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Weighted connected k-domination and weighted k-dominating clique in distance-hereditary graphs ☆

Hong-Gwa Yeh[a], Gerard J. Chang[b, *]

[a] Department of Mathematics, National Central University, Chungli 320, Taiwan
[b] Department of Applied Mathematics, National Chiao Tung University, 1001 Ta Hsueh Road, Hsinchu 30050, Taiwan

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Abstract

A graph is distance-hereditary if the distance between any two vertices in a connected induced subgraph is the same as in the original graph. This paper presents efficient algorithms for solving the weighted connected k-domination and the weighted k-dominating clique problems in distance-hereditary graphs. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

In a graph G, the *distance* $d_G(x, y)$ between two vertices x and y is the minimum length of an x–y path. A graph is *distance-hereditary* if the distance between any two vertices in a connected induced subgraph is the same as in the original graph. Distance-hereditary graphs were introduced by Howorka [14], who gave the first characterization of these graphs. Further characterizations and optimization problems in these graphs were then extensively studied in the literature, see the references. The purpose of this paper is to present efficient algorithms for solving the weighted connected k-domination and the weighted k-dominating clique problems in distance-hereditary graphs.

To consider these two problems on a graph G, every vertex v of G is associated with a nonnegative integer $k(v)$ (the *domination restriction function*) and a nonnegative weight $w(v)$ (the *weight function*). The weight $w(S)$ of a vertex set S is the sum of the weights of its elements. A vertex subset D is a *connected k-dominating set* of G if D induces a connected subgraph and every vertex v of G is *k-dominated* by some

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* Corresponding author. Tel.: +886-3-573-1945; fax: +886-3-542-2682.
E-mail address: gjchang@math.nctu.edu.tw (G.J. Chang).

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vertex x in D, i.e., $d_G(v,x) \leqslant k(v)$. The *weighted connected k-domination problem* is to determine the *weighted connected k-domination number* $\gamma_c(G,k,w)$, which is the minimum weight of a connected k-dominating set of G.

In the above definitions, if in addition the set D is a clique then we have the *weighted k-dominating clique problem* and the *weighted k-dominating clique number* $\gamma_{\text{clique}}(G,k,w)$. For the case when all $w(v)$ are 1, we have unweighted versions of the problems. For the case when all $k(v)$ are 1, we have domination for k-domination.

Domination and its variations, originates from location problems in operations research, have been extensively studied in the literature, see the references. In general, determining $\gamma_c(G)$ and $\gamma_{\text{clique}}(G)$ and $\mathcal{NP}$-complete [12, 15]. On the other hand, there are efficient algorithms for the connected k-domination and the k-dominating clique problems in strongly chordal graphs [5, 11] and dually chordal graphs [9, 1], which include strongly chordal graphs.

As for distance-hereditary graphs, D'Atri and Moscarini [8] first gave an $O(|V||E|)$-time algorithm for the connected domination problem; Yeh and Chang [21] developed a linear-time algorithm for the weighted connected domination problem; Brandstädt and Dragan [2] presented a linear-time algorithm for the connected k-domination problem. Also, Dragan [10] gave a linear-time algorithm for the k-dominating clique problem in distance-hereditary graphs. Combining their techniques, this paper presents efficient algorithms for the weighted connected k-domination and the weighted k-dominating clique problems in distance-hereditary graphs. For technical reasons, we define

$$\gamma_c(G,k,w,u) = \min\{w(D) : D \text{ is a connected } k\text{-dominating set with } u \in D\}$$

and

$$\gamma_{\text{clique}}(G,k,w,u) = \min\{w(D) : D \text{ is a } k\text{-dominating clique with } u \in D\}.$$

For convenience, we call a subset D of V a *$\gamma_c(k,u)$-set of G* if D is a connected k-dominating set of G such that $u \in D$.

2. Preliminaries for distance-hereditary graphs

Suppose A and B are two vertex sets in a graph $G=(V,E)$. $G[A]$ denotes the subgraph of G induced by A. The *neighborhood* $N_A(B)$ of B in A is the set of vertices in A that are adjacent to some vertex in B. The *closed neighborhood* $N_A[B]$ of B in A is $N_A(B) \cup B$. For simplicity, $N_A(v)$, $N_A[v]$, $N(B)$, and $N[B]$ stand for $N_A(\{v\})$, $N_A[\{v\}]$, $N_V(B)$, and $N_V[B]$, respectively.

The *hanging* h_u of a connected graph $G=(V,E)$ at a vertex $u \in V$ is the collection of sets (called *levels*) $L_0(u), L_1(u), \ldots, L_t(u)$ (or $L_0, L_1, \ldots, L_t$ if there is no ambiguity), where $t = \max_{v \in V} d_G(u,v)$ and $L_i(u) = \{v \in V : d_G(u,v) = i\}$ for $0 \leqslant i \leqslant t$. For any $1 \leqslant i \leqslant t$ and any vertex $v \in L_i$, let $N'(v) = N(v) \cap L_{i-1}$. A vertex $v \in L_i$ with $1 \leqslant i \leqslant t$ has a *minimal neighborhood* in L_{i-1} if $N'(x)$ is not a proper subset of $N'(v)$ for any $x \in L_i$.

