ELSEVIER

Computer Networks 36 (2001) 269–290

COMPUTER NETWORKS

www.elsevier.com/locate/comnet

Performance of hot billing mobile prepaid service

Ming-Feng Chang, Yi-Bing Lin *, Wei-Zu Yang

Dept. of Computer Science and Information Engineering, National Chiao Tung University, 1001 Ta Hsueh Road, Hsinchu, Taiwan, ROC

Received 19 September 2000; accepted 10 January 2001

Responsible Editor: I.F. Akyildiz

Abstract

Prepaid service has become an important mobile application with rapid growth for subscription rate in the recent years. Implementation of prepaid service may generate large network traffic that significantly affects the performance of a mobile network. This paper studies the hot billing solution for prepaid service. We investigate how the amount of prepaid credit and the frequency of call detail record (CDR) transmissions affect network signaling and potential bad debt that a service provider may bear. Our study suggests that a prepaid service provider should encourage customers to buy large prepaid credits by giving them discounts. Furthermore, based on call traffic, an optimal CDR transmission frequency can be found by using our modeling technique.

Keywords: Bad debt; Call detail record; Hot billing; Prepaid service center; Recharge

1. Introduction

Prepaid telecommunication services were offered in Europe and Asia in 1982 and became popular in the US in 1992 [1]. During the past few years, the mobile prepaid service has been growing exponentially all over the world. In the US, more than thirty prepaid solution vendors, such as Corsair Communications, Boston Communications Group and Vicorp, are competing for carrier business [13]. Today, more than 100 million prepaid cards have been issued [2], and revenues of more than US $650 million had been generated from the prepaid service in the US by the year 2000. In 1997, there were about 60 million GSM subscribers across the world and 8% of them subscribed to prepaid service. It is predicted that in 2001, the number of GSM subscribers will increase to 140 million and 25% of the customers will subscribe to the prepaid service [15]. Asian countries such as Philippine, Australia, Hong Kong, Singapore and Taiwan have already shown successful examples for prepaid services. For example, Islacom in Philippine launched prepaid service in November 1997 and has a comparable number of prepaid and postpaid customers now [14]. In Australia, Telstra started prepaid service with an initial capacity of 100,000 customers and has exceeded the capacity

* Corresponding author. Tel.: +886-3-573-1842; fax: +886-3-572-4176.
E-mail addresses: mfchang@csie.nctu.edu.tw (M.-F. Chang), liny@csie.nctu.edu.tw (Y.-B. Lin), wzyang@csie.nctu.edu.tw (W.-Z. Yang).

1389-1286/01/$ - see front matter
PII: S1389-1286(01)00148-7

in early 1999. In Taiwan, FarEastone reported that more than 47% of the customers subscribed to prepaid service in March 1999.

Several factors have contributed to the rapid growth of mobile prepaid service [2]. Firstly, the growth rate of cellular subscribers appears to decrease while the competition among carriers keeps inflaming. The service providers are searching for new ways of increasing revenues, reducing expenses and improving the customer satisfaction. Secondly, as the customer base grows to cover customers with poor credit, providing new services that can minimize or avoid fraud usage is becoming more and more critical today. Mobile prepaid service offers a desirable solution that satisfies the aforementioned requirements.

In the GSM prepaid service, a customer subscribes to the GSM service with a prepaid credit. This credit is either coded into the subscriber identity module (SIM) card or kept in the network [1]. In many areas, the initialization of prepaid services must be completed within a certain number of days after subscription. In Taiwan, prepaid service is available immediately after purchasing the service. Whenever the customer originates a prepaid call, the corresponding payment is decremented from the prepaid credit. Status report of the credit balance can be obtained from the SIM card or the network.

If the balance is depleted, the customer cannot originate calls, but may be allowed to receive phone calls for a period (e.g., 6 months). To recover the prepaid service, the balance needs to be recharged by purchasing a top-up card. The top-up card is like a lottery scratch card. When the seal is scratched off, a secret code appears. The customer dials a toll-free number and follows the instructions of an interactive voice response (IVR) to input the Mobile Station ISDN Number (MSISDN, i.e., the GSM phone number) and the secret code. The system will verify and refresh the account if it is a valid code. On the other hand, if the prepaid balance is not depleted at the end of a valid period, the balance is automatically reset to zero. After a certain amount of time, the unused prepaid credit may be considered abandoned and becomes the government's property.

Four solutions have been proposed to implement the prepaid services: hot billing approach, service node approach, intelligent network approach and handset-based approach [4]. The hot billing and the handset-based approaches provide solutions without major changes to the network infrastructure. Intelligent network solution offers real-time rating and real-time call control, but is not widely deployed today. The service node approach, which utilizes extra voice circuits and switching resources for prepaid calls, provides a variant to the intelligent network solution.

This paper studies the hot billing approach. The other three approaches are out of the scope of this paper. Details of these approaches can be found in [13]. We first describe the hot billing approach. Then, we investigate the performance of this approach by both analytical and simulation models. We assume that the reader is familiar with the GSM terms such as SIM, mobile switching center (MSC), home location register (HLR), authentication center (AuC), MSISDN and international mobile subscriber identity (IMSI). Details of these terms can be found in [6,9,11]. For the reader's benefit, Appendix A lists the notations used in this paper.

2. Hot billing solutions

Hot billing uses the call detail records (CDRs) produced by the wireless switch (i.e., MSC) to process the prepaid usage. The information in a CDR includes the type of service, date/time of usage, user identification and location information [8]. When calls are completed, the CDRs are generated and transported from the MSC to the prepaid service center. The balance of the customer's account is decremented according to the CDRs. As a customer uses up the prepaid credit, the HLR and the AuC are notified to prevent further service access and the prepaid service center instructs the network to route the next prepaid call attempt to an IVR to play an announcement indicating that the balance has been depleted. The IVR can also communicate with the customer to replenish the prepaid credit by using a top-up card, a credit/debit card or credit transfer from the bank account.

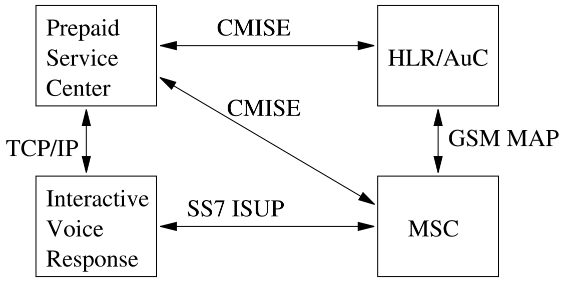


Fig. 1. Hot billing architecture for prepaid service.

The architecture of the hot billing approach is depicted in Fig. 1. The CDR for a prepaid call is created in the MSC based on the destination of the call and the connection time for the call. This call record can be sent from the MSC to the prepaid service center by using protocols such as common management information service element (CMISE) [5]. The same protocol can be used for communication between the prepaid service center and the HLR. The HLR communicates with the MSC by invoking GSM MAP service primitives [6]. The IVR generates automatic messages that allow the customer accounts to be queried and reloaded. The voice trunks between the IVR and the MSC are set up by SS7 ISUP (ISDN User Part) messages [3].

Fig. 2 illustrates the initialization of prepaid service. This procedure is described in the following steps:

Step 1. The customer subscribes to the prepaid service. The prepaid service center creates a subscriber data record including IMSI, MSISDN, prepaid credit and other authentication-related information.
Step 2. The prepaid service center sends the customer data to the HLR and activates the prepaid service.

The prepaid call origination procedure is illustrated in Fig. 3 with the following steps:

Step 1. When a customer originates a prepaid call, the IMSI is sent to the MSC.
Step 2. The MSC instructs the HLR to check if it is a valid service.

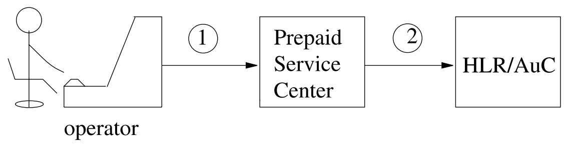


Fig. 2. Initialization of prepaid service.

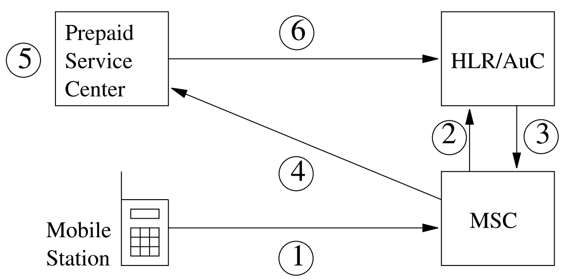


Fig. 3. The prepaid call origination procedure.

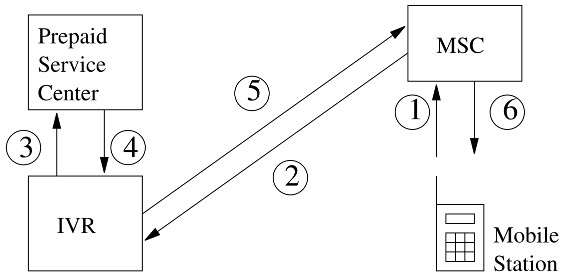


Fig. 4. The prepaid credit querying procedure.

Step 3. If the verification is successful, the HLR downloads the customer data and a prepaid tag to the MSC.
Step 4. When the call terminates, a billing record is created and sent to the prepaid service center.
Step 5. The prepaid service center decrements the prepaid credit based on the received billing record.
Step 6. If the balance is negative, the prepaid service center instructs the HLR to suspend the prepaid service or to delete the customer record.

A customer can query his/her current credit through the following steps (see Fig. 4):

Step 1. The customer makes a service query call that is typically free of charge.
Step 2. The MSC sends the request together with the MSISDN of the customer to the IVR and sets up a voice path to the IVR.
Steps 3 and 4. The IVR queries the prepaid service center for the balance information.
Steps 5 and 6. The IVR plays an announcement to answer the customer.

When the prepaid credit has been decremented below a threshold, the prepaid service center automatically calls the customer and plays a warning message that reminds the customer to recharge the prepaid credit. The customer may recharge the prepaid credit using the top-up card described in Section 1. This recharging procedure is similar to the credit query procedure illustrated in Fig. 4.

