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ON THE RELIABILITY OF THE ESTIMATED INCAPABILITY INDEX

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SUMMARY
Greenwich and Jahr-Schaffrath (1995) introduced the process incapability index $C_{pp} = C_{ip} + C_{ia}$, which provides an uncontaminated separation between information concerning the process precision (C_{ip}) and process accuracy (C_{ia}). In this paper, we consider the three indices, and investigate the statistical properties of their natural estimators. For the three indices, we obtain their UMVUEs and MLEs, and compare the reliability of the two estimators based on the relative mean square errors. In addition, we construct 90%, 95%, and 99% upper confidence limits, and the maximum values of $\hat{C}_{pp}$ for which the process is capable 90%, 95%, and 99% of the time. The results obtained in this paper are useful to the practitioners in choosing good estimators and making reliable decisions on judging process capability. Copyright © 2001 John Wiley & Sons, Ltd.

KEY WORDS: imprecision index; inaccuracy index; UMVUE; MLE; relative mean square error

1. INTRODUCTION

Greenwich and Jahr-Schaffrath [1] introduced the process incapability index C_{pp} to provide numerical measures on process performance for industrial applications. The index C_{pp} is a simple transformation of C^*_{pm}, a general form of the capability index C_{pm} considered by Chan *et al* [2], which provides an uncontaminated separation between information concerning the process precision and the process accuracy. The index C_{pp} is defined as follows:

$$C_{pp} = \frac{1}{C^{*2}_{pm}} = \left(\frac{\sigma}{D}\right)^2 + \left(\frac{\mu - T}{D}\right)^2$$

where μ is the process mean, σ is the process standard deviation, $D = \min\{(USL - T)/3, (T - LSL)/3\}$, USL and LSL are the upper and the lower specification limits, and T is the target value. If we define $C_{ip} = (\sigma/D)^2$, and $C_{ia} = [(\mu - T)/D]^2$, then C_{pp} can be expressed as $C_{pp} = C_{ip} + C_{ia}$. Since C_{ip} measures the process variation relative to the specification tolerance, it has been referred to as the process imprecision index. On the other hand, C_{ia} measures the relative process departure and has been referred to as the process inaccuracy index. We note that the mathematical relationships $C_{ip} = 1/(C_p)^2$, and $C_{ia} = 9(1-C_a)^2$ can be established, where C_p and C_a are two basic process capability indices considered by Kane [3] and Pearn *et al.* [4].

In this paper, we consider the three indices C_{ip}, C_{ia}, and C_{pp} and investigate the statistical properties of their natural estimators. For C_{ip}, we show that the natural estimator is the UMVUE, which is consistent and asymptotically efficient. We also obtain the MLE (maximum likelihood estimator), which has smaller mean square error than the UMVUE (uniformly minimum variance unbiased estimator), hence it is more reliable, particularly, for short production run applications. For C_{ia}, we show that the natural estimator is the MLE. We also obtain the UMVUE, which is shown to be more reliable than the MLE for applications with $n \geq 4$. We show that the UMVUE is consistent and asymptotically efficient. For C_{pp}, we show that the natural estimator is the MLE and also the UMVUE, which is consistent and asymptotically efficient. In addition, we construct tables of 90%, 95%, and 99% upper confidence limits for C_{pp} based on the UMVUE. We also construct tables of the maximum values of $\hat{C}_{pp}$ under $\mu = T$ for which the process is capable. The estimators we recommend have all

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the desired statistical properties, and are considered reliable.

2. ESTIMATING PROCESS IMPRECISION

To estimate the process imprecision, we consider the natural estimator $\hat{C}_{ip}$ defined as follows, where $S_{n-1} = [\sum_{i=1}^{n}(X_i - \bar{X})^2/(n-1)]^{1/2}$ is the conventional estimator of the process standard deviation σ,

$$\hat{C}_{ip} = \frac{1}{n-1}\sum_{i=1}^{n}\frac{(X_i - \bar{X})^2}{D^2} = \frac{S_{n-1}^2}{D^2}$$

The natural estimator $\hat{C}_{ip}$ can be rewritten as

$$\hat{C}_{ip} = \frac{C_{ip}}{n-1}\frac{(n-1)\hat{C}_{ip}}{C_{ip}} = \frac{C_{ip}}{n-1}\sum_{i=1}^{n}\frac{(X_i - \bar{X})^2}{\sigma^2}$$

If the process follows the normal distribution, then $\hat{C}_{ip}$ is distributed as $[C_{ip}/(n-1)]\chi_{n-1}^2$, where χ_{n-1}^2 is a chi-squared distribution with $(n-1)$ degrees of freedom. The probability density function of $\hat{C}_{ip}$ can be easily derived as

$$\begin{aligned} f(y) = {} & \{[(n-1)y/(2C_{ip})]^{(n-1)/2} \\ & \times \exp[-(n-1)y/(2C_{ip})]\} \\ & \times \{y\Gamma[(n-1)/2]\}^{-1}, \quad \text{for } y > 0 \end{aligned}$$

The rth moment, the expected value, the variance, and the mean squared error of $\hat{C}_{ip}$ can be calculated as follows:

$$\mathrm{E}(\hat{C}_{ip})^r = \frac{\Gamma\{[(n-1)/2]+r\}}{\Gamma[(n-1)/2]}\left(\frac{2C_{ip}}{n-1}\right)^r$$

$$\begin{aligned} \mathrm{E}(\hat{C}_{ip}) &= \left(\frac{C_{ip}}{n-1}\right)\mathrm{E}(\chi_{n-1}^2) \\ &= C_{ip} \end{aligned}$$

$$\begin{aligned} \mathrm{Var}(\hat{C}_{ip}) &= \left(\frac{C_{ip}}{n-1}\right)^2 \mathrm{Var}(\chi_{n-1}^2) \\ &= \left(\frac{C_{ip}}{n-1}\right)^2 2(n-1) \\ &= \frac{2C_{ip}^2}{n-1} \end{aligned}$$

$$\begin{aligned} \mathrm{MSE}(\hat{C}_{ip}) &= \mathrm{E}(\hat{C}_{ip} - C_{ip})^2 \\ &= \mathrm{Var}(\hat{C}_{ip}) + [\mathrm{E}(\hat{C}_{ip}) - C_{ip}]^2 \\ &= \frac{2C_{ip}^2}{n-1} \end{aligned}$$

If the process characteristic is normally distributed, then we can show that the natural estimator $\hat{C}_{ip}$ is the UMVUE of C_{ip}, which is consistent. We can also show that $\sqrt{n}(\hat{C}_{ip} - C_{ip})$ converges to $N(0, 2C_{ip}^2)$ in distribution, and that $\hat{C}_{ip}$ is asymptotically efficient (see the Appendix for the proofs). Thus, in real-world applications using $\hat{C}_{ip}$, which has all the desired statistical properties, as an estimate of C_{ip} would be reasonable.

We note that by multiplying the UMVUE $\hat{C}_{ip}$ by the constant $c_n = (n-1)/n$, we obtain the MLE of C_{ip}. We can show that the MLE $\hat{C}'_{ip}$ is consistent, and is asymptotically unbiased. We can also show that $\sqrt{n}(\hat{C}'_{ip} - C_{ip})$ converges to $N(0, 2C_{ip}^2)$ in distribution, and that $\hat{C}'_{ip}$ is asymptotically efficient. Since $c_n < 1$, then $\hat{C}'_{ip} = c_n\hat{C}_{ip}$ underestimates C_{ip} but with smaller variance. In fact, we may calculate

$$\mathrm{MSE}(\hat{C}'_{ip}) = [(2n-1)/n^2](C_{ip})^2$$

and obtain

$$\begin{aligned} & \mathrm{MSE}(\hat{C}_{ip}) - \mathrm{MSE}(\hat{C}'_{ip}) \\ & \quad = [(3n-1)/n^2(n-1)](C_{ip})^2 > 0, \quad \text{for all } n \end{aligned}$$

Therefore, the MLE $\hat{C}'_{ip}$ has smaller mean squared error than the UMVUE $\hat{C}_{ip}$, hence it is more reliable, particularly for short production run applications.

Tables 1(a) and 1(b) display the relative error of the UMVUE $\hat{C}_{ip}$, defined as $[\mathrm{MSE_R}(\hat{C}_{ip})]^{1/2} = \{\mathrm{E}[(\hat{C}_{ip} - C_{ip})/C_{ip}]^2\}^{1/2}$, for sample sizes $n = 2(1)50$, and 60(10)550, and some commonly used values of $C_{ip} =$ 1.00, 0.56, 0.44, 0.36, and 0.25, equivalent to $C_p =$ 1.00, 1.33, 1.50, 1.67, and 2.00, covering the range of the precision requirements for most applications.