The following properties of distance-hereditary graphs are useful in this paper.

Theorem 1 (D'Atri and Moscarini [8]). *A connected graph $G=(V,E)$ is distance-hereditary if and only if for every hanging $h_u=(L_0,L_1,\ldots,L_t)$ of G and every pair of vertices x and y in L_i with $1\leqslant i\leqslant t$ in the same component of $G[V-L_{i-1}]$, we have $N'(x)=N'(y)$.*

Theorem 2 (Hammer and Maffray [13, Fact 3.4]). *Suppose $h_u=(L_0,L_1,\ldots,L_t)$ is a hanging of a connected distance-hereditary graph at u. For each $1\leqslant i\leqslant t$, there exists a vertex $v\in L_i$ such that v has a minimal neighborhood in L_{i-1}. In addition, if v satisfies the above condition then $N_{V-N'(v)}(x)=N_{V-N'(v)}(y)$ for every pair of vertices x and y in $N'(v)$.*

3. The algorithm

We now establish theorems that are basis of the efficient algorithms for the weighted connected k-domination and the weighted k-dominating clique problems in distance-hereditary graphs.

Theorem 3. *Suppose $h_u=(L_0,L_1,\ldots,L_t)$ is a hanging of a connected distance-hereditary graph $G=(V,E)$ at u. Let k be a domination restriction function and D be a $\gamma_c(k,u)$-set of G. Then for each $x\in L_i$ with $i\geqslant 1$ and $k(x)\geqslant 1$, there exists a vertex $y\in D\cap L_j$ for some $j<i$ which k-dominates x.*

Proof. We first note that for each vertex $z\in L_p$, a shortest u–z path has the form $P: u=v_0,v_1,\ldots,v_p=z$, where $v_r\in L_r$ for $1\leqslant r\leqslant p$. Since D is a $\gamma_c(k,u)$-set of G, by the definition, there exists a vertex $y'\in D$ and an x–y' path $P_1: x=x_0,x_1,\ldots,x_d=y'$ such that $d\leqslant k(x)$. We may assume that $y'\in D\cap L_{j'}$ and y' is the only vertex of P_1 that is in D. If $j'<i$, then we may choose $y=y'$ to k-dominate x as desired. Next, we consider the case of $j'\geqslant i$. Since $G[D]$ is connected, there is a shortest u–y' path $P_2: u=y_0,y_1,\ldots,y_{j'}=y'$ in G using vertices only in D, where $y_r\in L_r$ for $0\leqslant r\leqslant j'$.

Let s be the smallest index such that $L_s\cap P_1\neq\emptyset$, say $x_r\in L_s\cap P_1$. Since $1\leqslant s\leqslant i\leqslant j'$, the vertices $x\in L_i$ and $x_r,y_s\in L_s$ are connected in $G[V-L_{s-1}]$ through the x–y_s walk:

$$x=x_0,x_1,\ldots,x_r,x_{r+1},\ldots,x_d=y'=y_{j'},y_{j'-1},\ldots,y_s.$$

For the case of $x_r=y'$, we have $s=i=j'$. By Theorem 1, $N'(x)=N'(y_s)$ and so x is adjacent to y_{s-1}. We may then choose $y=y_{s-1}\in D\cap L_{s-1}$ with $s-1<i$ to k-dominate x as desired. For the case of $x_r\neq y'$, we have $r\leqslant d-1$. Again, $N'(x_r)=N'(y_s)$ by Theorem 1 and so x_r is adjacent to y_{s-1}. Thus,

$$d_G(x,y_{s-1})\leqslant d_G(x,x_r)+1\leqslant(d-1)+1\leqslant k(x)$$

and so we may choose $y=y_{s-1}$ to k-dominate x as desired. This completes the proof of the theorem. □

Theorem 4. *Suppose $h_u = (L_0, L_1, \ldots, L_t)$ is a hanging of a connected distance-hereditary graph $G = (V, E)$ at u. Let k be a domination restriction function, w a nonnegative weight function, and x a vertex in L_t having a minimal neighborhood $B = N_{L_{t-1}}(x)$. Choose $y \in B$ with $w(y) = \min_{b \in B} w(b)$ and $z \in B$ with $k(z) = \min_{b \in B} k(b)$. Suppose $G' = G - x$, the weight function w' is the restriction of w on $V(G')$, and k' is defined by*

$$k'(v) = \begin{cases} 0 & \text{if } v = y \text{ with } k(z) > k(x) = 0, \\ k(x) - 1 & \text{if } v = y \text{ with } k(z) \geqslant k(x) \geqslant 1, \\ k(v) & \text{otherwise.} \end{cases}$$