If the prepaid credit depletes during a phone call, the credit becomes negative at the end of the phone call. The negative credit is potential bad debt. If the customer does not recharge the credit, this negative credit becomes a real bad debt of the service provider. To avoid bad debt, some approaches (such as service node) decrement the prepaid credit by seconds during a phone call. However, in the hot billing approach, sending these "real time" CDRs to the prepaid service center and processing these CDRs at the center may incur heavy overheads to the network. Thus, the CDRs are delivered and processed on a per-call basis and in some cases, on a multiple-call basis. In the hot billing approach, it is important to select the CDR sending frequency such that the sum of the CDR sending/processing cost and the bad debt is minimized. This paper utilizes analytical and simulation models to investigate the performance of hot billing. Our study provides the guidelines to select the CDR sending frequency and the amount for the initial prepaid credit.

3. The analytical model

In this section, we propose an analytical model to study the performance of the hot billing approach. In our model, a CDR is sent from the MSC to the prepaid service center for every m complete prepaid call, where $m \geqslant 1$. The prepaid service center decrements the customer's credit according to the CDRs received. When a customer's credit becomes negative, the customer is not allowed to make a phone call. We will estimate the number of CDRs transmitted and the amount of potential bad debt.

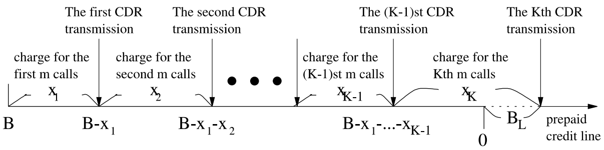


Fig. 5. The charges for the prepaid calls (events occurring in the "prepaid credit line").

Let B be the prepaid credit and K be the total number of CDRs sent to the prepaid service center when the prepaid service ends. Let x_i ($i = 1, 2, \ldots$) be the charge indicated in the ith CDR. If a CDR is sent per m call completion, then x_i represents the net charge for the $[m(i-1)+1]$st call to the $(m \times i)$th call. We assume that x_i is a random variable with the density function $f(x_i)$ and the expected value $E[x_i] = m/\gamma$ (i.e., the expected charge for a call is $1/\gamma$). Let B_L be the amount of potential bad debt. That is, $B_L = \sum_{i=1}^{K} x_i - B$. The amount B_L becomes a real bad debt if the customer does not recharge the prepaid credit. Fig. 5 illustrates the charges for the prepaid calls. The horizontal line is the "prepaid credit line" that illustrates the decrement of the prepaid credit due to the CDR transmission events (the vertical arrows).

The theory of a renewal process [12] can be applied to evaluate the hot billing approach. A renewal process is a counting process for which the interarrival times of events are independent and identically distributed. The transmission of CDRs in the hot billing approach is a renewal process, since the interarrival times between two CDR transmissions are independent and identically distributed. Hence, the number of CDR transmissions, the expected value and the second moment of potential bad debt can be estimated by the approximate solutions of a renewal process. For the large prepaid credit (i.e., B is sufficiently large), $E[K]$ can be approximated as

$$E[K] \approx \frac{B}{E[x_i]} + \frac{E[x_i^2]}{2(E[x_i])^2}. \quad (1)$$

The mean of B_L can be approximated as

$$E[B_L] \approx \frac{E[x_i^2]}{2E[x_i]}. \quad (2)$$

The second moment of B_L can be approximated as

$$E[B_L^2] \approx \frac{E[x_i^3]}{3E[x_i]}. \quad (3)$$

The approximate solutions suggest that $E[B_L]$ and $E[B_L^2]$ be independent of the prepaid credit as long as the credit is large enough. Later in this paper, we will derive an exact solution for the expected potential bad debt. The solution will show that the prepaid credit may affect the expected value and variance of the potential bad debt when the prepaid credit is small.

In PCS services, the call holding times are usually assumed to be exponentially or Erlang distributed [7,9,10]. Since a CDR is sent per m call completions and the charge of a call is proportional to its call holding time, we assume that x_i has an Erlang distribution with mean $E[x_i] = m/\gamma$ and variance $\mathrm{Var}[x_i] = m/(l\gamma^2)$ (i.e., the shape parameter and scale parameter of x_i are lm and $l\gamma$, respectively). From (1), $E[K]$ is approximated as

$$E[K] \approx \frac{2l\gamma B + lm + 1}{2lm}.$$

From (2), $E[B_L]$ is approximated as

$$E[B_{\mathrm{L}}] \approx \frac{lm+1}{2l\gamma}. \tag{4}$$

From (3), $E[B_{\mathrm{L}}^2]$ can be approximated as

$$E[B_{\mathrm{L}}^2] \approx \left[\frac{lm(lm+1)(lm+2)}{(l\gamma)^3}\right] \Big/ \left(\frac{3m}{\gamma}\right) = \frac{(lm+1)(lm+2)}{3l^2\gamma^2}. \tag{5}$$

From (4) and (5), the variance of the potential bad debt $\mathrm{Var}[B_{\mathrm{L}}]$ can be approximated as

$$\mathrm{Var}[B_{\mathrm{L}}] \approx \frac{(lm+1)(lm+2)}{3l^2\gamma^2} - \frac{(lm+1)^2}{4l^2\gamma^2} = \frac{(lm+1)(lm+5)}{12l^2\gamma^2}.$$

In the following subsections, we consider two cases for prepaid credit B. In case I, a customer is given a small prepaid credit and the customer does not recharge after the credit depletes. In case II, B is a random variable with an arbitrary distribution. Case II represents a customer who may recharge the prepaid card when the credit runs out.

3.1. Case I: small prepaid credit

Small prepaid credit may be provided to promote the prepaid service. In this case, the prepaid credit is a constant. Let y_n be the accumulated charge of the calls for the first n CDRs. That is, $y_n = \sum_{i=1}^{n} x_i$. Let $F_n(y) = \Pr\{y_n < y\}$ be the distribution function for y_n. Let $N(y) = \max\{n \mid y_n < y\}$. Then, $N(y)$ represents the number of CDRs transmitted in $(0, y]$ and is a renewal process. It is apparent that $K = N(B) + 1$. Let $\Pr\{N(B) = n\}$ be the probability that n CDRs have been sent to the prepaid service center just before the credit runs out. Then, the expected value $E[K]$ is derived as

$$E[K] = \sum_{n=0}^{\infty} n \Pr\{N(B) = n\} + 1 = \sum_{n=0}^{\infty} n[F_n(B) - F_{n+1}(B)] + 1 = \sum_{n=1}^{\infty} F_n(B) + 1. \tag{6}$$

We assume that the call charges are Erlang distributed with mean $E[x_i] = m/\gamma$ and variance $\mathrm{Var}[x_i] = m/(l\gamma^2)$. From (6), $E[K]$ can be expressed as

$$E[K] = \sum_{n=1}^{\infty}\left[1 - \sum_{j=0}^{lmn-1} \frac{(l\gamma B)^j \mathrm{e}^{-l\gamma B}}{j!}\right] + 1 = \mathrm{e}^{-l\gamma B}\left[\sum_{n=1}^{\infty}\sum_{j=lmn}^{\infty} \frac{(l\gamma B)^j}{j!}\right] + 1 = \mathrm{e}^{-z} G_m(z) + 1, \tag{7}$$

where $z = l\gamma B$ and $G_m(z) = \sum_{n=1}^{\infty} \sum_{j=lmn}^{\infty} z^j/j!$. From (7), a differential equation can be obtained as follows.

$$G_m(z) + \frac{\mathrm{d}G_m(z)}{\mathrm{d}z} + \frac{\mathrm{d}^2 G_m(z)}{\mathrm{d}z^2} + \cdots + \frac{\mathrm{d}^{lm-1} G_m(z)}{\mathrm{d}z^{lm-1}} = z\mathrm{e}^z. \tag{8}$$

Using the Laplace transform and following the process described in Appendix B, $E[K]$ can be expressed as

$$\begin{aligned} E[K] = {} & \left\{\left[1 - \sum_{i=1}^{lm-1}\left\{\left[(\omega^i - 1)^2 \prod_{1 \leqslant j \leqslant lm-1, j \neq i} (\omega^i - \omega^j)\right]^{-1} \prod_{1 \leqslant j \leqslant lm-1, j \neq i} (2 - \omega^j)\right\}\right] \Big/ \prod_{1 \leqslant j \leqslant lm-1} (2 - \omega^j)\right\} \\ & + \frac{\mathrm{e}^{l\gamma B} - 1}{(1-\omega)(1-\omega^2)\cdots(1-\omega^{lm-1})} + \sum_{i=1}^{lm-1}\left\{\left[(\omega^i - 1)^2 \prod_{1 \leqslant j \leqslant lm-1, j \neq i} (\omega^i - \omega^j)\right]^{-1} \mathrm{e}^{l\gamma B(\omega^i - 1)}\right\} + 1, \end{aligned} \tag{9}$$

where ω is the principle $(l \times m)$th root of 1 (i.e., $\omega = \cos(2\pi)/(lm) + \mathrm{i}\sin(2\pi)/(lm)$). Based on Wald's equation [12], the expected value $E[B_{\mathrm{L}}]$ can be expressed as

$$E[B_{\mathrm{L}}] = E\left(\sum_{i=1}^{K} x_i - B\right) = E[K]E[x_i] - B. \tag{10}$$

From (C.17) in Appendix C, $E[B_{\mathrm{L}}^2]$ is expressed as

$$\begin{aligned} E[B_{\mathrm{L}}^2] = {} & \sum_{n=0}^{\infty}\sum_{j=0}^{lm-1}[(j+2)(j+1)]\left[\frac{(l\gamma)^{lmn+j}B^{lmn+j+2}\mathrm{e}^{-l\gamma B}}{(lmn+j+2)!}\right] \\ & - 2lm\sum_{n=0}^{\infty}\sum_{j=0}^{lm}(j+1)\left[\frac{(l\gamma)^{lmn+j-1}B^{lmn+j+1}\mathrm{e}^{-l\gamma B}}{(lmn+j+1)!}\right] \\ & + \sum_{n=0}^{\infty}\sum_{j=0}^{lm+1}\left[\frac{lm(lm+1)}{(lmn+j)!}\right]\left[(l\gamma)^{lmn+j-2}B^{lmn+j}\mathrm{e}^{-l\gamma B}\right]. \end{aligned} \tag{11}$$

From (10) and (11), $\mathrm{Var}[B_{\mathrm{L}}]$ can be obtained.

