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Channel graphs of bit permutation networks

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Abstract

Channel graphs have been widely used in the study of blocking networks. In this paper, we show that a bit permutation network has a unique channel graph if and only if it is connected, and two connected bit permutation networks are isomorphic if and only if their channel graphs are isomorphic.

Keywords: Multistage interconnection network; Switching network; Channel graph

1. Introduction

Recently, Chang et al. [2] defined a class of $(m+1)$-*stage* d-*nary bit permutation networks* which are multistage interconnection networks using only $d \times d$ square switches. Such a network has N/d ($N = d^{n+1}$ is the network size) switches in a stage which are labeled by d-nary sequences of length n, and there is a link from switch x at stage i to switch y at stage $i+1$ if the bits of y can be obtained from the bits of x, except one, by a permutation depending only on i. More precisely, the network has vertices $(x_n, x_{n-1}, \ldots, x_1)_i$ in stage i, S_i, where $x_j \in \{0, 1, \ldots, d-1\}$ for $1 \leqslant j \leqslant n$ and $0 \leqslant i \leqslant m$. And, there exist m permutations $f_1, f_2, \ldots, f_m$ on $\{0, 1, \ldots, n\}$ with $f_i(0) \neq 0$ for $1 \leqslant i \leqslant m$ such that $(x_n, x_{n-1}, \ldots, x_1)_{i-1}$ is adjacent to $(x_{f_i(n)}, x_{f_i(n-1)}, \ldots, x_{f_i(1)})_i$, where $x_0 \in \{0, 1, \ldots, d-1\}$ and $1 \leqslant i \leqslant m$. We use $N_d(n, f_1, f_2, \ldots, f_m)$ to denote the network defined above. For any vertex $(x_n, x_{n-1}, \ldots, x_1)_i$ in the network, x_j is called the jth *coordinate* of the vertex. Note that the popular class of binary $(n+1)$th-stage networks including Omega, baseline, banyan, etc., their inverses and their k-extra-stage extensions are all bit permutation networks. The above-mentioned class of $(n+1)$th-

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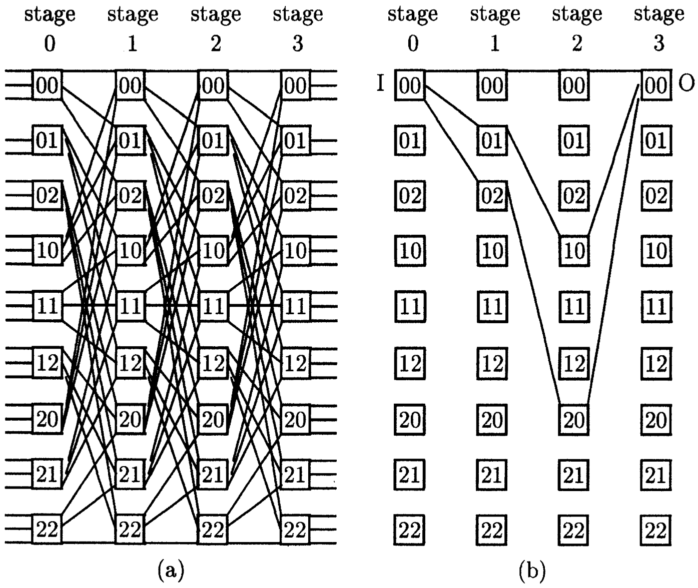

Fig. 1. A bit permutation network and its channel graph.

stage networks are known [1, 7] to be topologically equivalent and will be referred to as the *Omega-equivalent class*. Fig. 1(a) shows a tertiary four-stage shuffle exchange network $N_3(2, f_1, f_2, f_3)$ with $f_1 = f_2 = f_3 =$ the cyclic permutation $(0\,2\,1)$; i.e., switch $(x_2, x_1)_{i-1}$ is adjacent to switch $(x_1, x_0)_i$ for $1 \leqslant i \leqslant 3$.

Switches in the 0th (respectively, mth) stage of an $(m+1)$th-stage network are called *input* (respectively, *output*) switches. For a switch x^s in stage s and a switch x^t in stage $t \geqslant s$, an $x^s - x^t$ *channel-path* is a path $(x^s, x^{s+1}, \dots, x^t)$, where each x^j is in stage j for $s \leqslant j \leqslant t$. The *channel graph* $CG(I, O)$ between an input switch I and an output switch O is the union of all channel-paths in the network connecting the pair. If $CG(I, O)$ is independent of I and O, then it is called the *channel graph of the network*

Channel graphs have been widely used in the study of blocking networks (see [3] for a survey), and recently also used by Lea and Shyy [4–6] to determine the strict and rearrangeable nonblockingness of a network. In [5, 6], results were established for the k-extra-stage inverse banyan network, but implied to hold for all k-extra-stage Omega-equivalent networks. Hwang et al. [4] pointed out that the equivalence among the Omega-equivalent networks is not preserved under the extra-stage addition. However, the results of Lea and Shyy [5, 6] actually depend only on the channel graph of the network. Thus the question arises as to whether there exist nonisomorphic networks having isomorphic channel graphs; of course, the prior condition is that the channel graph of a k-extra-stage network always exists. In this paper we give an affirmative answer to the second question for the larger bit permutation class, and a negative answer to the first question.