Precision requirement:
- Capable: $0.56 \le C_{ip} \le 1.00$
- Satisfactory: $0.44 \le C_{ip} \le 0.56$
- Good: $0.36 \le C_{ip} \le 0.44$
- Excellent: $0.25 \le C_{ip} \le 0.36$
- Super: $C_{ip} \le 0.25$

The square root of the relative mean squared error is a direct measurement, which presents the expected relative error of the estimation from the true C_{ip}. We note that for UMVUE $\hat{C}_{ip}$, $[\mathrm{MSE_R}(\hat{C}_{ip})]^{1/2} = [2/(n-1)]^{1/2}$, which is a function of the sample size n only. Therefore, $[\mathrm{MSE_R}(\hat{C}_{ip})]^{1/2}$ values are the same for all C_{ip} values. For example, with $n = 300$ we have $[\mathrm{MSE_R}(\hat{C}_{ip})]^{1/2} = 0.0818$. Thus, for $n = 300$, we expect that the average error of $\hat{C}_{ip}$ would be no greater than 8.18% of the true C_p. Tables 2(a) and 2(b) display the relative error, $[\mathrm{MSE_R}(\hat{C}'_{ip})]^{1/2}$, of the MLE $\hat{C}'_{ip}$. We note that $[\mathrm{MSE_R}(\hat{C}'_{ip})]^{1/2} = [(2n-1)/n^2]^{1/2}$,

Table 1. $[\mathrm{MSE_R}(\hat{C}_{ip})]^{1/2}$ for various C_{ip}, and sample sizes (a) $n = 2(1)50$ and (b) $n = 60(10)550$

	C_{ip}						C_{ip}				
n	1.00	0.56	0.44	0.36	0.25	n	1.00	0.56	0.44	0.36	0.25
1	******	******	******	******	******	60	0.1841	0.1841	0.1841	0.1841	0.1841
2	1.4142	1.4142	1.4142	1.4142	1.4142	70	0.1703	0.1703	0.1703	0.1703	0.1703
3	1.0000	1.0000	1.0000	1.0000	1.0000	80	0.1591	0.1591	0.1591	0.1591	0.1591
4	0.8165	0.8165	0.8165	0.8165	0.8165	90	0.1499	0.1499	0.1499	0.1499	0.1499
5	0.7071	0.7071	0.7071	0.7071	0.7071	100	0.1421	0.1421	0.1421	0.1421	0.1421
6	0.6325	0.6325	0.6325	0.6325	0.6325	110	0.1355	0.1355	0.1355	0.1355	0.1355
7	0.5774	0.5774	0.5774	0.5774	0.5774	120	0.1296	0.1296	0.1296	0.1296	0.1296
8	0.5345	0.5345	0.5345	0.5345	0.5345	130	0.1245	0.1245	0.1245	0.1245	0.1245
9	0.5000	0.5000	0.5000	0.5000	0.5000	140	0.1200	0.1200	0.1200	0.1200	0.1200
10	0.4714	0.4714	0.4714	0.4714	0.4714	150	0.1159	0.1159	0.1159	0.1159	0.1159
11	0.4472	0.4472	0.4472	0.4472	0.4472	160	0.1122	0.1122	0.1122	0.1122	0.1122
12	0.4264	0.4264	0.4264	0.4264	0.4264	170	0.1088	0.1088	0.1088	0.1088	0.1088
13	0.4082	0.4082	0.4082	0.4082	0.4082	180	0.1057	0.1057	0.1057	0.1057	0.1057
14	0.3922	0.3922	0.3922	0.3922	0.3922	190	0.1029	0.1029	0.1029	0.1029	0.1029
15	0.3780	0.3780	0.3780	0.3780	0.3780	200	0.1003	0.1003	0.1003	0.1003	0.1003
16	0.3651	0.3651	0.3651	0.3651	0.3651	210	0.0978	0.0978	0.0978	0.0978	0.0978
17	0.3536	0.3536	0.3536	0.3536	0.3536	220	0.0956	0.0956	0.0956	0.0956	0.0956
18	0.3430	0.3430	0.3430	0.3430	0.3430	230	0.0935	0.0935	0.0935	0.0935	0.0935
19	0.3333	0.3333	0.3333	0.3333	0.3333	240	0.0915	0.0915	0.0915	0.0915	0.0915
20	0.3244	0.3244	0.3244	0.3244	0.3244	250	0.0896	0.0896	0.0896	0.0896	0.0896
21	0.3162	0.3162	0.3162	0.3162	0.3162	260	0.0879	0.0879	0.0879	0.0879	0.0879
22	0.3086	0.3086	0.3086	0.3086	0.3086	270	0.0862	0.0862	0.0862	0.0862	0.0862
23	0.3015	0.3015	0.3015	0.3015	0.3015	280	0.0847	0.0847	0.0847	0.0847	0.0847
24	0.2949	0.2949	0.2949	0.2949	0.2949	290	0.0832	0.0832	0.0832	0.0832	0.0832
25	0.2887	0.2887	0.2887	0.2887	0.2887	300	0.0818	0.0818	0.0818	0.0818	0.0818
26	0.2828	0.2828	0.2828	0.2828	0.2828	310	0.0805	0.0805	0.0805	0.0805	0.0805
27	0.2774	0.2774	0.2774	0.2774	0.2774	320	0.0792	0.0792	0.0792	0.0792	0.0792
28	0.2722	0.2722	0.2722	0.2722	0.2722	330	0.0780	0.0780	0.0780	0.0780	0.0780
29	0.2673	0.2673	0.2673	0.2673	0.2673	340	0.0768	0.0768	0.0768	0.0768	0.0768
30	0.2626	0.2626	0.2626	0.2626	0.2626	350	0.0757	0.0757	0.0757	0.0757	0.0757
31	0.2582	0.2582	0.2582	0.2582	0.2582	360	0.0746	0.0746	0.0746	0.0746	0.0746
32	0.2540	0.2540	0.2540	0.2540	0.2540	370	0.0736	0.0736	0.0736	0.0736	0.0736
33	0.2500	0.2500	0.2500	0.2500	0.2500	380	0.0726	0.0726	0.0726	0.0726	0.0726
34	0.2462	0.2462	0.2462	0.2462	0.2462	390	0.0717	0.0717	0.0717	0.0717	0.0717
35	0.2425	0.2425	0.2425	0.2425	0.2425	400	0.0708	0.0708	0.0708	0.0708	0.0708
36	0.2390	0.2390	0.2390	0.2390	0.2390	410	0.0699	0.0699	0.0699	0.0699	0.0699
37	0.2357	0.2357	0.2357	0.2357	0.2357	420	0.0691	0.0691	0.0691	0.0691	0.0691
38	0.2325	0.2325	0.2325	0.2325	0.2325	430	0.0683	0.0683	0.0683	0.0683	0.0683
39	0.2294	0.2294	0.2294	0.2294	0.2294	440	0.0675	0.0675	0.0675	0.0675	0.0675
40	0.2265	0.2265	0.2265	0.2265	0.2265	450	0.0667	0.0667	0.0667	0.0667	0.0667
41	0.2236	0.2236	0.2236	0.2236	0.2236	460	0.0660	0.0660	0.0660	0.0660	0.0660
42	0.2209	0.2209	0.2209	0.2209	0.2209	470	0.0653	0.0653	0.0653	0.0653	0.0653
43	0.2182	0.2182	0.2182	0.2182	0.2182	480	0.0646	0.0646	0.0646	0.0646	0.0646
44	0.2157	0.2157	0.2157	0.2157	0.2157	490	0.0640	0.0640	0.0640	0.0640	0.0640
45	0.2132	0.2132	0.2132	0.2132	0.2132	500	0.0633	0.0633	0.0633	0.0633	0.0633
46	0.2108	0.2108	0.2108	0.2108	0.2108	510	0.0627	0.0627	0.0627	0.0627	0.0627
47	0.2085	0.2085	0.2085	0.2085	0.2085	520	0.0621	0.0621	0.0621	0.0621	0.0621
48	0.2063	0.2063	0.2063	0.2063	0.2063	530	0.0615	0.0615	0.0615	0.0615	0.0615
49	0.2041	0.2041	0.2041	0.2041	0.2041	540	0.0609	0.0609	0.0609	0.0609	0.0609
50	0.2020	0.2020	0.2020	0.2020	0.2020	550	0.0604	0.0604	0.0604	0.0604	0.0604

Table 2. $[\mathrm{MSE_R}(\hat{C}'_{ip})]^{1/2}$ for various C_{ip}, and sample sizes (a) $n = 1(1)50$ and (b) $n = 60(10)550$