3.2. Case II: recharged credit

In the recharged credit case, a customer may recharge his/her prepaid card before the credit runs out. Assume that B has a density function $h(b)$ with the Laplace transform $h^*(s)$. As described in the previous section, we assume that the charge x_i $(i = 1, 2, \ldots)$ indicated in the ith CDR has an Erlang distribution with mean $E[x_i] = m/\gamma$ and variance $\mathrm{Var}[x_i] = m/(l\gamma^2)$. The probability that n CDRs have been sent to the prepaid service center before total credit runs out is expressed as

$$\begin{aligned} & \Pr\{y_n < b < y_n + x_{n+1}\} \\ & = \int_{b=0}^{\infty}\int_{y_n=0}^{b}\int_{x_{n+1}=b-y_n}^{\infty} h(b)\left[\frac{(l\gamma)^{lmn}}{(lmn-1)!}y_n^{lmn-1}\mathrm{e}^{-l\gamma y_n}\right]\left[\frac{(l\gamma)^{lm}}{(lm-1)!}x_{n+1}^{lm-1}\mathrm{e}^{-l\gamma x_{n+1}}\right]\mathrm{d}x_{n+1}\,\mathrm{d}y_n\,\mathrm{d}b \\ & = \sum_{j=0}^{lm-1}\left\{\int_{b=0}^{\infty} h(b)b^{lmn+j}\left[\frac{(l\gamma)^{lmn+j}}{(lmn-1)!j!}\mathrm{e}^{-l\gamma b}\right]\left[\sum_{i=0}^{j}(-1)^{i+j}\binom{j}{i}\frac{1}{lmn+j-i}\right]\mathrm{d}b\right\}. \end{aligned} \tag{12}$$

From (C.5) in Appendix B, we have

$$\sum_{i=0}^{j}(-1)^{i+j}\binom{j}{i}\left(\frac{1}{lmn+j-i}\right) = \frac{(lmn-1)!j!}{(lmn+j)!}. \tag{13}$$

Substituting (13) into (12), we have

$$\begin{aligned} \Pr\{y_n < b < y_n + x_{n+1}\} & = \sum_{j=0}^{lm-1}\left[\frac{(l\gamma)^{lmn+j}}{(lmn+j)!}\right]\left[\int_{b=0}^{\infty} h(b)b^{lmn+j}\mathrm{e}^{-l\gamma b}\mathrm{d}b\right] \\ & = \sum_{j=0}^{lm-1}\left\{\left[\frac{(-l\gamma)^{lmn+j}}{(lmn+j)!}\right]\left[\frac{\mathrm{d}^{(lmn+j)}h^*(s)}{\mathrm{d}s^{(lmn+j)}}\right]\Bigg|_{s=l\gamma}\right\}. \end{aligned} \tag{14}$$

If B has a Gamma distribution with the shape parameter α and the scale parameter β (i.e., the mean of the distribution is $E[B] = \alpha/\beta$ and the variance is $\mathrm{Var}[B] = \alpha/\beta^2$), then the Laplace transform of B is

$$h^*(s) = \left(\frac{\beta}{s+\beta}\right)^{\alpha}.$$

The $(lmn + j)$th order derivative for $h^*(s)$ is

$$h^{*(lmn+j)}(s) = \left[\frac{(-1)^{lmn+j}(\alpha + lmn + j - 1)!}{(\alpha - 1)!}\right]\left[\frac{\beta^{\alpha}}{(s+\beta)^{\alpha+lmn+j}}\right]. \tag{15}$$

Substituting (15) into (14), we have

$$\Pr\{y_n < b < y_n + x_{n+1}\} = \sum_{j=0}^{lm-1}\left\{\binom{\alpha + lmn + j - 1}{lmn + j}\left[\frac{(l\gamma)^{lmn+j}\beta^{\alpha}}{(l\gamma+\beta)^{lmn+j+\alpha}}\right]\right\}.$$

From (6), the expected value $E[K]$ can be expressed as

$$E[K] = \sum_{n=0}^{\infty}(n+1)\left\{\sum_{j=0}^{lm-1}\binom{\alpha + lmn + j - 1}{lmn + j}\left[\frac{(l\gamma)^{lmn+j}\beta^{\alpha}}{(l\gamma+\beta)^{lmn+j+\alpha}}\right]\right\}. \tag{16}$$

From (10), $E[B_{\mathrm{L}}]$ can be expressed as

$$\begin{aligned} E[B_{\mathrm{L}}] &= E\left[\sum_{i=1}^{K} x_i\right] - E[B] \\ &= \left(\frac{m}{\gamma}\right)\left\{\sum_{n=0}^{\infty}(n+1)\sum_{j=0}^{lm-1}\binom{\alpha + lmn + j - 1}{lmn + j}\left[\frac{(l\gamma)^{lmn+j}\beta^{\alpha}}{(l\gamma+\beta)^{lmn+j+\alpha}}\right]\right\} - \frac{\alpha}{\beta}. \end{aligned} \tag{17}$$

From Appendix D, $E[B_{\mathrm{L}}^2]$ can be expressed as

$$\begin{aligned} E[B_{\mathrm{L}}^2] &= \sum_{n=0}^{\infty}\sum_{j=0}^{lm-1}\left\{(j+1)(j+2)\left[\frac{(l\gamma)^{lmn+j}\beta^{\alpha}}{(l\gamma+\beta)^{\alpha+lmn+j+2}}\right]\left[\frac{\Gamma(\alpha+lmn+j+2)}{\Gamma(\alpha)(lmn+j+2)!}\right]\right\} \\ &\quad - 2lm\sum_{n=0}^{\infty}\sum_{j=0}^{lm}\left\{(j+1)\left[\frac{(l\gamma)^{lmn+j-1}\beta^{\alpha}}{(l\gamma+\beta)^{\alpha+lmn+j+1}}\right]\left[\frac{\Gamma(\alpha+lmn+j+1)}{\Gamma(\alpha)(lmn+j+1)!}\right]\right\} \\ &\quad + [lm(lm+1)]\sum_{n=0}^{\infty}\sum_{j=0}^{lm+1}\left\{\left[\frac{\beta^{\alpha}(l\gamma)^{lmn+j-2}}{(\beta+l\gamma)^{\alpha+lmn+j}}\right]\left[\frac{\Gamma(\alpha+lmn+j)}{\Gamma(\alpha)(lmn+j)!}\right]\right\}. \end{aligned} \tag{18}$$

The variance $\mathrm{Var}[B_{\mathrm{L}}]$ of the potential bad debt can be derived from (17) and (18). Different distributions (e.g., shift-Gamma) for the recharged credit can be derived with the similar approach. However, it will not be included in this paper.

4. Numerical examples

This section investigates the performance of the hot billing approach based on the analytical model developed in the previous section. Computer simulations have been conducted to validate the analytical results. Each simulation experiment was repeated 500,000 times to ensure stable results.

Tables 1–3 compare the results of simulation, exact solution and approximation for the large, small prepaid credit and recharged credit cases. To reflect the situation of prepaid service in Taiwan, the expected charge of a call is assumed to be NT\$36 dollars. The expected prepaid credit B considered in large and recharged credit cases is NT\$100, NT\$300, NT\$400 and NT\$500 dollars, respectively. For the small prepaid credit case, $B =$ NT\$6, NT\$12 and NT\$18. The tables indicate that the exact solutions are consistent

Table 1
Comparison of simulation, exact solution and approximation for large prepaid credit ($m = 1$, $E[x_i]$ = NT$36 and $\mathrm{Var}[x_i] = 432$)

B	Simulation	Exact solution	Approximation
$E[K]$			
100	3.444	3.444	3.444
300	9.000	9.000	9.000
400	11.774	11.778	11.778
500	14.551	14.556	14.556
$E[B_L]$			
100	24.028	24.000	24.000
300	23.955	24.000	24.000
400	23.949	24.000	24.000
500	24.005	24.000	24.000
$\mathrm{Var}[B_L]$			
100	385.321	384.000	384.000
300	382.482	384.000	384.000
400	382.443	384.000	384.000
500	386.892	384.000	384.000

Table 2
Comparison of simulation, exact solution and approximation for small prepaid credit ($m = 1$, $E[x_i]$ = NT$36 and $\mathrm{Var}[x_i] = 432$)

B	Simulation	Exact solution	Approximation
$E[K]$			
6	1.014	1.014	0.500
12	1.080	1.081	0.667
18	1.194	1.196	0.833
$E[B_L]$			
6	30.536	30.519	24.000
12	26.897	26.912	24.000
18	24.993	25.043	24.000
$\mathrm{Var}[B_L]$			
6	423.431	423.833	384.000
12	403.510	403.274	384.000
18	389.654	388.344	384.000

with the simulation results in all cases, while the approximate solution is good only in large prepaid credit case. Note that the discrepancy between the exact and approximate solution can be over 20% when the prepaid credit is small.

4.1. Effects of the variance $\mathrm{Var}[x_i]$ of CDR charges when B is large

Figs. 6 and 7 depict the effects of the variance $\mathrm{Var}[x_i]$ of CDR charges on $\mathrm{Var}[B_L]$ and $E[B_L]/E[B]$ for both large credit and recharged credit cases. In these two figures, a CDR record is sent to the prepaid service center for every call completion (i.e., $m = 1$) and the charge for a call has an Erlang distribution with shape parameter l and mean $E[x_i] = 1/\gamma$ = NT$36. In the recharged credit case, we consider Gamma prepaid credit distribution with the variances $\mathrm{Var}[B] = 10^4$, 9×10^4, 16×10^4 and 25×10^4 (i.e., the scale parameter β is 1/100, 1/300, 1/400 and 1/500), respectively. Both figures show that $E[B_L]$ and $\mathrm{Var}[B_L]$ increase as $\mathrm{Var}[x_i]$ increases.

