (ii) If $r_p \leq r < r_{p-1}$ and $l_q \leq l < l_{q-1}$, for $p, q = 3, 4, 5, \dots$, then $h(r, l) = \ln \lambda_{(p,q)}$, where $\lambda_{(p,q)}$ is the maximum root of

$$x^{p+q-2} - \sum_{i=0}^{p-2} x^i \sum_{j=0}^{q-2} x^j = 0. \tag{11}$$

- (iii) If $r_2 \leq r < (1)/(a + \beta)$ and $l_2 \leq l < (1)/(a + \beta)$, then $h(r, l) = 0$.

A table can be constructed based on the results of the above theorem to contrast the entropy between different r, l as in Fig. 1. Moreover, a three-dimensional graph can be designed as shown in Fig. 2.

The paper is organized as follows. In Sec. 2, we will consider the basic propositions of T and study the steady state solutions of (1) for some range of

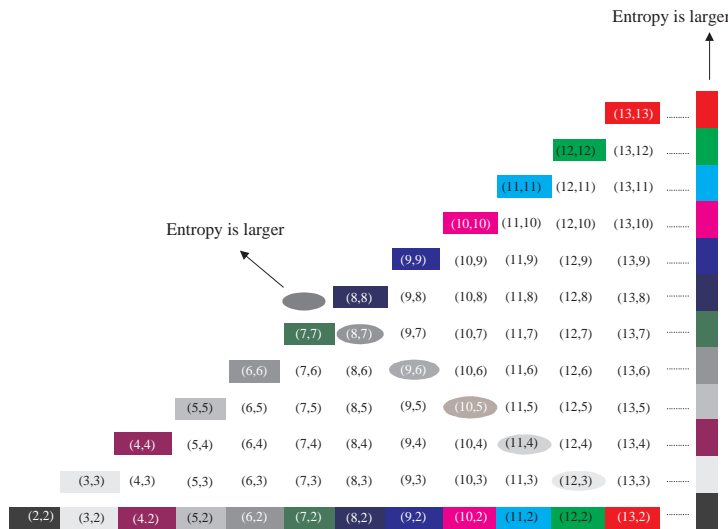


Fig. 1.

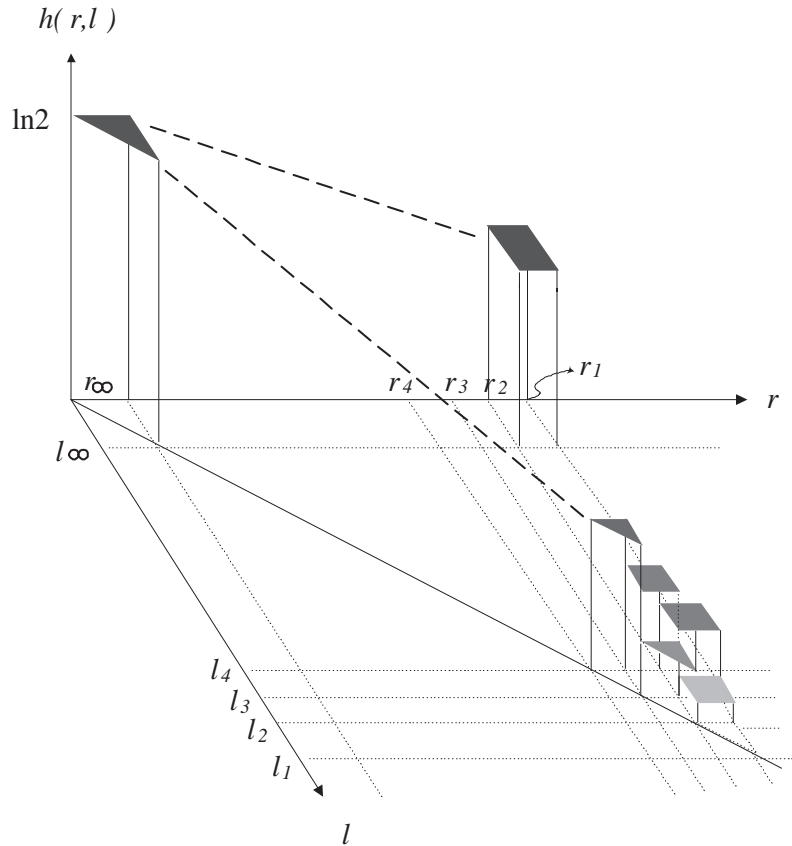


Fig. 2. $r_1 = l_1 = (1)/(\alpha + \beta)$.

parameters. Section 3 will prove the main theorem and construct the table in Fig. 1.