	C_{ip}						C_{ip}				
n	1.00	0.56	0.44	0.36	0.25	n	1.00	0.56	0.44	0.36	0.25
1	1.0000	1.0000	1.0000	1.0000	1.0000	60	0.1818	0.1818	0.1818	0.1818	0.1818
2	0.8660	0.8660	0.8660	0.8660	0.8660	70	0.1684	0.1684	0.1684	0.1684	0.1684
3	0.7454	0.7454	0.7454	0.7454	0.7454	80	0.1576	0.1576	0.1576	0.1576	0.1576
4	0.6614	0.6614	0.6614	0.6614	0.6614	90	0.1487	0.1487	0.1487	0.1487	0.1487
5	0.6000	0.6000	0.6000	0.6000	0.6000	100	0.1411	0.1411	0.1411	0.1411	0.1411
6	0.5528	0.5528	0.5528	0.5528	0.5528	110	0.1345	0.1345	0.1345	0.1345	0.1345
7	0.5151	0.5151	0.5151	0.5151	0.5151	120	0.1288	0.1288	0.1288	0.1288	0.1288
8	0.4841	0.4841	0.4841	0.4841	0.4841	130	0.1238	0.1238	0.1238	0.1238	0.1238
9	0.4581	0.4581	0.4581	0.4581	0.4581	140	0.1193	0.1193	0.1193	0.1193	0.1193
10	0.4359	0.4359	0.4359	0.4359	0.4359	150	0.1153	0.1153	0.1153	0.1153	0.1153
11	0.4166	0.4166	0.4166	0.4166	0.4166	160	0.1116	0.1116	0.1116	0.1116	0.1116
12	0.3997	0.3997	0.3997	0.3997	0.3997	170	0.1083	0.1083	0.1083	0.1083	0.1083
13	0.3846	0.3846	0.3846	0.3846	0.3846	180	0.1053	0.1053	0.1053	0.1053	0.1053
14	0.3712	0.3712	0.3712	0.3712	0.3712	190	0.1025	0.1025	0.1025	0.1025	0.1025
15	0.3590	0.3590	0.3590	0.3590	0.3590	200	0.0999	0.0999	0.0999	0.0999	0.0999
16	0.3480	0.3480	0.3480	0.3480	0.3480	210	0.0975	0.0975	0.0975	0.0975	0.0975
17	0.3379	0.3379	0.3379	0.3379	0.3379	220	0.0952	0.0952	0.0952	0.0952	0.0952
18	0.3287	0.3287	0.3287	0.3287	0.3287	230	0.0931	0.0931	0.0931	0.0931	0.0931
19	0.3201	0.3201	0.3201	0.3201	0.3201	240	0.0912	0.0912	0.0912	0.0912	0.0912
20	0.3123	0.3123	0.3123	0.3123	0.3123	250	0.0894	0.0894	0.0894	0.0894	0.0894
21	0.3049	0.3049	0.3049	0.3049	0.3049	260	0.0876	0.0876	0.0876	0.0876	0.0876
22	0.2981	0.2981	0.2981	0.2981	0.2981	270	0.0860	0.0860	0.0860	0.0860	0.0860
23	0.2917	0.2917	0.2917	0.2917	0.2917	280	0.0844	0.0844	0.0844	0.0844	0.0844
24	0.2857	0.2857	0.2857	0.2857	0.2857	290	0.0830	0.0830	0.0830	0.0830	0.0830
25	0.2800	0.2800	0.2800	0.2800	0.2800	300	0.0816	0.0816	0.0816	0.0816	0.0816
26	0.2747	0.2747	0.2747	0.2747	0.2747	310	0.0803	0.0803	0.0803	0.0803	0.0803
27	0.2696	0.2696	0.2696	0.2696	0.2696	320	0.0790	0.0790	0.0790	0.0790	0.0790
28	0.2649	0.2649	0.2649	0.2649	0.2649	330	0.0778	0.0778	0.0778	0.0778	0.0778
29	0.2603	0.2603	0.2603	0.2603	0.2603	340	0.0766	0.0766	0.0766	0.0766	0.0766
30	0.2560	0.2560	0.2560	0.2560	0.2560	350	0.0755	0.0755	0.0755	0.0755	0.0755
31	0.2519	0.2519	0.2519	0.2519	0.2519	360	0.0745	0.0745	0.0745	0.0745	0.0745
32	0.2480	0.2480	0.2480	0.2480	0.2480	370	0.0735	0.0735	0.0735	0.0735	0.0735
33	0.2443	0.2443	0.2443	0.2443	0.2443	380	0.0725	0.0725	0.0725	0.0725	0.0725
34	0.2407	0.2407	0.2407	0.2407	0.2407	390	0.0716	0.0716	0.0716	0.0716	0.0716
35	0.2373	0.2373	0.2373	0.2373	0.2373	400	0.0707	0.0707	0.0707	0.0707	0.0707
36	0.2341	0.2341	0.2341	0.2341	0.2341	410	0.0698	0.0698	0.0698	0.0698	0.0698
37	0.2309	0.2309	0.2309	0.2309	0.2309	420	0.0690	0.0690	0.0690	0.0690	0.0690
38	0.2279	0.2279	0.2279	0.2279	0.2279	430	0.0682	0.0682	0.0682	0.0682	0.0682
39	0.2250	0.2250	0.2250	0.2250	0.2250	440	0.0674	0.0674	0.0674	0.0674	0.0674
40	0.2222	0.2222	0.2222	0.2222	0.2222	450	0.0666	0.0666	0.0666	0.0666	0.0666
41	0.2195	0.2195	0.2195	0.2195	0.2195	460	0.0659	0.0659	0.0659	0.0659	0.0659
42	0.2169	0.2169	0.2169	0.2169	0.2169	470	0.0652	0.0652	0.0652	0.0652	0.0652
43	0.2144	0.2144	0.2144	0.2144	0.2144	480	0.0645	0.0645	0.0645	0.0645	0.0645
44	0.2120	0.2120	0.2120	0.2120	0.2120	490	0.0639	0.0639	0.0639	0.0639	0.0639
45	0.2096	0.2096	0.2096	0.2096	0.2096	500	0.0632	0.0632	0.0632	0.0632	0.0632
46	0.2074	0.2074	0.2074	0.2074	0.2074	510	0.0626	0.0626	0.0626	0.0626	0.0626
47	0.2052	0.2052	0.2052	0.2052	0.2052	520	0.0620	0.0620	0.0620	0.0620	0.0620
48	0.2031	0.2031	0.2031	0.2031	0.2031	530	0.0614	0.0614	0.0614	0.0614	0.0614
49	0.2010	0.2010	0.2010	0.2010	0.2010	540	0.0608	0.0608	0.0608	0.0608	0.0608
50	0.1990	0.1990	0.1990	0.1990	0.1990	550	0.0603	0.0603	0.0603	0.0603	0.0603

which is also a function of the sample size n only. Thus, $[\mathrm{MSE_R}(\hat{C}'_{\mathrm{ip}})]^{1/2}$ values are the same for all C_{ip} values.

For short run applications (such as accepting a supplier providing short production runs in QS-9000 certification), the difference between the two relative errors is considered significant for sample sizes $n \leq 35$, and we strongly recommend using the MLE $\hat{C}'_{\mathrm{ip}}$ rather than the UMVUE $\hat{C}_{\mathrm{ip}}$. For other applications with sample sizes $n > 35$, the difference between the two estimators is negligible (less than 0.52%).

3. ESTIMATING PROCESS INACCURACY

To estimate the process inaccuracy, we consider the natural estimator $\hat{C}_{\mathrm{ia}}$ defined as the following, where $\bar{X} = \sum_{i=1}^{n} X_i/n$ is the conventional estimator of the process mean μ. We note that the estimator $\hat{C}_{\mathrm{ia}}$ can also be written as a function of C_{in}:

$$\hat{C}_{\mathrm{ia}} = \frac{(\bar{X}-T)^2}{D^2} = \frac{C_{\mathrm{ip}}}{n}\frac{n\hat{C}_{\mathrm{ia}}}{C_{\mathrm{ip}}} = \frac{C_{\mathrm{ip}}}{n}\frac{n(\bar{X}-T)^2}{\sigma^2}$$

If the process characteristic is normally distributed, then the estimator $\hat{C}_{\mathrm{ia}}$ is distributed as $[C_{\mathrm{ip}}/n]\chi_1^2(\delta)$, where $\chi_1^2(\delta)$ is a non-central chi-squared distribution with one degree of freedom and non-centrality parameter $\delta = n(\mu - T)^2/\sigma^2$. Therefore, the probability density function of $\hat{C}_{\mathrm{ia}}$ can be expressed as

$$g(y) = \sum_{k=0}^{\infty}\left\{\frac{[(ny)/(2C_{\mathrm{ip}})]^{k+\frac{1}{2}}\exp[-(ny)/(2C_{\mathrm{ip}})]}{y\Gamma(k+\frac{1}{2})} \times \frac{(\delta/2)^k\exp(-\delta/2)}{\Gamma(k+1)}\right\}, \quad \text{for } y > 0$$

The rth moment, the expected value, the variance, and the mean squared error of $\hat{C}_{\mathrm{ia}}$, therefore, can be calculated as

$$\begin{aligned}
\mathrm{E}(\hat{C}_{\mathrm{ia}}^r) &= \left(\frac{C_{\mathrm{ip}}}{n}\right)^r \mathrm{E}[\chi_1^2(\delta)]^r \\
&= \sum_{k=0}^{\infty}\left\{\left(\frac{2C_{\mathrm{ip}}}{n}\right)^r \frac{\Gamma(k+\frac{1}{2}+r)}{\Gamma(k+\frac{1}{2})} \times \frac{(\delta/2)^k\exp(-\delta/2)}{\Gamma(k+1)}\right\} \\
\mathrm{E}(\hat{C}_{\mathrm{ia}}) &= \left(\frac{C_{\mathrm{ip}}}{n}\right)\mathrm{E}[\chi_1^2(\delta)] \\
&= \left(\frac{C_{\mathrm{ip}}}{n}\right)(1+\delta) \\
&= \frac{C_{\mathrm{ip}}}{n} + C_{\mathrm{ia}}
\end{aligned}$$

$$\begin{aligned}
\mathrm{Var}(\hat{C}_{\mathrm{ia}}) &= \left(\frac{C_{\mathrm{ip}}}{n}\right)^2 \mathrm{Var}[\chi_1^2(\delta)] \\
&= \left(\frac{C_{\mathrm{ip}}}{n}\right)^2 (2+4\delta) \\
&= \frac{4C_{\mathrm{ip}}C_{\mathrm{ia}}}{n} + \frac{2C_{\mathrm{ip}}^2}{n} \\
\mathrm{MSE}(\hat{C}_{\mathrm{ia}}) &= \mathrm{Var}(\hat{C}_{\mathrm{ia}}) + [\mathrm{E}(\hat{C}_{\mathrm{ia}}) - C_{\mathrm{ia}}]^2 \\
&= \frac{4C_{\mathrm{ip}}C_{\mathrm{ia}}}{n} + \frac{3C_{\mathrm{ip}}^2}{n^2}
\end{aligned}$$