Table 3
Comparison of simulation, exact solution and approximation for recharged credit ($m = 1$, $\alpha = 1/2$, $E[x_i] =$ NT$36 and $\mathrm{Var}[x_i] = 432$)

B	Simulation	Exact solution	Approximation
$E[K]$			
100	3.519	3.512	3.444
300	9.041	9.040	9.000
400	11.766	11.812	11.778
500	14.640	14.586	14.556
$E[B_L]$			
100	26.456	26.449	24.000
300	25.440	25.430	24.000
400	25.264	25.240	24.000
500	25.157	25.110	24.000
$\mathrm{Var}[B_L]$			
100	410.231	412.088	384.000
300	401.863	401.800	384.000
400	399.029	399.666	384.000
500	398.816	398.165	384.000

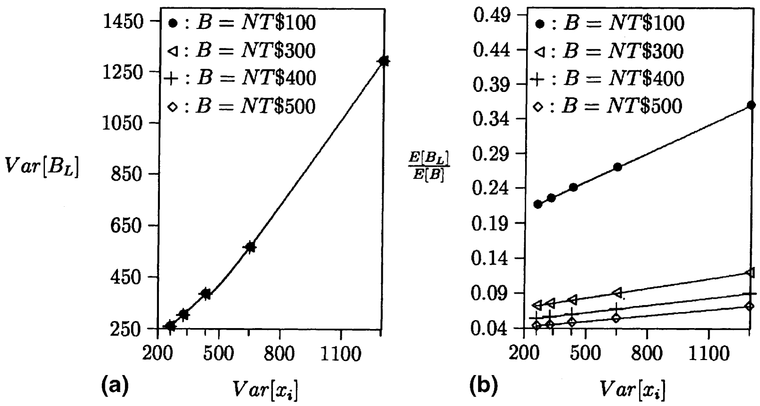


Fig. 6. Effect of the variance of call charges for large credit case ($m = 1$, $E[x_i] =$ NT$36): (a) variance of the potential bad debt; (b) potential loss per prepaid credit dollar.

The phenomenon that $E[B_L]$ increases as $\mathrm{Var}[x_i]$ increases is explained as follows. In the prepaid credit line of Fig. 5, the value 0 can be treated as an observer of the periods x_i. From an argument similar to the one for excess life theorem [12], longer x_i's are more likely to be observed by point "0" in the credit line. When $\mathrm{Var}[x_i]$ is large, there are more large x_i's and small x_i's. Thus, the value 0 is more likely to observe longer x_i as $\mathrm{Var}[x_i]$ increases. We conclude that if the call pattern of a prepaid customer is very irregular, it is more likely that the service provider will lose revenue due to bad debt.

Experience indicates that prepaid customers tend to purchase small prepaid credits to make the credit control easier and more economical. On the other hand, Figs. 6(b) and 7(b) indicate that large prepaid credit helps the service provider to reduce the potential bad debt per prepaid dollar (i.e., $E[B_L]/E[B]$). It also reduces the cost for distributing the prepaid cards and allows collecting the capitals quickly. Thus, based on the above analysis, a prepaid service provider should encourage the customers to buy large prepaid credits by giving them discounts.

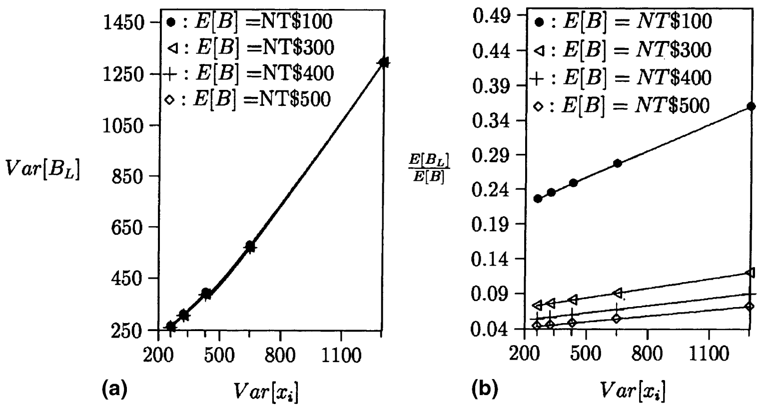


Fig. 7. Effect of the variance of call charges for the recharged credit case ($m = 1$, $E[x_i] =$ NT$36): (a) variance of the potential bad debt; (b) potential loss per prepaid credit dollar.

4.2. Effects of the variance $Var[x_i]$ of CDR charges when B is small

Fig. 8 shows the effects of the variance $\mathrm{Var}[x_i]$ of CDR charges on $E[B_L]$ when B is small. The mean charge of a CDR $E[x_i] =$ NT$36 and the variance $\mathrm{Var}[x_i]$ is varied as 648.0, 324.0 and 216.0. We assume that the CDR is sent whenever a call completes (i.e., $m = 1$). The small prepaid credit considered in our case ranges from NT$6 to NT$72. The results show that when B is small, most of the customers can only originate one call and the bad debt $E[B_L] \approx E[x_i] - B$. As the prepaid credit increases, the potential bad decreases and converges to the approximate value derived from the renewal theory when B is larger than $E[x_i]$. It is interesting to note that when $\mathrm{Var}[x_i]$ is small (i.e., the call pattern is regular), there exists an optimal prepaid credit such that the expected bad debt is minimal.

4.3. Effects of the prepaid credit variance

Fig. 9 shows the effects of the prepaid credit variance $\mathrm{Var}[B]$ on $E[B_L]$ and $\mathrm{Var}[B_L]$. We consider the case where every CDR is sent per single call completion (i.e., $m = 1$). We only present the results for the cases where the charge x_i for a CDR has an Erlang distribution with mean $E[x_i] =$ NT$36 and the variance $\mathrm{Var}[x_i] = 648$ (i.e., $l = 2$). Note that similar conclusions can be drawn for x_i with various variances. We

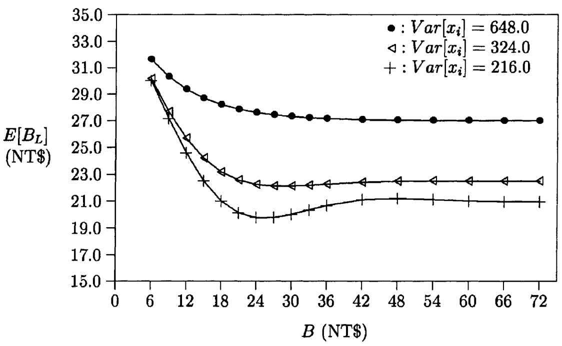


Fig. 8. Effect of the variance of call charges for small prepaid credit case ($m = 1, E[x_i] =$ NT$36).

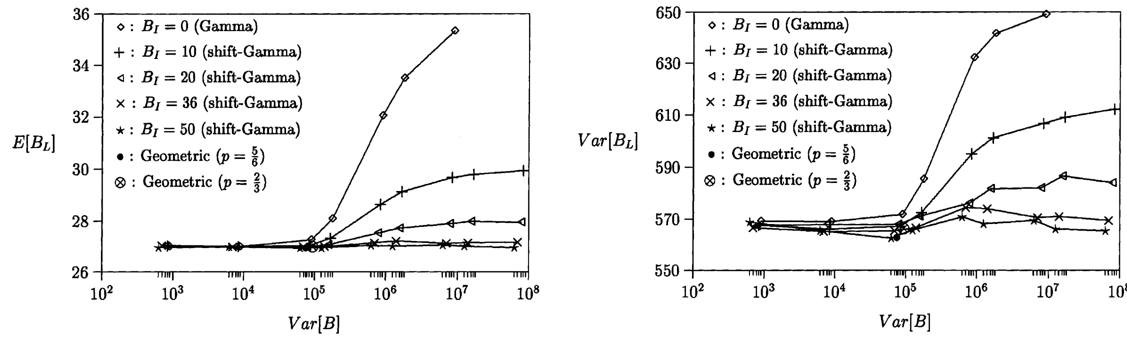


Fig. 9. Effect of the initial credit ($m = 1$, $E[x_i] =$ NT\$36, $E[B] =$ NT\$300).

assume that the prepaid credit B consists of two parts: the initial credit and the net recharged credit. At the beginning, the customer purchases an initial credit $B_{\rm I}$ and then recharges the credit several times. The net recharged credit is the sum of all recharged credits. We consider two cases where $E[B] =$ NT\$300. In the first case, the prepaid credit B has a shifted Gamma distribution. In this case, the net recharged credit is generated from a Gamma random number generator and the initial credit $B_{\rm I}$ is varied as NT\$0, NT\$10, NT\$20, NT\$36 and NT\$50. In the second case, the net recharged credit has a geometric distribution. We assume that every time the customer recharges the prepaid credit with probability p, and the amount of a single recharged credit is $B_{\rm r}$. Thus,

$$E[B] = B_{\rm I} + \left(\frac{p}{1-p}\right)B_{\rm r} \quad \text{and} \quad \mathrm{Var}[B] = \left[\frac{p}{(1-p)^2}\right]B_{\rm r}^2.$$

We illustrate two sets of input parameters for the geometric prepaid credit distribution: ($B_{\rm I} =$ NT\$50, $B_{\rm r} =$ NT\$50, $p = 5/6$) and ($B_{\rm I} =$ NT\$50, $B_{\rm r} =$ NT\$125, $p = 2/3$). Fig. 9 shows that for $B_{\rm I} \geqslant$ NT\$36, Var[B] has insignificant effect on $E[B_{\rm L}]$ and Var[$B_{\rm L}$] in all cases. The figure also indicates that Var[B] has effect on $B_{\rm L}$ when $B_{\rm I} <$ NT\$36. This phenomenon is referred to as the "small $B_{\rm I}$ effect". We conclude that for every prepaid credit value, the customer's recharging behavior does not affect the amount of the potential bad debt if the initial credit is larger than the cost for a single call. For prepaid service planning, a service provider can ignore the recharging behavior of customers.

4.4. Effects of multiple CDR transmissions

Fig. 10 shows the effect of multiple CDR transmissions on $E[K]$ and $E[B_{\rm L}]$ in the recharged credit cases where the charge for one call is exponentially distributed with mean NT\$36 and the expected value for the prepaid credits $E[B]$ is varied as NT\$100, NT\$300, NT\$400 and NT\$500. Similar results are observed for large credit case and are not presented here. In the recharged credit case, the prepaid credits have Gamma density functions with variances Var[B] $= 10^4$, 9×10^4, 16×10^4 and 25×10^4 (i.e., $\beta = 1/100$, $1/300$, $1/400$ and $1/500$), respectively. The figure shows the intuitive results that $E[K]$ decreases as m increases, while $E[B_{\rm L}]$ increases as m increases. Fig. 10(b) shows that $E[B_{\rm L}]$ increases as $E[B]$ decreases. The effect becomes significant when m is large. This phenomenon is similar to the small $B_{\rm I}$ effect observed in Fig. 9. When m increases, the ratio of prepaid credit to the charge indicated in a CDR ($E[B]/E[x_i]$) becomes smaller, which amplifies the small $B_{\rm I}$ effect.