2. Iteration Map

This section considers the one-dimensional map (7). If $\alpha = 0$, $a > 1$, $\beta > 0$, $z = 0$ and $m = 1$, then according to (5), the inverse function F of f is

$$F(v) = \begin{cases} \frac{1}{r}v - \frac{1}{r} + 1 & \text{if } v \geq 1, \\ v & \text{if } |v| \leq 1, \\ \frac{1}{l}v - 1 + \frac{1}{l} & \text{if } v \leq -1, \end{cases} \quad (12)$$

and, according to (7), the map T is

$$T(v) = \begin{cases} \frac{1}{\beta} \left(\frac{1}{r}v - \frac{1}{r} + 1 - av \right) & \text{if } v \geq 1, \\ \frac{1}{\beta}(1 - a)v & \text{if } |v| \leq 1, \\ \frac{1}{\beta} \left(\frac{1}{l}v - 1 + \frac{1}{l} - av \right) & \text{if } v \leq -1. \end{cases} \quad (13)$$

By elementary computation, the fixed points of T are

$$A = (A_1, A_2) = \left(\frac{1 - r}{1 - r(a + \beta)}, \frac{1 - r}{1 - r(a + \beta)} \right),$$

$$O = (O_1, O_2) = (0, 0),$$

$$D = (D_1, D_2) = \left(\frac{l - 1}{1 - l(a + \beta)}, \frac{l - 1}{1 - l(a + \beta)} \right). \quad (14)$$

Let B and C be the points $(1, T(1))$, $(-1, T(-1))$, i.e.

$$B = (B_1, B_2) = \left(1, \frac{1 - a}{\beta} \right), \quad (15)$$

$$C = (C_1, C_2) = \left(-1, \frac{a - 1}{\beta} \right). \quad (16)$$

Therefore, some graphs of T are shown in the following figures.

Now, we define the interval L by

$$L \equiv \left\{ v \in \mathbb{R} \mid \frac{l - 1}{1 - l(a + \beta)} \leq v \leq \frac{1 - r}{1 - r(a + \beta)} \right\}. \quad (17)$$

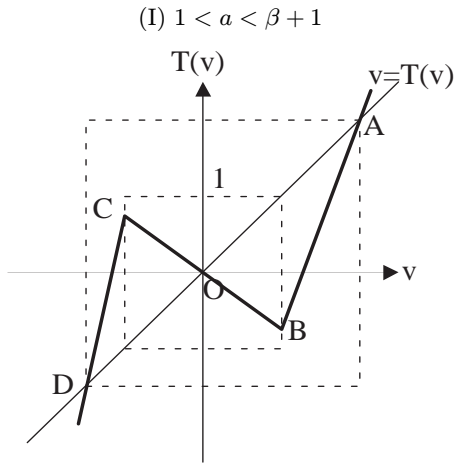


Fig. 3. Graph of T .

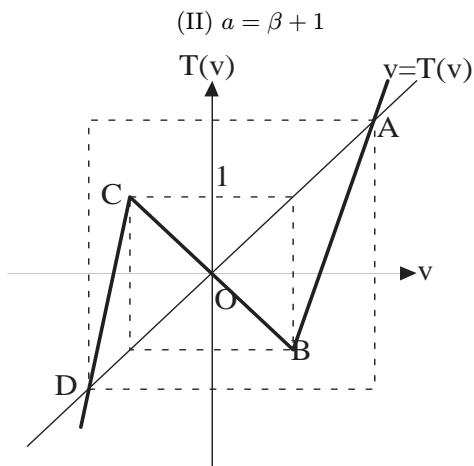


Fig. 4. Graph of T .

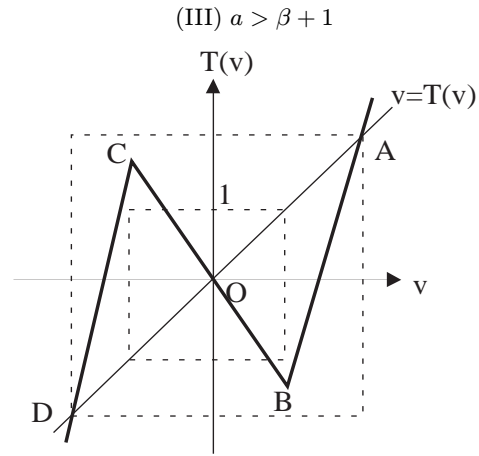


Fig. 5. $r > r_\infty$ and $l > l_\infty$.

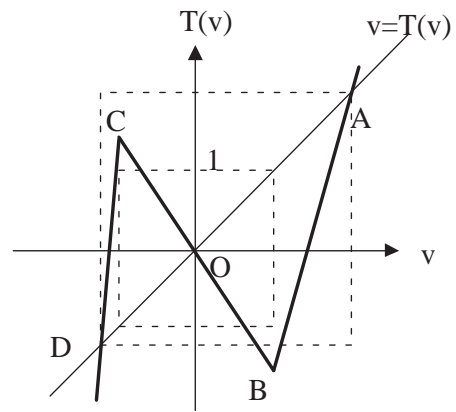


Fig. 6. $r > r_\infty$ and $0 < l < l_\infty$.