Since $\bar{X}$ is the MLE of μ, then by the invariance property of the MLE, the natural estimator $\hat{C}_{\mathrm{ia}}$ is the MLE of C_{ia}. Noting that $\mathrm{E}(\hat{C}_{\mathrm{ia}}) = C_{\mathrm{ia}} + (C_{\mathrm{ip}}/n)$, and $\mathrm{E}(\hat{C}_{\mathrm{ip}}) = C_{\mathrm{ip}}$, the corrected estimator $\tilde{C}_{\mathrm{ia}} = \hat{C}_{\mathrm{ia}} - (\hat{C}_{\mathrm{ip}}/n)$ must be unbiased for C_{ia}. We can show that $\tilde{C}_{\mathrm{ia}}$ is the UMVUE of C_{ia}, which is consistent. We can also show that $\sqrt{n}(\tilde{C}_{\mathrm{ia}} - C_{\mathrm{ia}})$ converges to $N(0, 4C_{\mathrm{ip}}C_{\mathrm{ia}})$ in distribution, and $\tilde{C}_{\mathrm{ia}}$ is asymptotically efficient (see the Appendix for the proofs). Thus, in real-world applications using the UMVUE $\tilde{C}_{\mathrm{ia}}$, which has all the desired statistical properties, as an estimate of C_{ia} would be reasonable.

We note that the MLE $\hat{C}_{\mathrm{ia}}$ has smaller variance than the UMVUE $\tilde{C}_{\mathrm{ia}}$. However, we can show that $\mathrm{MSE}(\tilde{C}_{\mathrm{ia}}) = 4C_{\mathrm{ip}}C_{\mathrm{ia}}/n + [2/n(n-1)](C_{\mathrm{ip}})^2$, and so $\mathrm{MSE}(\tilde{C}_{\mathrm{ia}}) - \mathrm{MSE}(\hat{C}_{\mathrm{ia}}) = [(3-n)/n^2(n-1)](C_{\mathrm{ip}})^2$, which is greater than 0 for $n = 2$, equal to 0 for $n = 3$, and less than 0 for $n \geq 4$. Therefore, the UMVUE $\tilde{C}_{\mathrm{ia}}$ has smaller mean squared error than the MLE $\hat{C}_{\mathrm{ia}}$, and is more reliable for applications with $n \geq 4$.

Tables 3(a) and 3(b) display the relative error, $[\mathrm{MSE_R}(\tilde{C}_{\mathrm{ia}})]^{1/2}$, of the UMVUE $\tilde{C}_{\mathrm{ia}}$ for $C_{\mathrm{ip}} = 1.00$, 0.56, 0.44, 0.36, 0.25, and $C_{\mathrm{ia}} = 2.25$. The value of C_{ia} is equivalent to $C_{\mathrm{a}} = 0.50$. The relative errors, $[\mathrm{MSE_R}(\tilde{C}_{\mathrm{ia}})]^{1/2}$, for $C_{\mathrm{ia}} = 5.06$ and 0.56 are available from the authors. We note that if the process is perfectly centered, then $C_{\mathrm{ia}} = 0.00$ (equivalently, $C_{\mathrm{a}} = 1.00$). For example, for $C_{\mathrm{ip}} = 1.00$, $C_{\mathrm{ia}} = 2.25$, and $n = 300$ we have $[\mathrm{MSE_R}(\tilde{C}_{\mathrm{ia}})]^{1/2} = 0.0770$. Thus, the average error (average relative deviation) of $\tilde{C}_{\mathrm{ia}}$ would be no greater than 7.70% of the true C_{ia}. Tables 4(a) and 4(b) display the relative error, $[\mathrm{MSE_R}(\hat{C}_{\mathrm{ia}})]^{1/2}$, of the MLE $\hat{C}_{\mathrm{ia}}$ for $C_{\mathrm{ip}} = 1.00$, 0.56, 0.44, 0.36, 0.25, and $C_{\mathrm{ia}} = 2.25$ (tables of $[\mathrm{MSE_R}(\hat{C}_{\mathrm{ia}})]^{1/2}$ for other values of C_{ia} are available from the authors). We note that for $n < 30$, the difference between the two relative errors (percentage of deviations) is significant. However, for $n > 30$, the difference between the two is negligible (less than 0.3%), and using either of the two estimators is equally reliable.

Table 3. $[\mathrm{MSE_R}(\tilde{C}_{ia})]^{1/2}$ for various C_{ip}, $C_{ia} = 2.25$ and (a) $n = 2(1)50$ and (b) $n = 60(10)550$

	C_{ip}						C_{ip}				
n	1.00	0.56	0.44	0.36	0.25	n	1.00	0.56	0.44	0.36	0.25
1	******	******	******	******	******	60	0.1725	0.1292	0.1149	0.1034	0.0861
2	1.0423	0.7500	0.6588	0.5879	0.4843	70	0.1596	0.1196	0.1063	0.0957	0.0797
3	0.8114	0.5951	0.5257	0.4710	0.3902	80	0.1493	0.1119	0.0994	0.0895	0.0746
4	0.6909	0.5103	0.4517	0.4053	0.3364	90	0.1407	0.1055	0.0937	0.0844	0.0703
5	0.6126	0.4541	0.4024	0.3613	0.3002	100	0.1335	0.1001	0.0889	0.0800	0.0667
6	0.5563	0.4133	0.3665	0.3292	0.2737	110	0.1273	0.0954	0.0848	0.0763	0.0636
7	0.5132	0.3819	0.3387	0.3044	0.2531	120	0.1218	0.0913	0.0812	0.0731	0.0609
8	0.4788	0.3567	0.3165	0.2845	0.2366	130	0.1170	0.0877	0.0780	0.0702	0.0585
9	0.4506	0.3359	0.2981	0.2680	0.2230	140	0.1128	0.0846	0.0752	0.0676	0.0564
10	0.4268	0.3184	0.2826	0.2541	0.2115	150	0.1089	0.0817	0.0726	0.0653	0.0544
11	0.4065	0.3034	0.2693	0.2422	0.2016	160	0.1055	0.0791	0.0703	0.0633	0.0527
12	0.3888	0.2903	0.2578	0.2318	0.1929	170	0.1023	0.0767	0.0682	0.0614	0.0511
13	0.3732	0.2788	0.2475	0.2226	0.1853	180	0.0994	0.0746	0.0663	0.0596	0.0497
14	0.3594	0.2685	0.2385	0.2145	0.1786	190	0.0968	0.0726	0.0645	0.0581	0.0484
15	0.3470	0.2593	0.2303	0.2071	0.1725	200	0.0943	0.0707	0.0629	0.0566	0.0471
16	0.3358	0.2510	0.2230	0.2005	0.1670	210	0.0921	0.0690	0.0614	0.0552	0.0460
17	0.3256	0.2435	0.2163	0.1945	0.1620	220	0.0899	0.0674	0.0599	0.0539	0.0450
18	0.3163	0.2366	0.2101	0.1890	0.1574	230	0.0880	0.0660	0.0586	0.0528	0.0440
19	0.3078	0.2302	0.2045	0.1839	0.1532	240	0.0861	0.0646	0.0574	0.0516	0.0430
20	0.2999	0.2243	0.1993	0.1793	0.1493	250	0.0844	0.0633	0.0562	0.0506	0.0422
21	0.2926	0.2189	0.1945	0.1749	0.1457	260	0.0827	0.0620	0.0551	0.0496	0.0413
22	0.2858	0.2138	0.1900	0.1709	0.1423	270	0.0812	0.0609	0.0541	0.0487	0.0406
23	0.2794	0.2091	0.1858	0.1671	0.1392	280	0.0797	0.0598	0.0531	0.0478	0.0398
24	0.2735	0.2047	0.1818	0.1636	0.1362	290	0.0783	0.0587	0.0522	0.0470	0.0392
25	0.2679	0.2005	0.1781	0.1603	0.1335	300	0.0770	0.0577	0.0513	0.0462	0.0385
26	0.2626	0.1966	0.1747	0.1571	0.1309	310	0.0758	0.0568	0.0505	0.0454	0.0379
27	0.2577	0.1929	0.1714	0.1542	0.1284	320	0.0746	0.0559	0.0497	0.0447	0.0373
28	0.2530	0.1894	0.1683	0.1514	0.1261	330	0.0734	0.0551	0.0489	0.0440	0.0367
29	0.2486	0.1861	0.1654	0.1488	0.1239	340	0.0723	0.0542	0.0482	0.0434	0.0362
30	0.2444	0.1830	0.1626	0.1463	0.1218	350	0.0713	0.0535	0.0475	0.0428	0.0356
31	0.2404	0.1800	0.1599	0.1439	0.1198	360	0.0703	0.0527	0.0469	0.0422	0.0351
32	0.2365	0.1771	0.1574	0.1416	0.1180	370	0.0693	0.0520	0.0462	0.0416	0.0347
33	0.2329	0.1744	0.1550	0.1394	0.1162	380	0.0684	0.0513	0.0456	0.0410	0.0342
34	0.2294	0.1718	0.1527	0.1374	0.1144	390	0.0675	0.0506	0.0450	0.0405	0.0338
35	0.2261	0.1693	0.1505	0.1354	0.1128	400	0.0667	0.0500	0.0444	0.0400	0.0333
36	0.2229	0.1670	0.1484	0.1335	0.1112	410	0.0659	0.0494	0.0439	0.0395	0.0329
37	0.2199	0.1647	0.1463	0.1317	0.1097	420	0.0651	0.0488	0.0434	0.0390	0.0325
38	0.2169	0.1625	0.1444	0.1299	0.1082	430	0.0643	0.0482	0.0429	0.0386	0.0322
39	0.2141	0.1604	0.1425	0.1282	0.1068	440	0.0636	0.0477	0.0424	0.0381	0.0318
40	0.2114	0.1584	0.1407	0.1266	0.1055	450	0.0629	0.0471	0.0419	0.0377	0.0314
41	0.2088	0.1564	0.1390	0.1251	0.1042	460	0.0622	0.0466	0.0414	0.0373	0.0311
42	0.2063	0.1545	0.1373	0.1236	0.1029	470	0.0615	0.0461	0.0410	0.0369	0.0308
43	0.2039	0.1527	0.1357	0.1221	0.1017	480	0.0609	0.0456	0.0406	0.0365	0.0304
44	0.2015	0.1510	0.1342	0.1207	0.1006	490	0.0602	0.0452	0.0402	0.0361	0.0301
45	0.1993	0.1493	0.1327	0.1194	0.0994	500	0.0596	0.0447	0.0398	0.0358	0.0298
46	0.1971	0.1476	0.1312	0.1181	0.0984	510	0.0591	0.0443	0.0394	0.0354	0.0295
47	0.1950	0.1461	0.1298	0.1168	0.0973	520	0.0585	0.0439	0.0390	0.0351	0.0292
48	0.1929	0.1445	0.1284	0.1156	0.0963	530	0.0579	0.0434	0.0386	0.0348	0.0290
49	0.1909	0.1430	0.1271	0.1144	0.0953	540	0.0574	0.0430	0.0383	0.0344	0.0287
50	0.1890	0.1416	0.1258	0.1132	0.0943	550	0.0569	0.0426	0.0379	0.0341	0.0284