Two costs are associated with the prepaid service: the CDR transmission cost and the bad debt. Fig. 10 indicates that the potential bad debt can be reduced by increasing the CDR transmission frequency. In other words, CDR transmission frequency and bad debt are conflicting factors. Consider a cost function

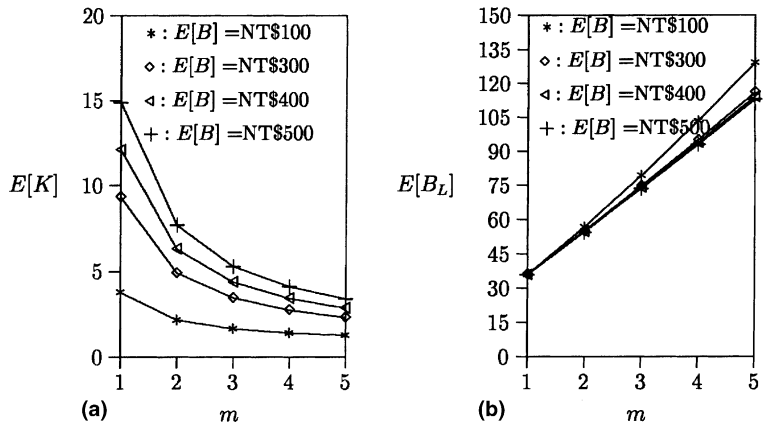


Fig. 10. Effect of m for the recharged credit case ($l = 1, E[x_i] =$ NT\$36): (a) number of CDR transmissions; (b) expected potential bad debt.

$C = E[K] + \phi E[B_L]$ where ϕ is a weighted cost that normalizes the bad debt with respect to the CDR transmission cost. The cost C provides the net effect of CDR transmissions and bad debt. The ϕ value is determined by two factors: the signaling cost C_s for a CDR transmission and the probability P_d that the potential bad debt becomes real bad debt. That is, $\phi = P_d/C_s$. According to OFTEL (cost analysis documents of OFTEL can be found in http://www.oftel.gov.uk/numbers/number.htm), the signaling cost is $C_s = 0.05$ pence $\approx$ NT\$0.025. If $0.1\% < P_d < 1\%$, then ϕ ranges from 0.04 to 0.4. Fig. 11 plots C as a function of ϕ and m with $l = 1$ and $E[x_i] =$ NT\$36 in large credit case. In this figure, the bullets in the curves represent the cost for the optimal m values. Consider the case when $B =$ NT\$500. For $\phi = 0.25$, the potential bad debt costs are high and $m = 2$ should be selected. For $\phi = 0.05$, the CDR transmission costs are high and $m = 4$ should be selected. If $\phi > 1$, $m = 1$ in all cases studied in this paper. Also, for the same ϕ value, the optimal m values increase as B increases. Therefore, if the prepaid credit is large (e.g., $B \geqslant$ NT\$500), a CDR should be transmitted after multiple call completions. Although the above result is intuitive, our analysis quantitatively computes the prepaid service overheads to select the optimal m values for specific network setup.

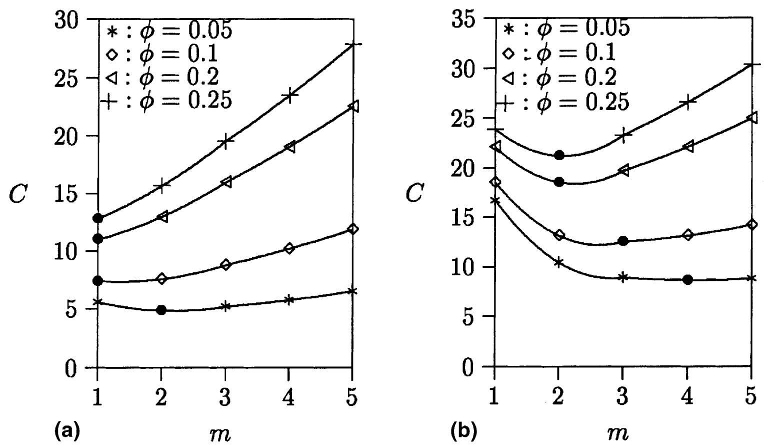


Fig. 11. The cost for large credit case ($l = 1, E[x_i] =$ NT\$36). (a) $B =$ NT\$100; (b) $B =$ NT\$500.

5. Conclusions

This paper studied the hot billing solution for prepaid service. We described the system architecture and the procedures for prepaid service initialization, call origination and credit recharging. An analytical model was proposed to analyze the performance in the large, small prepaid credit and the recharged credit cases. The analytical results were validated by simulation experiments. We observed the following results:

- If the call pattern of a prepaid customer is very irregular, it is more likely that the service provider will lose revenue due to bad debt.
- Large prepaid credit (i.e., the customer either purchases large initial credit or recharges the prepaid credit several times) helps the service provider to reduce the expected and the variance of the potential bad debt. Thus, the service provider may encourage the customers to buy large initial prepaid credit by giving them discounts.
- When the prepaid credit is small, the expected potential bad debt is affected by the amount of prepaid credit. If the call pattern is regular, there exists an optimal prepaid credit such that the expected bad debt is minimal.
- If the initial credit is larger than the cost for a single call, the customer's recharging behavior does not affect the amount of the potential bad debt.
- A cost function was used to determine the minimal cost for prepaid service. The minimal cost can be achieved by properly setting the CDR transmission frequency to balance the network traffic with the bad debt overhead. This optimal CDR transmission frequency can be determined by using our modeling technique.

Acknowledgements

Lin's work was sponsored in part by the MOE Program of Excellence Research under contract 89-E-FA04-4, CCL/ITRI, FarEastone, National Science Council under contract NSC 89-2213-E-009-203, the Lee and MTI Center for Networking Research, NCTU. Chang's work was sponsored in part by MOE Program of Excellence Research under contract 89-E-FA04-4, and National Science Council under contract NSC 89-2213-E-009-201.

Appendix A. Notations

The notations used in this paper are listed as below.

α	the shape parameter of the prepaid credit in the recharged credit case
B	the amount of the prepaid credit
B_I	the initial credit of a prepaid credit
B_L	the amount of potential bad debt
B_r	the amount of a single recharged credit when the prepaid credit is geometric distributed
β	the scale parameter of the prepaid credit in the recharged credit case
C	the net cost of the CDR transmissions and the bad debt
C_s	the signaling cost for a CDR transmission
$F_n(\cdot)$	the distribution function of y_n
$f(\cdot)$	the density function of x_i
$E[x_i] = 1/\gamma$	the expected charge of a call
$h(\cdot)$	the density function of B in the recharged case

$h^*(s)$	the Laplace transform of the prepaid credit distribution
K	the total number of CDRs sent to the prepaid center when the prepaid service ends
m	the number of complete prepaid calls accumulated in a CDR
lm	the shape parameter of x_i
$N(y)$	the largest n such that $y_n < y$
P_d	the probability that the potential bad debt becomes real bad debt
p	the probability that a prepaid customer will recharge his/her credit
ϕ	the weighted cost that normalizes the bad debt with respect to the CDR transmissions
x_i	the charge indicated in the ith CDR
y_n	the accumulated charge of the calls in the first n CDRs

Appendix B. Deriving $E[K]$ for the small prepaid credit case

This appendix derives $E[K]$ for the small prepaid credit case. Let $G_m^*(s)$ be the Laplace transform of $G_m(z)$. Then, (8) is re-written as

$$G_m^*(s) + sG_m^*(s) - G_m(0) + s^2 G_m^*(s) - sG_m(0) - \left.\frac{dG_m(z)}{dz}\right|_{z=0} + \cdots + s^{lm-1}G_m^*(s) - s^{lm-2}G_m(0) - \cdots - \left.\frac{d^{lm-2}G_m(z)}{dz^{lm-2}}\right|_{z=0} = \frac{1}{(s-1)^2}. \tag{B.1}$$

Note that

$$G_m(0) = \left.\frac{dG_m(z)}{dz}\right|_{z=0} = \cdots = \left.\frac{d^{lm-2}G_m(z)}{dz^{lm-2}}\right|_{z=0} = 0.$$

Thus, (B.1) can be re-written as

$$G_m^*(s) = \frac{1}{(s-1)^2(1+s+s^2+\cdots+s^{lm-1})} \tag{B.2}$$

$$= \frac{A_1}{s-1} + \frac{A_2 s + A_3}{(s-1)^2} + \frac{C_1}{s-\omega} + \frac{C_2}{s-\omega^2} + \cdots + \frac{C_{lm-1}}{s-\omega^{lm-1}} \tag{B.3}$$

$$= \frac{A_1 + A_2}{s-1} + \frac{A_2 + A_3}{(s-1)^2} + \frac{C_1}{s-\omega} + \frac{C_2}{s-\omega^2} + \cdots + \frac{C_{lm-1}}{s-\omega^{lm-1}},$$

where ω is the principle $(l \times m)$th root of 1 (i.e., $\omega = \cos(2\pi)/(lm) + \mathrm{i}\sin(2\pi)/(lm)$). Hence, $G_m(z)$ can be expressed as

$$G_m(z) = (A_1 + A_2)e^z + (A_2 + A_3)ze^z + C_1 e^{\omega z} + C_2 e^{(\omega^2 z)} + \cdots + C_{lm-1}e^{(\omega^{lm-1}z)}. \tag{B.4}$$

From (B.2) and (B.3), we have

$$1 = A_1(s-1)(s-\omega)(s-\omega^2)\cdots(s-\omega^{lm-1}) + (A_2 s + A_3)(s-\omega)(s-\omega^2)\cdots(s-\omega^{lm-1}) + C_1(s-1)^2(s-\omega^2)(s-\omega^3)\cdots(s-\omega^{lm-1}) + C_2(s-1)^2(s-\omega)(s-\omega^3)\cdots(s-\omega^{lm-1}) + \cdots + C_{lm-1}(s-1)^2(s-\omega)(s-\omega^2)\cdots(s-\omega^{lm-2}). \tag{B.5}$$

By letting $s = \omega^i$ $(i = 1, 2, \ldots, lm-1)$ in (B.5), we have

$$C_i = \left[(\omega^i - 1)^2 \prod_{1 \leqslant j \leqslant lm-1, j \neq i} (\omega^i - \omega^j)\right]^{-1}. \tag{B.6}$$

With $s = 1$ in (B.5), we have

$$A_2 + A_3 = \frac{1}{(1-\omega)(1-\omega^2)\cdots(1-\omega^{lm-1})}. \tag{B.7}$$

With $s = 2$ in (B.5), we have

$$A_1 + 2A_2 + A_3 = \frac{1 - \sum_{i=1}^{lm-1}\left[C_i \prod_{1\leqslant j\leqslant lm-1, j\neq i}(2-\omega^j)\right]}{\prod_{1\leqslant j\leqslant lm-1}(2-\omega^j)}. \tag{B.8}$$

From (B.7) and (B.8), we have

$$A_1 + A_2 = \frac{1 - \sum_{i=1}^{lm-1}\left[C_i \prod_{1\leqslant j\leqslant lm-1, j\neq i}(2-\omega^j)\right]}{\prod_{1\leqslant j\leqslant lm-1}(2-\omega^j)} - \frac{1}{(1-\omega)(1-\omega^2)\cdots(1-\omega^{lm-1})}. \tag{B.9}$$

From (7), (B.4), (B.6), (B.7) and (B.9), $E[K]$ is expressed as (9) in Section 3.1.