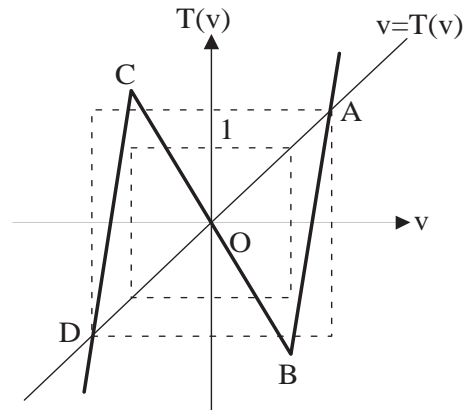


Fig. 7. Graph of T with $0 < r < r_\infty$ and $0 < l < l_\infty$.

It is easy to see the stability of the fixed points of T as follows.

Proposition 2.1

- (1) If $a \geq 1$ with $(1-r)/(1-r(a+\beta)) \geq (a-1)/\beta$ and $(l-1)/(1-l(a+\beta)) \leq (1-a)/\beta$ then $\Omega \equiv \{(s, t) \in \mathbb{R}^2 | s \in L \text{ and } t \in L\}$ is an invariant region of T .
- (2) If $1 < a < \beta + 1$, then both A and D are unstable, but O is stable.
- (3) If $a = \beta + 1$, then every point in L is eventually periodic with period 2 except for the fixed points.
- (4) If $a > \beta + 1$, then O, A, D are all unstable.

By applying Proposition 2.1, the chaotic behavior of trajectories of T only occurs when $a > \beta + 1$. Next, the stability results of steady state solutions of (1) are studied.

Definition 2.2. By letting $\bar{v} = \{\bar{v}_i\}_{i=-\infty}^{\infty}$ be the steady state solutions of (1), the linearized operator at \bar{v} is defined by

$$(\mathcal{L}(\bar{v})\zeta)_i = -\zeta_i + af'(\bar{v}_i)\zeta_i + \beta f'(\bar{v}_{i+1})\zeta_{i+1} \quad \text{for } \zeta \in \ell^2. \quad (18)$$

\bar{v} is called stable if all real parts of eigenvalues of \mathcal{L} are negative with eigenvectors in ℓ^2 and unstable otherwise.

Since the function f is not differentiable at $|\bar{v}_i| = 1$, (18) may not be well defined. Therefore, only $|\bar{v}_i| \neq 1$ is considered herein, subsequently leading to the following stability results.

Proposition 2.3. *Let $\bar{v} = \{\bar{v}_i\}_{i=-\infty}^{i=\infty}$ be the steady state solutions of (1). Assuming that $a > 1$, $\alpha = 0$ and $\beta > 0$ leads to*

- (i) *If $|\bar{v}_i| < 1$ for some $i \in Z$, then \bar{v} is unstable.*
- (ii) *If $r(a + \beta) < 1$, $l(a + \beta) < 1$ and $|\bar{v}_i| > 1$ for all $i \in Z$, then \bar{v} is stable.*

Proof. The assertion holds by Definition 2.2 directly. For details, see [Juang & Lin, 2000; Hsu & Lin, 1999a]. ■

3. Proof of Main Theorem

According to Propositions 2.1 and 2.3, we only have to consider $\{T^i(v)\}_{i=-\infty}^{i=\infty}$ for some $v \in L$ that satisfy

$$T^i(v) \in L \text{ and } |T^i(v)| > 1, \quad \text{for all } i \in Z. \quad (19)$$

The entropy function h can be computed to express whether the map has chaotic behavior. In particular, if the entropy is positive, then the map is called chaotic. Therefore, in this section, we attempt to compute the entropy of T at the set of all bounded stable orbits and see how the entropy h of T varies as r, l change.

We recall some definitions and some results of entropy for a dynamical system.

Definition 3.1. [Robinson, 1995]

- (i) Let $H : X \rightarrow X$ be a continuous map on the space X with metric d . A set $S \subset X$ is called (n, ε) -separated for H for a positive integer n and $\varepsilon > 0$ provided for every pair of distinct points $x, y \in S$, there is at least one k with $0 \leq k < n$ such that $d(H^k(x), H^k(y)) > \varepsilon$.
- (ii) The number of different orbits of length n (as measured by ε) is defined by

$$\gamma(n, \varepsilon, H) = \max\{\#\{S\} | S \subset X \text{ is } (n, \varepsilon) \text{ — separated set for } H\} \quad (20)$$

where $\#\{S\}$ is the number of elements in S .