Table 4. $[\mathrm{MSE_R}(\hat{C}_{ia})]^{1/2}$ for various C_{ip}, $C_{ia} = 2.25$ and (a) $n = 2(1)50$ and (b) $n = 60(10)550$

	C_{ip}						C_{ip}				
n	1.00	0.56	0.44	0.36	0.25	n	1.00	0.56	0.44	0.36	0.25
1	1.5396	1.0897	0.9525	0.8466	0.6939	60	0.1726	0.1293	0.1149	0.1034	0.0861
2	1.0184	0.7395	0.6514	0.5824	0.4811	70	0.1597	0.1197	0.1064	0.0957	0.0797
3	0.8114	0.5951	0.5257	0.4710	0.3902	80	0.1494	0.1119	0.0995	0.0895	0.0746
4	0.6939	0.5116	0.4526	0.4060	0.3368	90	0.1408	0.1055	0.0938	0.0844	0.0703
5	0.6158	0.4555	0.4034	0.3620	0.3006	100	0.1336	0.1001	0.0890	0.0800	0.0667
6	0.5592	0.4146	0.3673	0.3298	0.2740	110	0.1273	0.0954	0.0848	0.0763	0.0636
7	0.5158	0.3830	0.3395	0.3050	0.2535	120	0.1219	0.0914	0.0812	0.0731	0.0609
8	0.4811	0.3577	0.3172	0.2850	0.2369	130	0.1171	0.0878	0.0780	0.0702	0.0585
9	0.4526	0.3368	0.2987	0.2684	0.2232	140	0.1128	0.0846	0.0752	0.0676	0.0564
10	0.4286	0.3192	0.2832	0.2545	0.2117	150	0.1090	0.0817	0.0726	0.0653	0.0544
11	0.4081	0.3041	0.2698	0.2525	0.2018	160	0.1055	0.0791	0.0703	0.0633	0.0527
12	0.3902	0.2909	0.2582	0.2321	0.1931	170	0.1024	0.0767	0.0682	0.0614	0.0511
13	0.3745	0.2793	0.2479	0.2229	0.1855	180	0.0995	0.0746	0.0663	0.0596	0.0497
14	0.3606	0.2690	0.2388	0.2147	0.1787	190	0.0968	0.0726	0.0645	0.0581	0.0484
15	0.3481	0.2598	0.2306	0.2074	0.1726	200	0.0944	0.0707	0.0629	0.0566	0.0472
16	0.3368	0.2515	0.2232	0.2007	0.1671	210	0.0921	0.0690	0.0614	0.0552	0.0460
17	0.3265	0.2439	0.2165	0.1947	0.1621	220	0.0900	0.0674	0.0599	0.0540	0.0450
18	0.3172	0.2369	0.2104	0.1892	0.1575	230	0.0880	0.0660	0.0586	0.0528	0.0440
19	0.3086	0.2305	0.2047	0.1841	0.1533	240	0.0861	0.0646	0.0574	0.0517	0.0430
20	0.3006	0.2247	0.1995	0.1794	0.1494	250	0.0844	0.0633	0.0562	0.0506	0.0422
21	0.2933	0.2192	0.1947	0.1751	0.1458	260	0.0827	0.0620	0.0551	0.0496	0.0414
22	0.2864	0.2141	0.1901	0.1710	0.1424	270	0.0812	0.0609	0.0541	0.0487	0.0406
23	0.2800	0.2094	0.1859	0.1672	0.1393	280	0.0797	0.0598	0.0531	0.0478	0.0398
24	0.2740	0.2049	0.1820	0.1637	0.1363	290	0.0783	0.0587	0.0522	0.0470	0.0392
25	0.2684	0.2007	0.1783	0.1604	0.1336	300	0.0770	0.0578	0.0513	0.0462	0.0385
26	0.2632	0.1968	0.1748	0.1573	0.1310	310	0.0758	0.0568	0.0505	0.0454	0.0379
27	0.2582	0.1931	0.1715	0.1543	0.1285	320	0.0746	0.0559	0.0497	0.0447	0.0373
28	0.2535	0.1896	0.1684	0.1515	0.1262	330	0.0734	0.0551	0.0489	0.0440	0.0367
29	0.2490	0.1863	0.1655	0.1489	0.1240	340	0.0723	0.0542	0.0482	0.0434	0.0362
30	0.2448	0.1831	0.1627	0.1464	0.1219	350	0.0713	0.0535	0.0475	0.0428	0.0356
31	0.2408	0.1801	0.1600	0.1440	0.1199	360	0.0703	0.0527	0.0469	0.0422	0.0351
32	0.2369	0.1773	0.1575	0.1417	0.1180	370	0.0693	0.0520	0.0462	0.0416	0.0347
33	0.2333	0.1746	0.1551	0.1395	0.1162	380	0.0684	0.0513	0.0456	0.0410	0.0342
34	0.2298	0.1720	0.1528	0.1374	0.1145	390	0.0675	0.0506	0.0450	0.0405	0.0338
35	0.2264	0.1695	0.1506	0.1355	0.1128	400	0.0667	0.0500	0.0445	0.0400	0.0333
36	0.2232	0.1671	0.1485	0.1336	0.1112	410	0.0659	0.0494	0.0439	0.0395	0.0329
37	0.2202	0.1648	0.1464	0.1317	0.1097	420	0.0651	0.0488	0.0434	0.0390	0.0325
38	0.2172	0.1626	0.1445	0.1300	0.1083	430	0.0643	0.0482	0.0429	0.0386	0.0322
39	0.2144	0.1605	0.1426	0.1283	0.1069	440	0.0636	0.0477	0.0424	0.0381	0.0318
40	0.2117	0.1585	0.1408	0.1267	0.1055	450	0.0629	0.0415	0.0419	0.0377	0.0314
41	0.2091	0.1565	0.1391	0.1251	0.1042	460	0.0622	0.0466	0.0415	0.0373	0.0311
42	0.2066	0.1546	0.1374	0.1236	0.1030	470	0.0615	0.0461	0.0410	0.0369	0.0308
43	0.2041	0.1528	0.1358	0.1222	0.1018	480	0.0609	0.0457	0.0406	0.0365	0.0304
44	0.2018	0.1511	0.1342	0.1208	0.1006	490	0.0603	0.0452	0.0402	0.0361	0.0301
45	0.1995	0.1494	0.1327	0.1194	0.0995	500	0.0596	0.0447	0.0398	0.0358	0.0298
46	0.1973	0.1477	0.1313	0.1181	0.0984	510	0.0591	0.0443	0.0394	0.0354	0.0295
47	0.1952	0.1462	0.1299	0.1168	0.0973	520	0.0585	0.0439	0.0390	0.0351	0.0292
48	0.1931	0.1446	0.1285	0.1156	0.0963	530	0.0579	0.0434	0.0386	0.0348	0.0290
49	0.1911	0.1431	0.1272	0.1144	0.0953	540	0.0574	0.0430	0.0383	0.0344	0.0287
50	0.1892	0.1417	0.1259	0.1133	0.0944	550	0.0569	0.0426	0.0379	0.0341	0.0284