Then the following statements hold:

(1) *$\gamma_c(G, k, w, u) = \gamma_c(G', k', w', u) + w(x)$ when $k(x) = 0$, and $\gamma_c(G, k, w, u) = \gamma_c(G', k', w', u)$ otherwise.*

(2) *Suppose G' has a k'-dominating clique containing vertex u. If $k(x) \neq 0$, then $\gamma_{\text{clique}}(G, k, w, u) = \gamma_{\text{clique}}(G', k', w', u)$.*

Proof. We only give the proof for (1). The proof for (2) is similar and thus omitted.

Suppose D is a minimum weighted $\gamma_c(k, u)$-set of G. Since $G[D]$ is connected and $u \in D$, every vertex of $D - \{u\}$ is adjacent to a vertex of D in a lower level. This, together with Theorem 3, implies that we may assume $x \notin D$ when $k(x) \geqslant 1$, and that $D - \{x\}$ is a k-connected dominating set of G'. For the case of $k(z) > k(x) = 0$, B contains some vertex in D and so, by Theorem 2, we may assume that $y \in D$. For the case of $k(z) \geqslant k(x) \geqslant 1$, by Theorem 3, x is within distance $k(x)$ in G from a vertex $x' \in D$ in some level $j < t$. When $j = t - 1$, we may assume $x' = y$; and in any case, y is within distance $k(x) - 1$ in G' from x'. According to the definition, k' is the same as k except the above two cases; and so, by Theorem 3, $D - \{x\}$ is a connected k'-dominating set of G'. Therefore, $\gamma_c(G', k', w', u) \leqslant \gamma_c(G, k, w, u) - w(x)$ when $k(x) = 0$, and $\gamma_c(G', k', w', u) \leqslant \gamma_c(G, k, w, u)$ otherwise.

On the other hand, suppose D' is a connected k'-dominating set of G'. For the case of $k(x) = 0$, according to the definition of k', either $k'(y) = 0$ or $k'(z) = k(z) = 0$. Hence, B contains a vertex of D'. Then $D \cup \{x\}$ is a connected k-dominating set of G and so $\gamma_c(G, k, w, u) \leqslant \gamma_c(G', k', w', u) + w(x)$. For the case of $k(x) \geqslant 1$, according to the definition of k', B contains a vertex within distance $k(x) - 1$ in G' from a vertex $x' \in D'$. Then x is within distance $k(x)$ in G from $x' \in D'$. Therefore, D' is a k-connected dominating set of G and so $\gamma_c(G, k, w, u) \leqslant \gamma_c(G', k', w', u)$.

The above inequalities imply the result in (1). □

Based on Theorem 4, we have the following $O(|V||E|)$-time algorithm for computing $\gamma_c(G, k, w)$ of a connected distance-hereditary graph $G = (V, E)$. The algorithm can also be modified to solve the weighted k-dominating clique problem in the same time complexity.

begin

let $V = \{u_1, u_2, \ldots, u_n\}$;

$\gamma_c(G,k,w) \leftarrow \infty$;
for $i=1$ **to** n **do**
begin
  for $j=1$ **to** n **do** $\bar{k}(u_j)=k(u_j)$;
  $\bar{k}(u_i) \leftarrow 0$;
  $p \leftarrow w(u_i)$; / * p stands for $\gamma_c(G,k,w,u_i)$ * /
  determine the hanging $h_{u_i}=(L_0,L_1,\ldots,L_t)$ of G at u_i;
  for $j=t$ **downto** 1 **do**
  begin
    sort $L_j=\{x_1,x_2,\ldots,x_\ell\}$ such that
    $|N'(x_{p_1})| \leqslant |N'(x_{p_2})| \leqslant \cdots \leqslant |N'(x_{p_\ell})|$;
    for $r=1$ **to** ℓ **do**
    begin
      let $y,z \in B=N'(x_{p_r})$ such that
      $w(y)=\min_{b\in B} w(b)$ and $\bar{k}(z)=\min_{b\in B}\bar{k}(b)$;
      if $\bar{k}(z) > \bar{k}(x_{p_r})=0$ **then** $\bar{k}(y) \leftarrow 0$;
      if $\bar{k}(z) \geqslant \bar{k}(x_{p_r}) \geqslant 1$ **then** $\bar{k}(y) \leftarrow \bar{k}(x_{p_r})-1$;
      if $\bar{k}(x_{p_r})=0$ **then** $p \leftarrow p+w(x_{p_r})$;
    end
  end
  $\gamma_c(G,k,w) \leftarrow \min\{\gamma_c(G,k,w),\, p\}$
end
end.

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