2. The main results

A sequence $(k_1, k_2, \ldots, k_m)$ is *canonical* if $\{k_1, k_2, \ldots, k_m\} = \{1, 2, \ldots, r\}$ for some positive integer $r \leqslant n$, and for $k \in \{1, 2, \ldots, r-1\}$, the first appearance of k always precedes that of $k+1$ in the sequence. A canonical sequence $(k_1, k_2, \ldots, k_m)$ induces a bit permutation network, denoted by $N_d(n, k_1, k_2, \ldots, k_m)$, which is $N_d(n, f_1, f_2, \ldots, f_m)$ with f_i being the permutation $(0\, k_i)$ for $1 \leqslant i \leqslant m$. Note that in this case, a vertex in stage $i-1$ is adjacent to a vertex in stage i if and only if all of their coordinates are identical except possibly the k_ith one. It was shown [2] that $(m+1)$th-stage d-nary bit permutation networks can be characterized by canonical sequences.

Theorem 1 (Chang et al. [2]). *Any $N_d(n, f_1, f_2, \ldots, f_m)$ is isomorphic to $N_d(n, k_1, k_2, \ldots, k_m)$ for some canonical sequence over $\{1, 2, \ldots, n\}$. Moreover, two $(m+1)$th-stage d-nary bit permutation networks are isomorphic if and only if their corresponding canonical sequences are the same.*

Lemma 2. *Suppose $x = (x_n, x_{n-1}, \ldots, x_1)_s$ is a switch in stage s and $y = (y_n, y_{n-1}, \ldots, y_1)_t$ a switch in stage $t \geqslant s$, and D is the set of subscripts i where x_i and y_i differ. Then, there is an x–y channel-path in $N_d(n, k_1, k_2, \ldots, k_m)$ if and only if $D \subseteq \{k_{s+1}, k_{s+2}, \ldots, k_t\}$.*

Proof. Immediate from the definition of $N_d(n, k_1, k_2, \ldots, k_m)$. $\square$

Lemma 3. *$N_d(n, k_1, k_2, \ldots, k_m)$ is connected if and only if $\{k_1, k_2, \ldots, k_m\} = \{1, 2, \ldots, n\}$.*

Proof. Suppose the network is connected. Then there is a path from $(0, 0, \ldots, 0)_0$ to $(1, 1, \ldots, 1)_m$. Since a move from a vertex to a neighbor can only change on the k_ith coordinate, any j in $\{1, 2, \ldots, n\}$ is some k_i, i.e., $\{k_1, k_2, \ldots, k_m\} = \{1, 2, \ldots, n\}$.

Conversely, suppose $\{k_1, k_2, \ldots, k_m\} = \{1, 2, \ldots, n\}$. For any two switches x and y, let x' (respectively, y') be the input (respectively, output) switch whose coordinates are the same as x (respectively, y). By Lemma 2, there exist x'–x, x'–y' and y–y' channel-paths. Thus, the network is connected. $\square$

Note that for $\{k_1, k_2, \ldots, k_m\} = \{1, 2, \ldots, n\}$, $\{1, 2, \ldots, n\}$ is the disjoint union of sets $\{k_1, k_2, \ldots, k_i\} \cap \{k_{i+1}, k_{i+2}, \ldots, k_m\}$, $\{1, 2, \ldots, n\} - \{k_1, k_2, \ldots, k_i\}$, $\{1, 2, \ldots, n\} - \{k_{i+1}, k_{i+2}, \ldots, k_m\}$.

Lemma 4. *Suppose x is an input switch and y an output switch in a connected network $N_d(n, k_1, k_2, \ldots, k_m)$. Then, the vertex set of the channel graph $CG(x, y)$ is $\bigcup_{0 \leqslant i \leqslant m} \{z \in S_i : z_j = x_j$ for $j \in \{1, 2, \ldots, n\} - \{k_1, k_2, \ldots, k_i\}$ and $z_j = y_j$ for $j \in \{1, 2, \ldots, n\} - \{k_{i+1}, k_{i+2}, \ldots, k_m\}\}$.*

Proof. The lemma follows from Lemma 2 and the fact that z is a vertex of $CG(x, y)$ if and only if there exist x–z and z–y channel-paths. $\square$

Theorem 5. *The channel graph of a bit permutation network exists if and only if it is connected.*