4. ESTIMATING PROCESS INCAPABILITY

To estimate the process incapability (a combined measure of process imprecision and process inaccuracy), we consider the natural estimator $\hat{C}_{pp}$ defined as the following, where $\bar{X} = \sum_{i=1}^{n} X_i/n$, which can also be written as a function of C_{ip}:

$$\hat{C}_{pp} = \frac{1}{n}\sum_{i=1}^{n}\frac{(X_i-\bar{X})^2}{D^2} + \frac{(\bar{X}-T)^2}{D^2} = \frac{C_{ip}}{n}\frac{n\hat{C}_{pp}}{C_{ip}} = \frac{C_{ip}}{n}\sum_{i=1}^{n}\frac{(X_i-T)^2}{\sigma^2}$$

If the process characteristic is normally distributed, then the estimator $\hat{C}_{pp}$ is distributed as $[C_{ip}/n]\chi_n^2(\delta)$, where $\chi_n^2(\delta)$ is a non-central chi-squared distribution with n degrees of freedom and non-centrality parameter $\delta = n(\mu - T)^2/\sigma^2 = nC_{ia}/C_{ip}$. Therefore, the probability density function of $\hat{C}_{pp}$ can be expressed as

$$h(y) = \sum_{k=0}^{\infty}\left\{\frac{[(ny)/(2C_{ip})]^{k+(n/2)}\exp[-(ny)/(2C_{ip})]}{y\Gamma(k+(n/2))} \times \frac{(\delta/2)^k\exp(-\delta/2)}{\Gamma(k+1)}\right\}, \quad \text{for } y > 0$$

The rth moment (hence the expected value, the variance, and the mean squared error) of $\hat{C}_{pp}$, therefore can be calculated as follows:

$$\mathrm{E}(\hat{C}_{pp}^r) = \left(\frac{C_{ip}}{n}\right)^r \mathrm{E}[\chi_n^2(\delta)]^r = \sum_{k=0}^{\infty}\left\{\left(\frac{2C_{ip}}{n}\right)^r \frac{\Gamma(k+(n/2)+r)}{\Gamma(k+(n/2))} \times \frac{(\delta/2)^k\exp(-\delta/2)}{\Gamma(k+1)}\right\}$$

$$\mathrm{E}(\hat{C}_{pp}) = \left(\frac{C_{ip}}{n}\right)\mathrm{E}[\chi_n^2(\delta)] = \frac{C_{ip}}{n}(n+\delta) = C_{ip} + C_{ia} = C_{pp}$$

$$\mathrm{Var}(\hat{C}_{pp}) = \left(\frac{C_{ip}}{n}\right)^2 \mathrm{Var}[\chi_n^2(\delta)] = \left(\frac{C_{ip}}{n}\right)^2(2n+4\delta) = \frac{2C_{ip}}{n}(C_{ia}+C_{pp})$$

$$\mathrm{MSE}(\hat{C}_{pp}) = \mathrm{Var}(\hat{C}_{pp}) + [\mathrm{E}(\hat{C}_{pp}) - C_{pp}]^2 = \frac{2C_{ip}}{n}(C_{ia}+C_{pp})$$

If the process characteristic follows the normal distribution, then we can show that $\hat{C}_{pp}$ is the MLE, which is also the UMVUE of C_{pp}. We can also show that $\hat{C}_{pp}$ is consistent, $\sqrt{n}(\hat{C}_{pp} - C_{pp})$ converges to $N(0, 2C_{ip}C_{ia} + 2C_{ip}C_{pp})$ in distribution, and $\hat{C}_{pp}$ is asymptotically efficient (see the Appendix for the proofs). Since the estimator has all the desired statistical properties, in practice using $\hat{C}_{pp}$ to estimate process incapability would be reasonable.

5. DECISION MAKING

Under the normality assumption, $n\hat{C}_{pp}/(C_{pp} - C_{ia})$ is distributed as $\chi_n^2(\delta)$, a non-central chi-squared distribution with n degrees of freedom and non-centrality parameter $\delta = n(\mu - T)^2/\sigma^2 = nC_{ia}/C_{ip}$. Let $W = W(X_1, X_2, \ldots, X_n)$ be a statistic calculated from the sample data satisfying $\mathrm{P}\{C_{pp} \le W\} = 1 - \alpha$, where the confidence level $1 - \alpha$ does not depend on C_{pp}. Then, W is a $100(1-\alpha)\%$ upper confidence limit for C_{pp}. We note that

$$\begin{aligned}\mathrm{P}\{C_{pp} \le W\} &= \mathrm{P}\{C_{pp} - C_{ia} \le W - C_{ia}\}\\ &= \mathrm{P}\{1/(C_{pp} - C_{ia}) \ge 1/(W - C_{ia})\}\\ &= \mathrm{P}\{n\hat{C}_{pp}/(\hat{C}_{pp} - C_{ia}) \ge n\hat{C}_{pp}/(W - C_{ia})\}\\ &= \mathrm{P}\{\chi_n^2(\delta) \ge n\hat{C}_{pp}/(W - C_{ia})\}\\ &= 1 - \alpha\end{aligned}$$

Therefore, $n\hat{C}_{pp}/(W - C_{ia}) = \chi_n^2(\alpha, \delta)$, where $\chi_n^2(\alpha, \delta)$ is the (lower) αth percentile of the $\chi_n^2(\delta)$ distribution. A $100(1 - \alpha)\%$ upper confidence limit on C_{pp} can be written in terms of $\hat{C}_{pp}$ as $W = C_{ia} + [n\hat{C}_{pp}/\chi_n^2(\alpha, \delta)]$.

Tables 5(a), 6(a), and 7(a) give 90%, 95%, and 99% upper confidence limits for C_{pp} under $\mu = T$ with n given, and $\hat{C}_{pp}$ calculated from the sample data. On the other hand, $\hat{C}_{pp} = \chi_n^2(\alpha, \delta)(W - C_{ia})/n$ depends on W and C_{ia}. In the special case when $\mu = T$ and W equals the recommended maximum value for C_{pp}, the probability that $C_{pp} \le W$ would be either 1 or 0 if C_{pp} were known. In practice, since C_{pp} is unknown, we take a random sample of size n and calculate $\hat{C}_{pp}$.

Suppose that a process is capable if $\hat{C}_{pp} \le \chi_n^2(\alpha, \delta)(C_0 - C_{ia})/n$, where C_0 is the recommended maximum value, and we claim that the process is capable for at least $100(1-\alpha)\%$ of the time. Therefore,

Table 5. (a) The 90% upper confidence limits for C_{pp} under $\mu = T$, with given $\hat{C}_{pp}$. (b) The maximum value of $\hat{C}_{pp}$ under $\mu = T$ for which the process is capable ($C_{pp} \leq C_0$) 90% of the time

(a)

	Sample size n											
$\hat{C}_{pp}$	5	10	15	20	25	30	35	40	45	50	55	60
0.25	0.776	0.514	0.439	0.402	0.379	0.364	0.353	0.344	0.337	0.332	0.327	0.323
0.36	1.118	0.740	0.632	0.579	0.546	0.524	0.508	0.496	0.486	0.478	0.471	0.465
0.44	1.366	0.904	0.772	0.707	0.668	0.641	0.621	0.606	0.594	0.584	0.575	0.568
0.56	1.739	1.151	0.983	0.900	0.850	0.816	0.790	0.771	0.756	0.743	0.732	0.723
1.00	3.105	2.055	1.755	1.607	1.518	1.456	1.411	1.377	1.349	1.323	1.308	1.291

	Sample size n											
$\hat{C}_{pp}$	70	80	90	100	110	120	130	140	150	160	170	180
0.25	0.316	0.311	0.307	0.304	0.301	0.298	0.296	0.294	0.292	0.291	0.289	0.288
0.36	0.455	0.448	0.442	0.437	0.433	0.429	0.426	0.423	0.421	0.419	0.417	0.415
0.44	0.557	0.548	0.540	0.534	0.529	0.525	0.521	0.518	0.515	0.512	0.509	0.507
0.56	0.708	0.697	0.688	0.680	0.673	0.668	0.669	0.659	0.655	0.651	0.648	0.646
1.00	1.265	1.245	1.228	1.214	1.203	1.193	1.184	1.176	1.169	1.163	1.158	1.153