Appendix C. Deriving $E[B_{\mathrm{L}}^2]$ for the small prepaid credit case

This appendix derives $E[B_{\mathrm{L}}^2]$ for the small prepaid credit case. From the assumptions in Section 3.1,

$$\begin{aligned}
E[B_{\mathrm{L}}^2] &= \sum_{n=0}^{\infty}\int_{y_n=0}^{B}\int_{x_{n+1}=B-y_n}^{\infty}\left\{(x_{n+1}+y_n-B)^2\left[\frac{(l\gamma)^{lmn}}{(lmn-1)!}y_n^{lmn-1}\mathrm{e}^{-l\gamma y_n}\right]\right.\\
&\qquad\left.\times\left[\frac{(l\gamma)^{lm}}{(lm-1)!}x_{n+1}^{lm-1}\mathrm{e}^{-l\gamma x_{n+1}}\right]\right\}\mathrm{d}x_{n+1}\,\mathrm{d}y_n\\
&= \sum_{n=0}^{\infty}\int_{y_n=0}^{B}\int_{x_{n+1}=B-y_n}^{\infty}\left\{(x_{n+1}^2+y_n^2+B^2-2Bx_{n+1}-2By_n+2x_{n+1}y_n)\right.\\
&\qquad\left.\times\left[\frac{(l\gamma)^{lmn}}{(lmn-1)!}y_n^{lmn-1}\mathrm{e}^{-l\gamma y_n}\right]\left[\frac{(l\gamma)^{lm}}{(lm-1)!}x_{n+1}^{lm-1}\mathrm{e}^{-l\gamma x_{n+1}}\right]\right\}\mathrm{d}x_{n+1}\,\mathrm{d}y_n\\
&= D_1+D_2+D_3+D_4+D_5+D_6,
\end{aligned} \tag{C.1}$$

where the first item of the right-hand side in (C.1) is

$$\begin{aligned}
D_1 &= \sum_{n=0}^{\infty}\int_{y_n=0}^{B}\int_{x_{n+1}=B-y_n}^{\infty}\left\{x_{n+1}^2\left[\frac{(l\gamma)^{lmn}}{(lmn-1)!}y_n^{lmn-1}\mathrm{e}^{-l\gamma y_n}\right]\left[\frac{(l\gamma)^{lm}}{(lm-1)!}x_{n+1}^{lm-1}\mathrm{e}^{-l\gamma x_{n+1}}\right]\right\}\mathrm{d}x_{n+1}\,\mathrm{d}y_n\\
&= \sum_{n=0}^{\infty}\sum_{j=0}^{lm+1}\left\{\left[\frac{(l\gamma)^{lmn+j-2}}{(lmn-1)!j!}\right]\left[lm(lm+1)B^{lmn+j}\mathrm{e}^{-l\gamma B}\right]\left[\sum_{i=0}^{j}(-1)^{i+j}\binom{j}{i}\left(\frac{1}{lmn+j-i}\right)\right]\right\}.
\end{aligned} \tag{C.2}$$

The third item of the right-hand side in (C.2) can be re-written as

$$\sum_{i=0}^{j}(-1)^{i+j}\binom{j}{i}\left(\frac{1}{lmn+j-i}\right) = \left(\frac{1}{lmn}\right)\sum_{i=0}^{j}(-1)^{i}\binom{j}{i}\left(\frac{lmn}{lmn+i}\right). \tag{C.3}$$

For the derivation purpose, we define $\Omega(\theta, j)$ as

$$\Omega(\theta,j) = \sum_{i=0}^{j}(-1)^i\binom{j}{i}\left(\frac{\theta}{\theta+i}\right) = \sum_{i=0}^{j}(-1)^i\left[\binom{j-1}{i}+\binom{j-1}{i-1}\right]\left(\frac{\theta}{\theta+i}\right)$$
$$= \Omega(\theta,j-1) - \left(\frac{\theta}{\theta+1}\right)\Omega(\theta+1,j-1) = \binom{\theta+j}{j}^{-1}. \qquad (C.4)$$

From (C.4) and let $\theta = lmn$, (C.3) is re-written as

$$\sum_{i=0}^{j}(-1)^{i+j}\binom{j}{i}\left(\frac{1}{lmn+j-i}\right) = \frac{(lmn-1)!j!}{(lmn+j)!}. \qquad (C.5)$$

Substituting (C.5) into (C.2), we have

$$D_1 = \sum_{n=0}^{\infty}\sum_{j=0}^{lm+1}\left[\frac{lm(lm+1)}{(lmn+j)!}\right](l\gamma)^{lmn+j-2}B^{lmn+j}\mathrm{e}^{-l\gamma B}. \qquad (C.6)$$

The second item of the right-hand side in (C.1) is

$$D_2 = \sum_{n=0}^{\infty}\int_{y_n=0}^{B}\int_{x_{n+1}=B-y_n}^{\infty}\left\{y_n^2\left[\frac{(l\gamma)^{lmn}}{(lmn-1)!}y_n^{lmn-1}\mathrm{e}^{-l\gamma y_n}\right]\left[\frac{(l\gamma)^{lm}}{(lm-1)!}x_{n+1}^{lm-1}\mathrm{e}^{-l\gamma x_{n+1}}\right]\right\}\mathrm{d}x_{n+1}\,\mathrm{d}y_n$$
$$= \sum_{n=0}^{\infty}\sum_{j=0}^{lm-1}\left[\frac{(l\gamma)^{lmn+j}}{(lmn-1)!j!}B^{lmn+j+2}\mathrm{e}^{-l\gamma B}\right]\left[\left(\frac{1}{lmn+2}\right)\Omega(lmn+2,j)\right]. \qquad (C.7)$$

From (C.4), (C.7) is re-written as

$$D_2 = \sum_{n=0}^{\infty}\sum_{j=0}^{lm-1}\left[\frac{lmn(lmn+1)}{(lmn+j+2)!}\right](l\gamma)^{lmn+j}B^{lmn+j+2}\mathrm{e}^{-l\gamma B}. \qquad (C.8)$$

The third item of the right-hand side in (C.1) is

$$D_3 = \sum_{n=0}^{\infty}\int_{y_n=0}^{B}\int_{x_{n+1}=B-y_n}^{\infty}\left\{B^2\left[\frac{(l\gamma)^{lmn}}{(lmn-1)!}y_n^{lmn-1}\mathrm{e}^{-l\gamma y_n}\right]\left[\frac{(l\gamma)^{lm}}{(lm-1)!}x_{n+1}^{lm-1}\mathrm{e}^{-l\gamma x_{n+1}}\right]\right\}\mathrm{d}x_{n+1}\,\mathrm{d}y_n$$
$$= B^2\sum_{n=0}^{\infty}\sum_{j=0}^{lm-1}\left[\frac{(l\gamma)^{lmn+j}}{(lmn-1)!j!}B^{lmn+j}\mathrm{e}^{-l\gamma B}\right]\left[\left(\frac{1}{lmn}\right)\Omega(lmn,j)\right]. \qquad (C.9)$$

From (C.4), (C.9) is re-written as

$$D_3 = \sum_{n=0}^{\infty}\sum_{j=0}^{lm-1}\left[\frac{1}{(lmn+j)!}\right](l\gamma)^{lmn+j}B^{lmn+j+2}\mathrm{e}^{-l\gamma B}. \qquad (C.10)$$

The fourth item of the right-hand side in (C.1) is

$$D_4 = -2B\sum_{n=0}^{\infty}\int_{y_n=0}^{B}\int_{x_{n+1}=B-y_n}^{\infty}\left\{\left[\frac{(l\gamma)^{lmn}}{(lmn-1)!}y_n^{lmn-1}\mathrm{e}^{-l\gamma y_n}\right]\left[\frac{(l\gamma)^{lm}}{(lm-1)!}x_{n+1}^{lm}\mathrm{e}^{-l\gamma x_{n+1}}\right]\right\}\mathrm{d}x_{n+1}\,\mathrm{d}y_n$$
$$= -2lm\sum_{n=0}^{\infty}\sum_{j=0}^{lm}\left[\frac{(l\gamma)^{lmn+j-1}}{(lmn-1)!j!}B^{lmn+j+1}\mathrm{e}^{-l\gamma B}\right]\left[\left(\frac{1}{lmn}\right)\Omega(lmn,j)\right]. \qquad (C.11)$$

From (C.4), (C.11) is re-written as

$$D_4 = -2lm \sum_{n=0}^{\infty} \sum_{j=0}^{lm} \left[\frac{1}{(lmn+j)!} \right] (l\gamma)^{lmn+j-1} B^{lmn+j+1} \mathrm{e}^{-l\gamma B}. \tag{C.12}$$

The fifth item of the right-hand side in (C.1) is

$$\begin{aligned} D_5 &= -2B \sum_{n=0}^{\infty} \int_{y_n=0}^{B} \int_{x_{n+1}=B-y_n}^{\infty} \left\{ \left[\frac{(l\gamma)^{lmn}}{(lmn-1)!} y_n^{lmn} \mathrm{e}^{-l\gamma y_n} \right] \left[\frac{(l\gamma)^{lm}}{(lm-1)!} x_{n+1}^{lm-1} \mathrm{e}^{-l\gamma x_{n+1}} \right] \right\} \mathrm{d}x_{n+1}\, \mathrm{d}y_n \\ &= -2 \sum_{n=0}^{\infty} \sum_{j=0}^{lm-1} \left[\frac{(l\gamma)^{lmn+j}}{(lmn-1)!j!} B^{lmn+j+2} \mathrm{e}^{-l\gamma B} \right] \left[\left(\frac{1}{lmn+1} \right) \Omega(lmn+1, j) \right]. \end{aligned} \tag{C.13}$$