Proof. The existence of the channel graph of a network necessarily implies the connectivity of the network. Now, suppose $CG(x,y)$ and $CG(x',y')$ are two channel graphs of a connected bit permutation network $N_d(n,k_1,k_2,\dots,k_m)$. Define a function $f: V(CG(x,y)) \to V(CG(x',y'))$ according to Lemma 4 by $f(z)=z'$, where $z,z' \in S_i$ and

$$z'_j = \begin{cases} z_j & \text{if } j \in \{k_1,k_2,\dots,k_i\} \cap \{k_{i+1},k_{i+2},\dots,k_m\}, \\ x'_j & \text{if } j \in \{1,2,\dots,n\} - \{k_1,k_2,\dots,k_i\}, \\ y'_j & \text{if } j \in \{1,2,\dots,n\} - \{k_{i+1},k_{i+2},\dots,k_m\}. \end{cases}$$

For any two distinct vertices u and v in $V(CG(x,y)) \cap S_i$, there exists some $j \in \{k_1,k_2,\dots,k_i\} \cap \{k_{i+1},k_{i+2},\dots,k_m\}$ such that $u_j \neq v_j$. Then, $u'_j = u_j \neq v_j = v'_j$ and so $u' \neq v'$. This proves that f is a one-to-one function. By Lemma 4, $|V(CG(x,y))| = |V(CG(x',y'))|$ and then f is a bijection.

For any edge $uw \in E(CG(x,y))$ with $u \in S_{i-1}$ and $w \in S_i$, $u_j = w_j$ for all $j \neq k_i$. Then, $u'_j = u_j = w_j = w'_j$ or $u'_j = x'_j = w'_j$ or $u'_j = y'_j = w'_j$, depending on $j \in \{k_1,k_2,\dots,k_i\} \cap \{k_{i+1},k_{i+2},\dots,k_m\}$ or $j \in \{1,2,\dots,n\} - \{k_1,k_2,\dots,k_i\}$ or $j \in \{1,2,\dots,n\} - \{k_{i+1},k_{i+2},\dots,k_m\}$. Hence, $u'w'$ is an edge of $E(CG(x',y'))$. Therefore, $CG(x,y)$ and $CG(x',y')$ are isomorphic. □

Theorem 6. *Two connected bit permutation networks are isomorphic if and only if their channel graphs are isomorphic.*

Proof. The "only if" part is trivial. We prove the "if" part. Let $N_d(n,k_1,k_2,\dots,k_m)$ and $N_d(n,k_1^*,k_2^*,\dots,k_m^*)$ be two nonisomorphic connected networks. Then, by Theorem 1, there exists a smallest i such that $k_i \neq k_i^*$. By the definition of a canonical sequence, $k_i \in \{k_1,k_2,\dots,k_{i-1}\}$ or $k_i^* \in \{k_1^*,k_2^*,\dots,k_{i-1}^*\}$, say, the first case holds. Then, there exist $j<i$ such that $k_j = k_i$ and $k_i^* \notin \{k_j^*,k_{j+1}^*,\dots,k_{i-1}^*\}$. Consider the switches $x=(0,0,\dots,0)_{j-1}$ and $y=(0,0,\dots,0)_i$ in stages $j-1$ and i, respectively. By Lemma 2, in the network $N_d(n,k_1,k_2,\dots,k_m)$, there exist two internal vertex-disjoint x–y channel-paths, whose vertices have coordinates 0 except the k_jth coordinate of each internal vertex of second path is 1. However, since $k_i^* \notin \{k_j^*,k_{j+1}^*,\dots,k_{i-1}^*\}$, it is impossible to find two vertices x^* and y^* in stage j and i, respectively, such that there exist two disjoint x^*–y^* channel-paths isomorphic to the preceding ones in $N_d(n,k_1^*,k_2^*,\dots,k_m^*)$. This proves the theorem. □

References

[1] J.-C. Bermond, J.M. Fourneau, A. Jean-Marie, Equivalence of multistage interconnection networks, Inform. Process Lett. 26 (1) (1987) 45–50.
[2] G.J. Chang, F.K. Hwang, L.D. Tong, Characterizing bit permutation networks, Networks, to appear.
[3] F.K. Hwang, Superior channel graphs, Proc. 9th Internat. Teletraffic Congr., Torremolinos, Spain, 1979.

[4] C.-T. Lea, Multi-$\mathrm{Log}_2 N$ networks and their applications in high speed electronic and photonic switching systems, IEEE Trans. Commun. 38 (1990) 1740–1749.
[5] C.-T. Lea, D.-J. Shyy, Tradeoff of horizontal decomposition versus vertical stacking in rearrangeable nonblocking networks, IEEE Trans. Commun. 39 (1991) 899–904.
[6] D.-J. Shyy, C.-T. Lea, $\mathrm{Log}_2(N, m, p)$ strictly nonblocking networks, IEEE Trans. Commun. 39 (1991) 1502–1510.
[7] C.-L. Wu, T.Y. Feng, On a class of multistage interconnection networks, IEEE Trans. Comput. C-29 (8) (1980) 694–702.