(b)

	Sample size n											
C_0	5	10	15	20	25	30	35	40	45	50	55	60
0.25	0.081	0.122	0.142	0.156	0.165	0.172	0.177	0.182	0.185	0.188	0.191	0.194
0.36	0.116	0.175	0.205	0.224	0.237	0.247	0.255	0.261	0.267	0.271	0.275	0.279
0.44	0.142	0.214	0.251	0.274	0.290	0.302	0.312	0.320	0.326	0.332	0.336	0.341
0.56	0.180	0.272	0.319	0.348	0.369	0.385	0.397	0.407	0.415	0.422	0.428	0.434
1.00	0.322	0.487	0.570	0.622	0.659	0.687	0.708	0.726	0.741	0.754	0.765	0.774

	Sample size n											
C_0	70	80	90	100	110	120	130	140	150	160	170	180
0.25	0.198	0.201	0.204	0.206	0.208	0.210	0.211	0.213	0.214	0.215	0.216	0.217
0.36	0.285	0.289	0.293	0.296	0.299	0.302	0.304	0.306	0.308	0.309	0.311	0.312
0.44	0.348	0.354	0.358	0.362	0.366	0.369	0.372	0.374	0.376	0.378	0.380	0.382
0.56	0.443	0.450	0.456	0.461	0.466	0.470	0.473	0.476	0.479	0.481	0.484	0.486
1.00	0.790	0.803	0.814	0.824	0.832	0.839	0.845	0.850	0.855	0.860	0.864	0.868

the factor $\chi_n^2(\alpha, \delta)(C_0 - C_{ia})/n$ is the maximum value of the estimated incapability index $\hat{C}_{pp}$ in order that the process is considered capable at least $100(1-\alpha)\%$ of the time. Tables 5(b), 6(b), and 7(b) give the maximum values of $\hat{C}_{pp}$ in the case with $\mu = T$ for the process to be considered capable (i.e., $C_{pp} \leq C_0$) 90%, 95%, and 99% of the time.

Suppose that the requirement for a process to be capable is that $C_{pp} \leq 1.00$. We take a random sample of size n, and calculate $\hat{C}_{pp}$. Using Table 6(b) based on the random sample of $n = 30$, for example, we obtain $C_0 = 0.616$. Thus, if the calculated $\hat{C}_{pp} \leq 0.616$, then we claim that the process is capable at least 95% of the time.

6. CONCLUSION

Greenwich and Jahr-Schaffrath [1] introduced the process incapability index $C_{pp} = C_{ip} + C_{ia}$, which provides an uncontaminated separation between information concerning the process precision (C_{ip}) and process accuracy (C_{ia}). In this note, we consider the three indices, and investigate the statistical properties of their natural estimators. For the three indices, we obtain their UMVUEs and MLEs. For each index, we compare the reliability of the two estimators based on their relative errors (square root of the relative mean squared error). In addition, we construct 90%, 95%, and 99% upper confidence limits,

Table 6. (a) The 95% upper confidence limits for C_{pp} under $\mu = T$, with given $\hat{C}_{pp}$. (b) The maximum value of $\hat{C}_{pp}$ under $\mu = T$ for which the process is capable ($C_{pp} \le C_0$) 95% of the time

(a)

	Sample size n											
$\hat{C}_{pp}$	5	10	15	20	25	30	35	40	45	50	55	60
0.25	1.091	0.634	0.516	0.461	0.428	0.406	0.389	0.377	0.368	0.360	0.353	0.347
0.36	1.571	0.914	0.744	0.664	0.616	0.584	0.561	0.543	0.529	0.518	0.508	0.500
0.44	1.921	1.117	0.909	0.811	0.753	0.714	0.686	0.664	0.647	0.633	0.621	0.611
0.56	2.444	1.421	1.157	1.032	0.958	0.908	0.872	0.845	0.823	0.805	0.791	0.778
1.00	4.365	2.538	2.066	1.843	1.711	1.622	1.558	1.509	1.470	1.438	1.412	1.389
	Sample size n											
$\hat{C}_{pp}$	70	80	90	100	110	120	130	140	150	160	170	180
0.25	0.338	0.331	0.325	0.321	0.317	0.313	0.311	0.308	0.306	0.304	0.302	0.300
0.36	0.487	0.477	0.469	0.462	0.456	0.451	0.447	0.443	0.440	0.437	0.435	0.432
0.44	0.595	0.583	0.573	0.565	0.558	0.552	0.547	0.542	0.538	0.534	0.531	0.528
0.56	0.758	0.742	0.729	0.719	0.710	0.702	0.696	0.690	0.685	0.680	0.676	0.672
1.00	1.353	1.325	1.302	1.283	1.267	1.254	1.242	1.232	1.223	1.214	1.207	1.200

(b)

	Sample size n											
C_0	5	10	15	20	25	30	35	40	45	50	55	60
0.25	0.057	0.099	0.121	0.136	0.146	0.154	0.160	0.166	0.170	0.174	0.177	0.180
0.36	0.082	0.142	0.174	0.195	0.210	0.222	0.231	0.239	0.245	0.250	0.255	0.259
0.44	0.101	0.173	0.213	0.239	0.257	0.271	0.282	0.292	0.299	0.306	0.312	0.317
0.56	0.128	0.221	0.271	0.304	0.327	0.345	0.359	0.371	0.381	0.389	0.397	0.403
1.00	0.229	0.394	0.484	0.543	0.584	0.616	0.642	0.663	0.680	0.695	0.708	0.720
	Sample size n											
C_0	70	80	90	100	110	120	130	140	150	160	170	180
0.25	0.185	0.189	0.192	0.195	0.197	0.199	0.201	0.203	0.204	0.206	0.207	0.208
0.36	0.266	0.272	0.277	0.281	0.284	0.287	0.290	0.292	0.294	0.296	0.298	0.300
0.44	0.325	0.332	0.338	0.343	0.347	0.351	0.354	0.357	0.360	0.362	0.365	0.367
0.56	0.414	0.423	0.430	0.436	0.442	0.447	0.451	0.455	0.458	0.461	0.464	0.467
1.00	0.739	0.755	0.768	0.779	0.789	0.798	0.805	0.812	0.818	0.823	0.829	0.833

and the maximum values of $\hat{C}_{pp}$ for which the process is capable. The results obtained in this paper are useful to the practitioners in choosing good estimators and making reliable decisions on judging process capability.

APPENDIX

Theorem 1. *If the process characteristic is normally distributed, then:*

(a) $\hat{C}_{ip}$ *is the UMVUE of* C_{ip}*;*
(b) $\hat{C}_{ip}$ *is consistent;*
(c) $\sqrt{n}(\hat{C}_{ip} - C_{ip})$ *converges to* $N(0, 2C_{ip}^2)$ *in distribution;*
(d) $\hat{C}_{ip}$ *is asymptotically efficient.*

Proof. (a) Since S_{n-1}^2 is a sufficient and complete statistic for σ^2, and the unbiased estimator $\hat{C}_{ip}$ is a function S_{n-1}^2 of only, then by the Lehmann–Scheffe Theorem, $\hat{C}_{ip}$ is the UMVUE. (b) For all $\varepsilon > 0$,

$$P\{|\hat{C}_{ip} - C_{ip}| > \varepsilon\} < E(\hat{C}_{ip} - C_{ip})^2/\varepsilon^2$$

Since

$$E(\hat{C}_{ip} - C_{ip})^2 = \mathrm{Var}(\hat{C}_{ip}) = 2C_{ip}^2/(n-1)$$

converges to zero, then $\hat{C}_{ip}$ must be consistent. (c) Greenwich and Jahr-Schaffrath [1] showed that, under general conditions, $\sqrt{n}(\hat{C}_{ip} - C_{ip})$ converges to $N(0, \sigma_{ip}^2)$ in distribution, where $\sigma_{ip}^2 = (\mu_4 - \sigma^4)/D^4$. The result follows directly since for a normal

Table 7. (a) The 99% upper confidence limits for C_{pp} under $\mu = T$, with given $\hat{C}_{pp}$. (b) The maximum value of $\hat{C}_{pp}$ under $\mu = T$ for which the process is capable ($C_{pp} \leq C_0$) 99% of the time

(a)

	Sample size n											
$\hat{C}_{pp}$	5	10	15	20	25	30	35	40	45	50	55	60
0.25	2.255	0.977	0.717	0.605	0.542	0.502	0.473	0.451	0.434	0.421	0.410	0.400
0.36	3.247	1.407	1.033	0.872	0.781	0.722	0.681	0.650	0.625	0.606	0.590	0.576
0.44	3.969	1.720	1.262	1.065	0.955	0.883	0.832	0.794	0.764	0.741	0.721	0.704
0.56	5.051	2.189	1.606	1.356	1.215	1.123	1.059	1.011	0.973	0.943	0.917	0.896
1.00	9.020	3.909	2.868	2.421	2.169	2.006	1.891	1.805	1.737	1.683	1.638	1.601