From (C.4), (C.13) is re-written as

$$D_5 = -2 \sum_{n=0}^{\infty} \sum_{j=0}^{lm-1} \left[\frac{lmn}{(lmn+j+1)!} \right] (l\gamma)^{lmn+j} B^{lmn+j+2} \mathrm{e}^{-l\gamma B}. \tag{C.14}$$

The sixth item of the right-hand side in (C.1) is

$$\begin{aligned} D_6 &= 2 \sum_{n=0}^{\infty} \int_{y_n=0}^{B} \int_{x_{n+1}=B-y_n}^{\infty} \left\{ (x_{n+1} y_n) \left[\frac{(l\gamma)^{lmn}}{(lmn-1)!} y_n^{lmn-1} \mathrm{e}^{-l\gamma y_n} \right] \left[\frac{(l\gamma)^{lm}}{(lm-1)!} \; x_{n+1}^{lm-1} \mathrm{e}^{-l\gamma x_{n+1}} \right] \right\} \mathrm{d}x_{n+1}\, \mathrm{d}y_n \\ &= 2lm \sum_{n=0}^{\infty} \sum_{j=0}^{lm} \left[\frac{(l\gamma)^{lmn+j-1}}{(lmn-1)!j!} B^{lmn+j+1} \mathrm{e}^{-l\gamma B} \right] \left[\left(\frac{1}{lmn+1} \right) \Omega(lmn+1, j) \right]. \end{aligned} \tag{C.15}$$

From (C.4), (C.15) is re-written as

$$D_6 = 2lm \sum_{n=0}^{\infty} \sum_{j=0}^{lm} \left[\frac{lmn}{(lmn+j+1)!} \right] (l\gamma)^{lmn+j-1} B^{lmn+j+1} \mathrm{e}^{-l\gamma B}. \tag{C.16}$$

From (C.1), (C.6), (C.8), (C.10), (C.12), (C.14) and (C.16), $E[B_\mathrm{L}^2]$ is expressed as

$$\begin{aligned} E[B_\mathrm{L}^2] &= \sum_{n=0}^{\infty} \sum_{j=0}^{lm-1} [(j+2)(j+1)] \left[\frac{(l\gamma)^{lmn+j} B^{lmn+j+2} \mathrm{e}^{-l\gamma B}}{(lmn+j+2)!} \right] \\ &\quad - 2lm \sum_{n=0}^{\infty} \sum_{j=0}^{lm} (j+1) \left[\frac{(l\gamma)^{lmn+j-1} B^{lmn+j+1} \mathrm{e}^{-l\gamma B}}{(lmn+j+1)!} \right] \\ &\quad + \sum_{n=0}^{\infty} \sum_{j=0}^{lm+1} \left[\frac{lm(lm+1)}{(lmn+j)!} \right] \left[(l\gamma)^{lmn+j-2} B^{lmn+j} \mathrm{e}^{-l\gamma B} \right]. \end{aligned} \tag{C.17}$$

Appendix D. Deriving $E[B_\mathrm{L}^2]$ for the recharged credit case

This appendix derives $E[B_\mathrm{L}^2]$ for the recharged credit case. From the assumptions in Section 3.2,

$$\begin{aligned} E[B_\mathrm{L}^2] &= \sum_{n=0}^{\infty} \int_{b=0}^{\infty} \int_{y_n=0}^{b} \int_{x_{n+1}=b-y_n}^{\infty} \left\{ (y_n + x_{n+1} - b)^2 \left[\frac{\beta^\alpha}{\Gamma(\alpha)} b^{\alpha-1} \mathrm{e}^{-\beta b} \right] \right. \\ &\qquad \left. \times \left[\frac{(l\gamma)^{lmn}}{(lmn-1)!} y_n^{lmn-1} \mathrm{e}^{-l\gamma y_n} \right] \left[\frac{(l\gamma)^{lm}}{(lm-1)!} x_{n+1}^{lm-1} \mathrm{e}^{-l\gamma x_{n+1}} \right] \right\} \mathrm{d}x_{n+1}\, \mathrm{d}y_n\, \mathrm{d}b \\ &= E_1 + E_2 + E_3 + E_4 + E_5 + E_6, \end{aligned} \tag{D.1}$$

where the first item of the right-hand side in (D.1) is

$$E_1 = \sum_{n=0}^{\infty} \int_{b=0}^{\infty} \int_{y_n=0}^{b} \int_{x_{n+1}=b-y_n}^{\infty} \left\{ y_n^2 \left[\frac{\beta^\alpha}{\Gamma(\alpha)} b^{\alpha-1} \mathrm{e}^{-\beta b} \right] \times \left[\frac{(l\gamma)^{lmn}}{(lmn-1)!} y_n^{lmn-1} \mathrm{e}^{-l\gamma y_n} \right] \left[\frac{(l\gamma)^{lm}}{(lm-1)!} x_{n+1}^{lm-1} \mathrm{e}^{-l\gamma x_{n+1}} \right] \right\} \mathrm{d}x_{n+1}\, \mathrm{d}y_n\, \mathrm{d}b$$

$$= \sum_{n=0}^{\infty} \sum_{j=0}^{lm-1} \left\{ \left[\frac{\beta^\alpha (l\gamma)^{lmn+j}}{\Gamma(\alpha)(lmn-1)!j!} \right] \left[\frac{\Gamma(\alpha+lmn+j+2)}{(\beta+l\gamma)^{\alpha+lmn+j+2}} \right] \left[\left(\frac{1}{lmn+2} \right) \Omega(lmn+2, j) \right] \right\}. \quad \text{(D.2)}$$

From (C.4), (D.2) is expressed as

$$E_1 = \sum_{n=0}^{\infty} lmn(lmn+1) \sum_{j=0}^{lm-1} \left\{ \left[\frac{\beta^\alpha (l\gamma)^{lmn+j}}{(\beta+l\gamma)^{\alpha+lmn+j+2}} \right] \left[\frac{\Gamma(\alpha+lmn+j+2)}{\Gamma(\alpha)(lmn+j+2)!} \right] \right\}. \quad \text{(D.3)}$$

The second item of the right-hand side in (D.1) is

$$E_2 = \sum_{n=0}^{\infty} \int_{b=0}^{\infty} \int_{y_n=0}^{b} \int_{x_{n+1}=b-y_n}^{\infty} \left\{ x_{n+1}^2 \left[\frac{\beta^\alpha}{\Gamma(\alpha)} b^{\alpha-1} \mathrm{e}^{-\beta b} \right] \times \left[\frac{(l\gamma)^{lmn}}{(lmn-1)!} y_n^{lmn-1} \mathrm{e}^{-l\gamma y_n} \right] \left[\frac{(l\gamma)^{lm}}{(lm-1)!} x_{n+1}^{lm-1} \mathrm{e}^{-l\gamma x_{n+1}} \right] \right\} \mathrm{d}x_{n+1}\, \mathrm{d}y_n\, \mathrm{d}b$$

$$= \left[\frac{lm(lm+1)}{(l\gamma)^2} \right] \sum_{n=0}^{\infty} \sum_{j=0}^{lm+1} \left\{ \left[\frac{\beta^\alpha (l\gamma)^{lmn+j}}{(l\gamma+\beta)^{\alpha+lmn+j}} \right] \left[\frac{\Gamma(\alpha+lmn+j)}{\Gamma(\alpha)(lmn-1)!j!} \right] \left[\left(\frac{1}{lmn} \right) \Omega(lmn, j) \right] \right\}. \quad \text{(D.4)}$$

From (C.4), (D.4) is re-written as

$$E_2 = lm(lm+1) \sum_{n=0}^{\infty} \sum_{j=0}^{lm+1} \left\{ \left[\frac{\beta^\alpha (l\gamma)^{lmn+j-2}}{(\beta+l\gamma)^{\alpha+lmn+j}} \right] \left[\frac{\Gamma(\alpha+lmn+j)}{\Gamma(\alpha)(lmn+j)!} \right] \right\}. \quad \text{(D.5)}$$

The third item of the right-hand side in (D.1) is

$$E_3 = \sum_{n=0}^{\infty} \int_{b=0}^{\infty} \int_{y_n=0}^{b} \int_{x_{n+1}=b-y_n}^{\infty} \left\{ b^2 \left[\frac{\beta^\alpha}{\Gamma(\alpha)} b^{\alpha-1} \mathrm{e}^{-\beta b} \right] \times \left[\frac{(l\gamma)^{lmn}}{(lmn-1)!} y_n^{lmn-1} \mathrm{e}^{-l\gamma y_n} \right] \left[\frac{(l\gamma)^{lm}}{(lm-1)!} x_{n+1}^{lm-1} \mathrm{e}^{-l\gamma x_{n+1}} \right] \right\} \mathrm{d}x_{n+1}\, \mathrm{d}y_n\, \mathrm{d}b$$

$$= \sum_{n=0}^{\infty} \sum_{j=0}^{lm-1} \left\{ \left[\frac{\beta^\alpha (l\gamma)^{lmn+j}}{\Gamma(\alpha)(lmn-1)!j!} \right] \left[\frac{\Gamma(\alpha+lmn+j+2)}{(\beta+l\gamma)^{\alpha+lmn+j+2}} \right] \left[\left(\frac{1}{lmn} \right) \Omega(lmn, j) \right] \right\}. \quad \text{(D.6)}$$

From (C.4), (D.6) is re-written as

$$E_3 = \sum_{n=0}^{\infty} \sum_{j=0}^{lm-1} \left\{ \left[\frac{\beta^\alpha (l\gamma)^{lmn+j}}{(\beta+l\gamma)^{\alpha+lmn+j+2}} \right] \left[\frac{\Gamma(\alpha+lmn+j+2)}{\Gamma(\alpha)(lmn+j)!} \right] \right\}. \quad \text{(D.7)}$$