- (iii) The topological entropy of H is defined as

$$h(H) = \lim_{\varepsilon \rightarrow 0, \varepsilon > 0} \limsup_{n \rightarrow \infty} \frac{\ln \gamma(n, \varepsilon, H)}{n}. \quad (21)$$

- (iv) An interval J_1 H -covers an interval J_2 provided that $H(J_1) \supset J_2$. We write $J_1 \rightarrow J_2$.

Proposition 3.2. [Robinson, 1995]. *Let A be a transition matrix on N symbols. Let $H : X \rightarrow X$ be a continuous map on the space X with metric d and $\sigma_A : \Sigma_A \rightarrow \Sigma_A$ be a subshift of finite type. If H is topologically conjugate to σ_A , then the entropy of H is equal to*

$$h(H) = \ln \lambda_1 \quad (22)$$

where λ_1 is the real eigenvalue of A such that $\lambda_1 \geq |\lambda_j|$ for all other eigenvalues λ_j of A .

Proposition 3.2 indicates that a subshift of finite type can be found such that T is topologically conjugate to the subshift. The subshift can be constructed by finding some subintervals of $L \setminus (-1, 1)$ with covering relation as in the proof of the main theorem later.

Proof of main theorem. Assume that $\beta = 1$ and the general cases can be similarly discussed. If $0 < r \leq r_\infty$ and $0 < l \leq l_\infty$, then C and B are not in L . Under these circumstances, the behavior of map T resembles that of the logistic map. Therefore, there exists an invariant Cantor set in L such that T is topologically conjugate to a one-side Bernoulli shift of two symbols. The entropy of the one-side Bernoulli shift of two symbols is $\ln 2$, according to why the entropy of the map T is $\ln 2$. That is, if $0 < r \leq r_\infty$ and $0 < l \leq l_\infty$ then $h(r, l) = \ln 2$.

To prove the case $r > r_\infty$ and $l > l_\infty$. Let $R^+(r) = (R_1^+(r), R_2^+(r))$ and $R^-(r) = (R_1^-(r), R_2^-(r))$ be the intersecting points of \overline{AB} with $T(v) = 1$ and $T(v) = -1$, respectively. Let $L^+(l) = (L_1^+(l), L_2^+(l))$ and $L^-(l) = (L_1^-(l), L_2^-(l))$ be the intersecting points of \overline{CD} with $T(v) = 1$ and $T(v) = -1$, respectively. By simple computation, we have

$$R_1^+(r) = \frac{1}{1 - ra}, R_1^-(r) = \frac{1 - 2r}{1 - ra}, \quad (23)$$

$$L_1^+(l) = \frac{2l - 1}{1 - la} \text{ and } L_1^-(l) = \frac{-1}{1 - la}. \quad (24)$$

Then, the continuity of $T(v; r, l)$ with respect to r, l makes it easy to prove that for any positive integers p, q with $p \geq 2, q \geq 2$, there exists an unique $r_p > 0$ and $l_q > 0$ such that $\{T^i(a - 1; r_p, l_q)\}_{i=-\infty}^{\infty}$ is a $p + q$ periodic orbit.

Indeed, r_p and l_q satisfy

$$T^{p-1}(a - 1; r_p, l_q) = 1, \tag{25}$$

$$T^{p+q-1}(a - 1; r_p, l_q) = -1. \tag{26}$$

Restated, $(v, T(v; r_p, l_q))$ maps C to B after p iterations; $(v, T(v; r_p, l_q))$ maps B to C after q iterations. When $p = \infty, (v, T(v; r_\infty, l_q))$ maps C to A . When $q = \infty, (v, T(v; r_p, l_\infty))$ maps B to D , where r_∞ and l_∞ are given by (8), i.e. $A_2 = C_2$ and $B_2 = D_2$. Since

$$\begin{aligned} \{T(v) = C_2 = a - 1\} \cap \overline{AB} \\ = \left(\frac{ra - 2r - 1}{1 - ra}, a - 1\right), \end{aligned}$$

and

$$\begin{aligned} \{T(v) = B_2 = 1 - a\} \cap \overline{CD} \\ = \left(\frac{2l - la - 1}{1 - la}, 1 - a\right), \end{aligned}$$

denote $\Omega_{r,l}$ by

$$\begin{aligned} \Omega_{r,l} = \{(v, T(v)) \mid \frac{2l - la - 1}{1 - la} \leq v \\ \leq \frac{ra - 2r - 1}{1 - ra} \text{ and } |T(v)| \leq a - 1\}. \tag{27} \end{aligned}$$

Obviously, $\Omega_{r,l} \subset \Omega$. By Proposition 2.1, every trajectory of T on $\Omega \setminus \Omega_{r,l}$ will tend to A or D backwards. Therefore, trajectories are all that need to be considered of T on $\Omega_{r,l}$. Figure 8 illustrates the 5-periodic orbit of $T(v; r_3, l_2)$. The $2p$ -periodic orbit of $T(v; r_p, l_p)$ with $p \geq 2$ is given in [Hsu & Lin, 1999a].