	Sample size n											
$\hat{C}_{pp}$	70	80	90	100	110	120	130	140	150	160	170	180
0.25	0.385	0.374	0.364	0.357	0.351	0.345	0.340	0.336	0.333	0.330	0.327	0.324
0.36	0.555	0.538	0.525	0.514	0.505	0.497	0.490	0.484	0.479	0.475	0.471	0.467
0.44	0.678	0.657	0.641	0.628	0.617	0.607	0.599	0.592	0.586	0.580	0.575	0.571
0.56	0.863	0.837	0.816	0.799	0.785	0.773	0.763	0.754	0.746	0.738	0.732	0.726
1.00	1.540	1.494	1.457	1.427	1.402	1.381	1.362	1.346	1.331	1.319	1.307	1.297

(b)

	Sample size n											
C_0	5	10	15	20	25	30	35	40	45	50	55	60
0.25	0.028	0.064	0.087	0.103	0.115	0.125	0.132	0.139	0.144	0.149	0.153	0.156
0.36	0.040	0.092	0.126	0.149	0.166	0.179	0.190	0.199	0.207	0.214	0.220	0.225
0.44	0.049	0.113	0.153	0.182	0.203	0.219	0.233	0.244	0.253	0.261	0.269	0.275
0.56	0.062	0.143	0.195	0.231	0.258	0.279	0.296	0.310	0.322	0.333	0.342	0.350
1.00	0.111	0.256	0.349	0.413	0.461	0.498	0.529	0.554	0.576	0.594	0.610	0.625

	Sample size n											
C_0	70	80	90	100	110	120	130	140	150	160	170	180
0.25	0.162	0.167	0.172	0.175	0.178	0.181	0.184	0.186	0.188	0.190	0.191	0.193
0.36	0.234	0.241	0.247	0.252	0.257	0.261	0.264	0.268	0.270	0.273	0.275	0.278
0.44	0.286	0.294	0.302	0.308	0.314	0.319	0.323	0.327	0.330	0.334	0.337	0.339
0.56	0.364	0.375	0.384	0.392	0.399	0.406	0.411	0.416	0.421	0.425	0.428	0.432
1.00	0.649	0.669	0.686	0.701	0.713	0.724	0.734	0.743	0.751	0.758	0.765	0.771

distribution, $\mu_4 = 3\sigma^4$ and $C_{ip} = (\sigma/D)^2$. (d) Under the normality assumption, the information matrix can be calculated as follows. Since the information lower bound is achieved, then $\hat{C}_{ip}$ must be asymptotically efficient:

$$I(\theta) = I(\mu, \sigma) = \begin{bmatrix} 1/\sigma^2 & 0 \\ 0 & 1/(2\sigma^4) \end{bmatrix},$$

$$\begin{bmatrix} \dfrac{\partial C_{ip}}{\partial \mu} \dfrac{\partial C_{ip}}{\partial \sigma^2} \end{bmatrix} \frac{I^{-1}(\theta)}{n} \begin{bmatrix} \dfrac{\partial C_{ip}}{\partial \mu} \\ \dfrac{\partial C_{ip}}{\partial \sigma^2} \end{bmatrix} = \frac{2C_{ip}^2}{n}$$

Theorem 2. *If the process characteristic is normally distributed, then:*

(a) $\tilde{C}_{ia}$ *is the UMVUE of* C_{ia}*;*
(b) $\tilde{C}_{ia}$ *is consistent;*
(c) $\sqrt{n}(\tilde{C}_{ia} - C_{ia})$ *converges to* $N(0, 4C_{ip}C_{ia})$ *in distribution;*
(d) $\tilde{C}_{ia}$ *is asymptotically efficient.*

Proof. (a) Noting that $\bar{X}$ is a sufficient and complete statistic for μ, and that the unbiased estimator $\tilde{C}_{ia}$ is a function of $\bar{X}$ only. By the Lehmann–Scheffe Theorem, $\tilde{C}_{ia}$ is the UMVUE of C_{ia}. (b) For any $\varepsilon > 0$,

$$P\{|\tilde{C}_{ia} - C_{ia}| > \varepsilon\} < E(\tilde{C}_{ia} - C_{ia})^2/\varepsilon^2$$

Since

$$E(\tilde{C}_{ia} - C_{ia})^2 = 4C_{ip}C_{ia}/n + [2C_{ip}^2/(n^2 - n)]$$

converges to zero, then $\tilde{C}_{ia}$ must be consistent. (c) Greenwich and Jahr-Schaffrath [1] showed that under general conditions $\sqrt{n}(\hat{C}_{ia} - C_{ia})$ converges to $N(0, \sigma_{ia}^2)$ in distribution, where $\sigma_{ia}^2 = 4(\mu - T)^2\sigma^2/D^4$. Under the normality assumption, $\sqrt{n}(\hat{C}_{ia} - C_{ia})$ must converge to $N(0, 4C_{ip}C_{ia})$ in distribution. Since $(\tilde{C}_{ia} - \hat{C}_{ia})$ converges to zero in probability, then by Slutsky's Theorem,

$$\sqrt{n}(\tilde{C}_{ia} - C_{ia}) = \sqrt{n}(\tilde{C}_{ia} - \hat{C}_{ia}) + \sqrt{n}(\hat{C}_{ia} - C_{ia})$$

converges to $N(0, 4C_{ip}C_{ia})$ in distribution. (d) Under the normality assumption, the information matrix can be calculated as follows. Since the information lower bound is achieved, then the estimator $\tilde{C}_{ia}$ must be asymptotically efficient:

$$I(\theta) = I(\mu, \sigma) = \begin{bmatrix} 1/\sigma^2 & 0 \\ 0 & 1/(2\sigma^4) \end{bmatrix},$$

$$\begin{bmatrix} \dfrac{\partial C_{ia}}{\partial \mu} & \dfrac{\partial C_{ia}}{\partial \sigma^2} \end{bmatrix} \frac{I^{-1}(\theta)}{n} \begin{bmatrix} \dfrac{\partial C_{ia}}{\partial \mu} \\ \dfrac{\partial C_{ia}}{\partial \sigma^2} \end{bmatrix} = \frac{4C_{ip}^2 C_{ia}}{n}$$

Theorem 3. *If the process characteristic is normally distributed, then:*

(a) $\hat{C}_{pp}$ *is the MLE of* C_{pp}*;*
(b) $\hat{C}_{pp}$ *is the UMVUE of* C_{pp}*;*
(c) $\hat{C}_{pp}$ *is consistent;*
(d) $\sqrt{n}(\hat{C}_{pp} - C_{pp})$ *converges to* $N(0, 2C_{ip}C_{ia} + 2C_{ip}C_{pp})$ *in distribution;*
(e) $\hat{C}_{pp}$ *is asymptotically efficient.*

Proof. (a) Since $(\bar{X}, S_n^2)$ is the MLE of (μ, σ^2), where $S_n^2 = \sum_{i=1}^n (X_i - \bar{X})^2/n$, and $\hat{C}_{pp} = (S_n^2/D^2) + [(\bar{X} - T)^2/D^2]$, then by the invariance property of the MLE, $\hat{C}_{pp}$ is the MLE of C_{pp}. (b) We note that $(\bar{X}, S_n^2)$ is sufficient and complete for (μ, σ^2). Since the unbiased estimator $\hat{C}_{pp}$ is a function of $(\bar{X}, S_n^2)$ only, then by the Lehmann–Scheffe theorem $\hat{C}_{pp}$ is the UMVUE. (c) For all $\varepsilon > 0$, $P\{|\hat{C}_{pp} - C_{pp}| > \varepsilon\} < E(\hat{C}_{pp} - C_{pp})^2/\varepsilon^2$. Since $E(\hat{C}_{pp} - C_{pp})^2 = \text{Var}(\hat{C}_{pp}) = 2C_{ip}(C_{ia} + C_{pp})/n$ converges to zero, then $\hat{C}_{pp}$ must be consistent. (d) Greenwich and Jahr-Schaffrath [1] showed that under general conditions $\sqrt{n}(\hat{C}_{pp} - C_{pp})$ converges to $N(0, \sigma_{pp}^2)$ in distribution, where

$$\sigma_{pp}^2 = [4(\mu - T)^2\sigma^2/D^4] + [4\mu_3(\mu - T)/D^4] + [(\mu_4 - \sigma^4)/D^4]$$

Therefore, $\sqrt{n}(\hat{C}_{pp} - C_{pp})$ converges to $N(0, 2C_{ip}C_{ia} + 2C_{ip}C_{pp})$ in distribution if the process is normal. (e) Under the normality assumption, the information matrix can be calculated as follows. Since the information lower bound is achieved, then the estimator $\hat{C}_{pp}$ must be asymptotically efficient:

$$I(\theta) = I(\mu, \sigma) = \begin{bmatrix} 1/\sigma^2 & 0 \\ 0 & 1/(2\sigma^4) \end{bmatrix},$$

$$\begin{bmatrix} \dfrac{\partial C_{pp}}{\partial \mu} & \dfrac{\partial C_{pp}}{\partial \sigma^2} \end{bmatrix} \frac{I^{-1}(\theta)}{n} \begin{bmatrix} \dfrac{\partial C_{pp}}{\partial \mu} \\ \dfrac{\partial C_{pp}}{\partial \sigma^2} \end{bmatrix} = \frac{2C_{ip}}{n}(C_{ia} + C_{pp})$$

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