The fourth item of the right-hand side in (D.1) is

$$E_4 = -2\sum_{n=0}^{\infty}\int_{b=0}^{\infty}\int_{y_n=0}^{b}\int_{x_{n+1}=b-y_n}^{\infty}\left\{\left[\frac{\beta^\alpha}{\Gamma(\alpha)}b^\alpha \mathrm{e}^{-\beta b}\right]\times\left[\frac{(l\gamma)^{lmn}}{(lmn-1)!}y_n^{lmn}\mathrm{e}^{-l\gamma y_n}\right]\left[\frac{(l\gamma)^{lm}}{(lm-1)!}x_{n+1}^{lm-1}\mathrm{e}^{-l\gamma x_{n+1}}\right]\right\}\mathrm{d}x_{n+1}\,\mathrm{d}y_n\,\mathrm{d}b$$
$$= -2\sum_{n=0}^{\infty}\sum_{j=0}^{lm-1}\left\{\left[\frac{\beta^\alpha(l\gamma)^{lmn+j}}{(l\gamma+\beta)^{\alpha+lmn+j+2}}\right]\left[\frac{\Gamma(\alpha+lmn+j+2)}{\Gamma(\alpha)(lmn-1)!j!}\right]\left[\left(\frac{1}{lmn+1}\right)\Omega(lmn+1,j)\right]\right\}. \quad \text{(D.8)}$$

From (C.4), (D.8) is

$$E_4 = -2lm\sum_{n=0}^{\infty}n\left\{\sum_{j=0}^{lm-1}\left[\frac{\beta^\alpha(l\gamma)^{lmn+j}}{(\beta+l\gamma)^{\alpha+lmn+j+2}}\right]\left[\frac{\Gamma(\alpha+lmn+j+2)}{(lmn+j+1)!\Gamma(\alpha)}\right]\right\}. \quad \text{(D.9)}$$

The fifth item of the right-hand side in (D.1) is

$$E_5 = -2\sum_{n=0}^{\infty}\int_{b=0}^{\infty}\int_{y_n=0}^{b}\int_{x_{n+1}=b-y_n}^{\infty}\left\{\left[\frac{\beta^\alpha}{\Gamma(\alpha)}b^\alpha \mathrm{e}^{-\beta b}\right]\times\left[\frac{(l\gamma)^{lmn}}{(lmn-1)!}y_n^{lmn-1}\mathrm{e}^{-l\gamma y_n}\right]\left[\frac{(l\gamma)^{lm}}{(lm-1)!}x_{n+1}^{lm}\mathrm{e}^{-l\gamma x_{n+1}}\right]\right\}\mathrm{d}x_{n+1}\,\mathrm{d}y_n\,\mathrm{d}b$$
$$= -2lm\sum_{n=0}^{\infty}\sum_{j=0}^{lm}\left\{\left[\frac{\beta^\alpha(l\gamma)^{lmn+j-1}}{(\beta+l\gamma)^{\alpha+lmn+j+1}}\right]\left[\frac{\Gamma(\alpha+lmn+j+1)}{\Gamma(\alpha)(lmn-1)!j!}\right]\left[\left(\frac{1}{lmn}\right)\Omega(lmn,j)\right]\right\}. \quad \text{(D.10)}$$

From (C.4), (D.10) is re-written as

$$E_5 = -2lm\sum_{n=0}^{\infty}\sum_{j=0}^{lm}\left\{\left[\frac{\beta^\alpha(l\gamma)^{lmn+j-1}}{(\beta+l\gamma)^{\alpha+lmn+j+1}}\right]\left[\frac{\Gamma(\alpha+lmn+j+1)}{\Gamma(\alpha)(lmn+j)!}\right]\right\}. \quad \text{(D.11)}$$

The sixth item of the right-hand side in (D.1) is

$$E_6 = 2\sum_{n=0}^{\infty}\int_{b=0}^{\infty}\int_{y_n=0}^{b}\int_{x_{n+1}=b-y_n}^{\infty}\left\{\left[\frac{\beta^\alpha}{\Gamma(\alpha)}b^{\alpha-1}\mathrm{e}^{-\beta b}\right]\times\left[\frac{(l\gamma)^{lmn}}{(lmn-1)!}y_n^{lmn}\mathrm{e}^{-l\gamma y_n}\right]\left[\frac{(l\gamma)^{lm}}{(lm-1)!}x_{n+1}^{lm}\mathrm{e}^{-l\gamma x_{n+1}}\right]\right\}\mathrm{d}x_{n+1}\,\mathrm{d}y_n\,\mathrm{d}b$$
$$= 2lm\sum_{n=0}^{\infty}\sum_{j=0}^{lm}\left\{\left[\frac{\beta^\alpha(l\gamma)^{lmn+j-1}}{(\beta+l\gamma)^{\alpha+lmn+j+1}}\right]\left[\frac{\Gamma(\alpha+lmn+j+1)}{\Gamma(\alpha)(lmn-1)!j!}\right]\left[\left(\frac{1}{lmn+1}\right)\Omega(lmn+1,j)\right]\right\}. \quad \text{(D.12)}$$

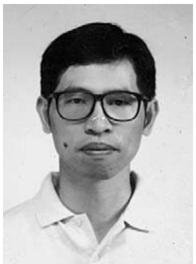
From (C.4), (D.12) is re-written as

$$E_6 = 2(lm)^2\sum_{n=0}^{\infty}n\left\{\sum_{j=0}^{lm}\left[\frac{\beta^\alpha(l\gamma)^{lmn+j-1}}{(\beta+l\gamma)^{\alpha+lmn+j+1}}\right]\left[\frac{\Gamma(\alpha+lmn+j+1)}{\Gamma(\alpha)(lmn+j+1)!}\right]\right\}. \quad \text{(D.13)}$$

From (D.1), (D.3), (D.5), (D.7), (D.9), (D.11) and (D.13), $E[B_\mathrm{L}^2]$ is expressed as (18) in Section 3.2.

References

[1] A. Arteta, Prepaid billing technologies – which one is for you? Billing World, February 1998, pp. 54–61.
[2] M. Bill, Convergence creates growth and promise for prepaid, Telecommunications 31 (5) (1997) 69–71.
[3] J.G. Bosse, in: Signaling in Telecommunication Networks, Wiley, New York, 1998.
[4] D. Edwards, Pay now, call later, Telecommunications, March 1998.
[5] ETSI/TC, European digital cellular telecommunications system (phase 2); Common aspects of PLMN network management, Technical Report Recommendation GSM 12.01, ETSI, 1993.
[6] ETSI/TC, Mobile application part (MAP) specification, version 4.8.0, Technical Report Recommendation GSM 09.02, ETSI, 1994.
[7] R. Guerin, Channel occupancy time distribution in a cellular radio system, IEEE Transactions on Vehicular Technology 36 (1987) 89–99.
[8] INFOCOMM, Intelligent prepaid system (IPS) functional description. Technical Report PDIPS-00-00001-FD, INFOCOMM, 1998.
[9] Y.-B. Lin, Performance modeling for mobile telephone networks, IEEE Network Magazine 11 (6) (1997) 63–68.
[10] Y.-B. Lin, I. Chlamtac, Effects of Erlang call holding times on PCS call completion, IEEE Transactions on Vehicular Technology 48 (3) (1999) 815–823.
[11] M. Mouly, M.-B. Pautet, The GSM system for mobile communications, M. Mouly, 49 rue Louise Bruneau, F-91120 Palaiseau, France, 1992.
[12] S.M. Ross, Stochastic Processes, Wiley, New York, 1983.
[13] F. Slavick, Counting down prepaid services, Billing World, September 1998, pp. 44–50.
[14] J. Stokes, The prepaid frenzy, Mobile Communications in Asia, February 1999, p. 19.
[15] J. Wei, The prepaid service of telecommunications UN and Singapore as an example, Taiwan Communication Magazine, August 1998, pp. 47–50.



Ming-Feng Chang received his B.S. and M.S. degrees in electrical engineering from the National Taiwan University in 1982 and 1984, respectively, and his Ph.D. degree in computer science from the University of Illinois at Urbana-Champaign in 1991. He is currently an Associate Professor in the Department of Computer Science and Information Engineering, Chiao-Tung University, Taiwan, Republic of China. His research interests include Internet communication, mobile computing and VLSI system design.

Yi-Bing Lin received his BSEE degree from the National Cheng Kung University in 1983, and his Ph.D. degree in Computer Science from the University of Washington in 1990. From 1990 to 1995, he was with the Applied Research Area at Bell Communications Research (Bellcore), Morristown, NJ. In 1995, he was appointed as a professor in the Department of Computer Science and Information Engineering (CSIE), National Chiao Tung University (NCTU). In 1996, he was appointed as the Deputy Director of Microelectronics and Information Systems Research Center, NCTU. During 1997–1999, he was elected as Chairman of CSIE, NCTU. His current research interests include design and analysis of personal communications services network, mobile computing, distributed simulation, and performance modeling. Dr. Lin is an associate editor of IEEE Network, an editor of IEEE J-SAC: Wireless Series, an editor of IEEE Personal Communications Magazine, an editor of Computer Networks, an area editor of ACM Mobile Computing and Communication Review, a columnist of ACM Simulation Digest, an editor of International Journal of Communications Systems, an editor of ACM/Baltzer Wireless Networks, an editor of Computer Simulation Modeling and Analysis, an editor of Journal of Information Science and Engineering, Program Chair for the Eighth Workshop on Distributed and Parallel Simulation, General Chair for the Ninth Workshop on Distributed and Parallel Simulation. Program Chair for the Second International Mobile Computing Conference, Guest Editor for the ACM/Baltzer MONET special issue on Personal Communications, a Guest Editor for IEEE Transactions on Computers special issue on Mobile Computing, and a Guest Editor for IEEE Communications Magazine special issue on Active, Programmable, and Mobile Code Networking. Lin is the author of the book Wireless and Mobile Network Architecture (co-author with Imrich Chlamtac; published by Wiley). Lin received 1998 and 2000 Outstanding Research Awards from National Science Council, ROC, and 1998 Outstanding Youth Electrical Engineer Award from CIEE, ROC. Lin is an Adjunct Research Fellow of Academia Sinica. Lin's email address is liny@csie.nctu.edu.tw.

Wei-Zu Yang received his M.S. degree from the Department of Computer Science and Information Engineering, National Chiao Tung University in 1992. He is currently a Ph.D. candidate in the Chiao Tung University. His research interests include performance modeling of PCS and ATM networks.