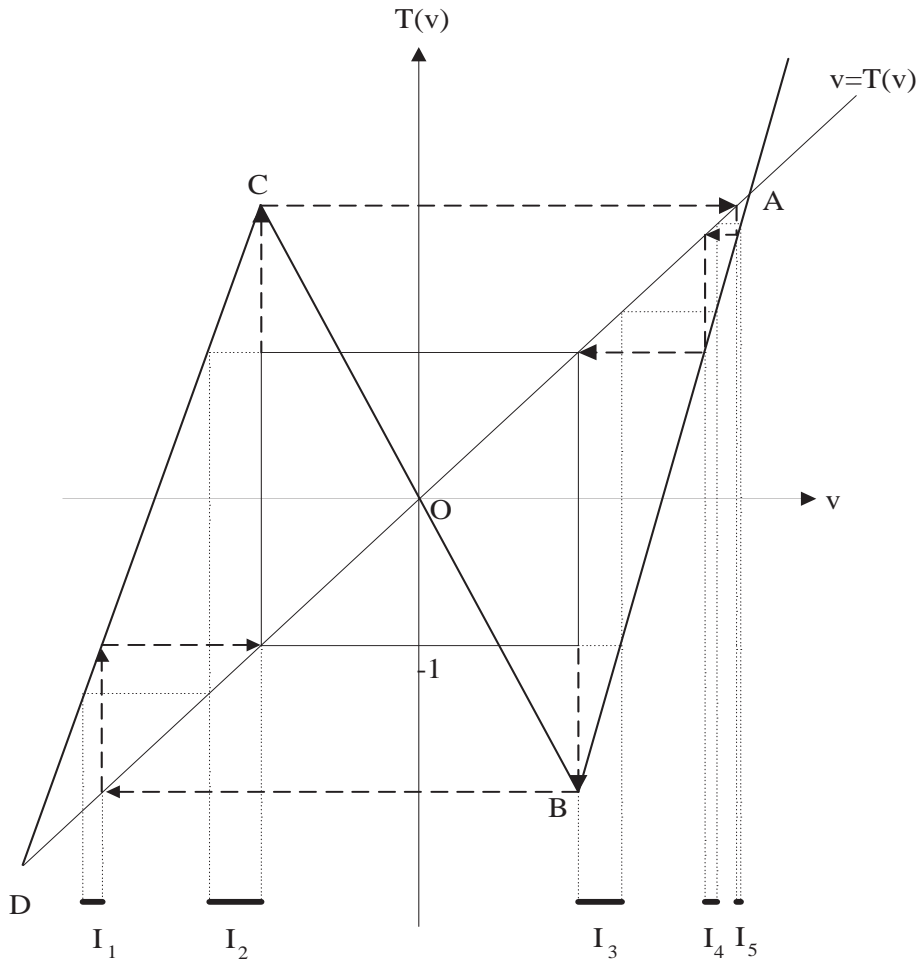


Fig. 8. Graph of $T(a - 1; r_3, l_2)$ and its stable subintervals.

Since the characteristic polynomial of the transition matrix $A(p, q)$ with $q < p$ and $p < q$ are the same, assume that $q < p$ in the following process.

We define the $(p + q)$ -stable subintervals with $p, q \geq 3, p > q$ and $r = r_p, l = l_q$ by

$$I_q = [-1, L_1^+], I_{q-k_1} = [T^{-k_1-1}(L_2^+), T^{-k_1}(L_2^-)]$$

for $k_1 = 1, 2, \dots, q - 1,$

$$I_{q+1} = [-1, R_1^-], I_{q+k_2} = [T^{-k_2+1}(R_2^+), T^{-k_2}(R_2^-)]$$

for $k_2 = 2, 3, \dots, p.$

Obviously, the $(p + q)$ -stable subintervals have the following covering relation:

$$I_1 \rightarrow I_2 \rightarrow I_3 \cdots \rightarrow I_q,$$

$$I_q \rightarrow I_{q+\hat{k}_1} \quad \text{for } \hat{k}_1 = 1, \dots, p - 1,$$

$$I_{q+1} \rightarrow I_{q-\hat{k}_2} \quad \text{for } \hat{k}_2 = 0, 1, \dots, q - 2$$

$$I_{p+q} \rightarrow I_{p+q-1} \rightarrow \cdots \rightarrow I_p \rightarrow I_{p-1} \rightarrow \cdots \rightarrow I_{q+1}.$$

Therefore, the transition matrix $A(p, q)$ of the stable subintervals is given by

$$A(p, q = [a_{ij}(p, q)]) = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & \cdots & 0 & 0 & 0 & 0 \\ \vdots & & & & & & \vdots & & & \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 1 & \cdots \\ 0 & 1 & 1 & 1 & \cdots & 1 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 & \cdots \\ \vdots & & & & & & \vdots & & & \\ 0 & 0 & 0 & 0 & \cdots & \cdots & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & \cdots & 0 & 0 & 1 & 0 \end{pmatrix} \quad (28)$$

Lemma 3.3. *If $r_p < r < r_{p-1}$ and $l_q < l < l_{q-1}$, then the corresponding transition matrix is the same as (28).*

Proof. Pulling back from B and C to find other points C' and B' , see Fig. 9. T will map $\overline{CC'}$ and $\overline{BB'}$ into $\{v | |T^i(v)| < 1 \text{ for some } i \in Z\}$, so $\overline{CC'}$ and $\overline{BB'}$ are not considered. Then the subintervals have the same covering relations as r_p and l_q . Therefore, the corresponding transition matrix is the same as (28). The proof is complete. ■

This study defines spaces Σ_{p+q} and Σ_A by

$$\Sigma_{p+q} = \{1, 2, \dots, q, q + 1, \dots, p + q\}^N, \quad (29)$$

$$\Sigma_A = \{\mathbf{s} \in \Sigma_{p+q} | a_{s_k s_{k+1}} = 1$$

for $k = 0, 1, 2, \dots\}$ (30)

with a metric Σ_{p+q} by

$$d(\mathbf{s}, \mathbf{t}) = \sum_{k=0}^{\infty} \frac{\delta(s_k, t_k)}{3^k} \quad (31)$$

for $\mathbf{s} = (s_0, s_1, s_2, \dots)$ and $\mathbf{t} = (t_0, t_1, t_2, \dots)$, where

$$\delta(i, j) = \begin{cases} 0 & \text{if } i = j, \\ 1 & \text{if } i \neq j. \end{cases} \quad (32)$$

Define a subshift map on Σ_A by $\sigma_A(\mathbf{s}) = \mathbf{t}$, where $t_k = s_{k+1}$, i.e. $\sigma_A(s_0, s_1, \dots) = (s_1, s_2, \dots)$. Then, by Proposition 3.2, we have

Lemma 3.4. *If $r_p \leq r < r_{p-1}$ and $l_q \leq l < l_{q-1}$, then there exists an invariant set $\Lambda_{p+q} \subseteq \Omega_{r,l}$ such that T is topologically conjugate to the subshift of $p + q$ symbols with transition matrix as in (28). Restated, T is topologically conjugate to the space (Σ_A, σ_A) .*

Lemma 3.5. *The characteristic polynomial $P(x; p, q)$ of the transition matrix $A(p, q)$ is*

$$P(x; p, q) = x^{p+q-2} - \sum_{i=0}^{p-2} x^i \sum_{j=0}^{q-2} x^j. \quad (33)$$

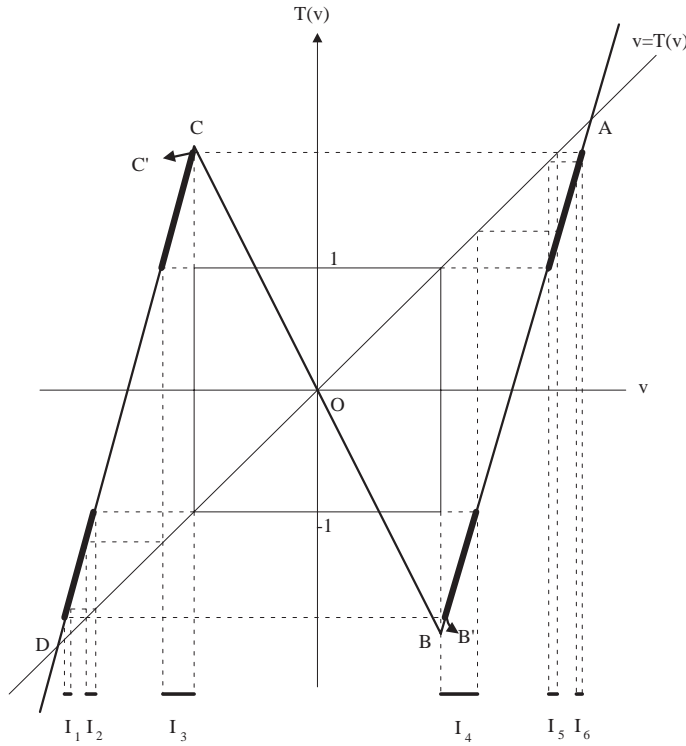


Fig. 9. $r_3 < r < r_2$ and $l_3 < l < l_2$.

Proof. Only the special case is computed when $(p, q) = (6, 4)$. For other p, q , $P(x; p, q)$ can be computed by induction.

$$\det[A(6, 4)] = \begin{vmatrix} -x & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -x & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -x & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -x & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & -x & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -x & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -x & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -x & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -x & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -x \end{vmatrix} \tag{34}$$

$$= \begin{vmatrix} -x & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -x & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -x & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -x & 1 & 1 & 1 & 1 & 0 & x \\ 0 & 1 & 1 & 1 & -x & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -x & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -x & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -x & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -x & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -x \end{vmatrix} \tag{35}$$

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$$= -xP(5, 4) + x \begin{vmatrix} -x & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -x & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -x & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & -x & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -x & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -x & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -x & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -x \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{vmatrix} \tag{36}$$

$$= -xP(5, 4) + x(-x) \begin{vmatrix} -x & 1 & 0 \\ 0 & -x & 1 \\ 1 & 1 & 1 \end{vmatrix} \tag{37}$$

Let

$$M_4 = \begin{vmatrix} -x & 1 & 0 \\ 0 & -x & 1 \\ 1 & 1 & 1 \end{vmatrix} = x^2 + x + 1,$$

then (36) is

$$P(x; 6, 4) = xP(x; 5, 4) + x^2|M_4|. \tag{38}$$

Repeat the same process from (34) to (37). It is easy to see that $P(x; 5, 4) = \det[A(5, 4)] = -xP(x; 4, 4) + x^2|M_4|$. Hence,

$$P(x; 6, 4) = x^2P(x; 4, 4) - (x^3 + x^2)|M_4|. \tag{39}$$

Induction produces

$$P(x; p, 4) = x^{p-4}P(x; 4, 4) - \sum_{i=2}^{p-3} x^i|M_4|. \tag{40}$$

Again, by induction

$$P(x; p, q) = x^{p-q}P(x; q, q) - \sum_{i=2}^{p-q+1} x^i|M_q|, \tag{41}$$

where $M_q = (-1)^q \sum_{j=0}^{q-2} x^j$. By [Hsu & Lin, 1999a], $P(x; q, q) = x^{2q-2} - (\sum_{i=0}^{q-2} x^i)^2$, then by elementary computation (33) is proven. ■

By Proposition 3.2, the entropy of T is $h(r_p, l_q) = \ln \lambda_{(p,q)}$, where $\lambda_{(p,q)}$ is the maximum root of $P(x; p, q)$.

Remark 3.6. Adjusting r when $r_p \leq r < r_{p-1}$ but $0 < l \leq l_\infty$ such that T maps C to B' after p iteration. Then, new subintervals with special covering

relations and transition matrix can be found. Similar to Lemmas 3.4 and 3.5, the entropy function h can be computed. It can be discussed similarly when $0 < r \leq r_\infty$ but $l_q \leq l < l_{q-1}$.

Corollary 3.7. *Let $p, q \geq 2$ and $p_1, q_1 \geq 2$ then:*

- (1) *If $p + q = p_1 + q_1$ and $p - q < p_1 - q_1$ then $h(r_p, l_q) > h(r_{p_1}, l_{q_1})$.*
- (2) *If $p - q = p_1 - q_1$ and $p + q > p_1 + q_1$ then $h(r_p, l_q) > h(r_{p_1}, l_{q_1})$.*
- (3) *If $q = q_1$ and $p > p_1$ then $h(r_p, l_q) > h(r_{p_1}, l_{q_1})$.*

Proof

- (1) We may assume $(p - q) + 2 = p_1 - q_1$, then $q - q_1 = 1$ and

$$\begin{aligned} P(x; p, q) &= x^{p-q}P(x; q, q) - \sum_{i=2}^{p-q+1} x^i|M_1| \\ &= x^{p_1-q_1-2}P(x; q, q) \\ &\quad - \sum_{i=2}^{p_1-q_1-1} x^i|M_q|. \end{aligned}$$

Similar to the process from (34) to (37) and by induction, we have

$$\begin{aligned} P(x; q, q) &= x^2P(x; q - 1, q - 1) \\ &\quad - (x^2 + 2x^3 + \dots + 2x^q). \tag{42} \end{aligned}$$

Then, by elementary computation

$$P(x; p, q) = P(x; p_1, q_1) - \sum_{i=q}^n x^i.$$

Obviously, $P(\lambda_{(p_1, q_1)}; p, q) < 0$, so $\lambda_{(p, q)} > \lambda_{(p_1, q_1)}$. Hence $h(r_p, l_q) > h(r_{p_1}, l_{q_1})$.

- (2) By assuming that $q_1 = q - 1$ and $p_1 - q_1 = p - q$, then

$$P(x; p, q) = x^{p-q}P(x; q, q) - \sum_{i=2}^{p-q+1} x^i |M_q|.$$

By (42) and elementary computation

$$P(x; p, q) = x^{p-q}P(x; p_1, q_1) - \sum_{i=q}^p x^i - x^2 |M_{q_1}|(1 + x).$$

Obviously, $P(\lambda_{(p_1, q_1)}; p, q) < 0$, so $\lambda_{(p, q)} > \lambda_{(p_1, q_1)}$. Hence $h(r_p, l_q) > h(r_{p_1}, l_{q_1})$.

- (3) By assuming that $p_1 = p - 1$ and $q = q_1$ leads to

$$\begin{aligned} p(x; p, q) &= x^{p-q}P(x; q, q) - \sum_{i=2}^{p-q+1} x^i |M_q| \\ &= xP(x; p_1, q_1) - \sum_{i=2}^q x^i. \end{aligned}$$

Obviously, $P(\lambda_{(p_1, q_1)}; p, q) < 0$, so $\lambda_{(p, q)} > \lambda_{(p_1, q_1)}$. Hence $h(r_p, l_q) > h(r_{p_1}, l_{q_1})$.

According to Corollary 3.7, a table can be constructed as shown in Fig. 1 to contrast the entropy between different p, q . The proof of the main theorem is complete. ■

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Appendix

This study demonstrates that $\{r_p\}$ and $\{l_q\}$ are decreasing sequences in p and q , respectively.

Figure 10 reveals that $d = a - 2$. Since $d = R_1^+ - 1$, then

$$\frac{r_2}{1 - r_2 a} = a - 2, \text{ i.e. } r_2(a^2 - a) = a - 2.$$

On the other hand, Fig. 11 reveals that $d(1 + r_3) = a - 2$, hence

$$\begin{aligned} \frac{r_3}{1 - r_3 a}(1 + r_3) &= a - 2, \\ \text{i.e. } r_3^2 a + r_3(a^2 - a) &= a - 2. \end{aligned}$$

By induction, we have

$$a \sum_{i=2}^{p-1} r_p^i + r_p(a^2 - a) = a - 2,$$

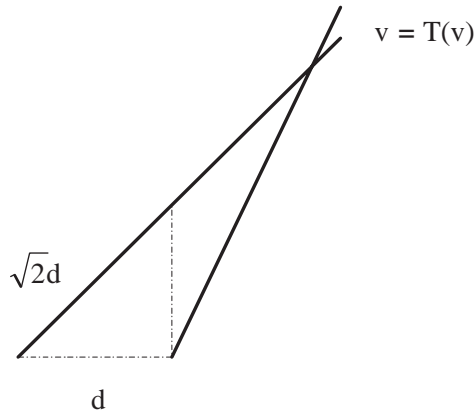


Fig. 10. $T(v)$ with slope $1/r_2 - a$.

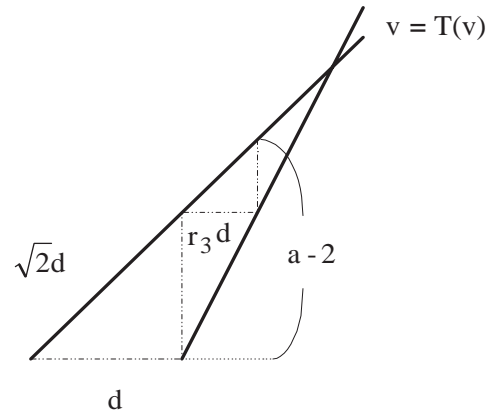


Fig. 11. $T(v)$ with slope $1/r_3 - a$.

and

$$\begin{aligned}
 a \sum_{i=2}^p r_{p+1}^i + r_{p+1}(a^2 - a) &= a - 2 \\
 &= a \sum_{i=2}^{p-1} r_p^i + r_p(a^2 - a).
 \end{aligned}$$

Therefore,

$$r_{p+1}^p a + [r_{p+1} - r_p] a \cdot \eta(r_{p+1}, r_p) = 0,$$

where

$$\eta(r_{p+1}, r_p) = \sum_{i=2}^{p-1} \left(\sum_{j=0}^{i-1} r_{p+1}^j r_p^{i-j-1} \right) + a - 1.$$

Since $r_{p+1}^p > 0$ and $\eta(r_{p+1}, r_p) > 0$, so $r_{p+1} - r_p < 0$. That is, $r_{p+1} < r_p$ for all $p \geq 2$. Similarly, $l_{q+1} < l_q$ for all $q \geq 2$. The proof is complete. ■

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