



Lower bounds for wide-sense nonblocking Clos network

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Abstract

The 3-stage Clos network is generally considered the most basic multistage interconnecting network (MIN). The nonblocking property of such network has been extensively studied in the past. However, there are only a few lower bound results regarding wide-sense nonblocking. We show that in the classical circuit switching environment, to guarantee wide-sense nonblocking for large r , $2n - 1$ center switches are necessary, where r is the number of input switches and n is the number of inlets of each input switch. For the multirate environment, we show that for large r , any 3-stage Clos network needs at least $3n - 2$ center switches to guarantee wide-sense nonblocking. Our proof works even for the two-rate environment. © 2001 Elsevier Science B.V. All rights reserved.

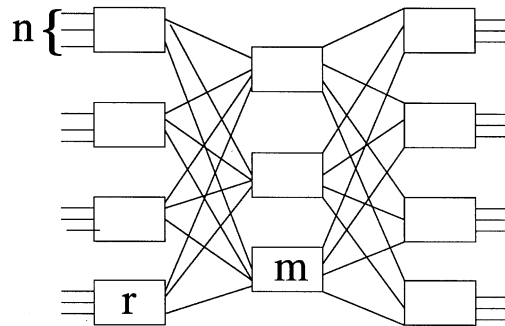
1. Introduction

The 3-stage Clos network is generally considered the most basic multistage interconnecting network (MIN). It is symmetric with respect to the center stage. The first stage, or the *input stage*, has $rn \times m$ crossbar switches; the center stage has $mr \times r$ crossbar switches and the final stage, or the *output stage*, has $rm \times n$ crossbar switches. The n inlets (outlets) on each input (output) switch are the *inputs (outputs)* of the network. There is exactly one link between every center switch and every input (output) switch. We use $C(n, m, r)$ to denote a 3-stage Clos network. An example of $C(3, 3, 4)$ is shown in Fig. 1.

In the classical circuit switching environment, i.e., every link can only serve one connection request, three types of nonblocking properties have been extensively studied.

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Fig. 1. $C(3,3,4)$.

They are *strictly nonblocking* (SNB), *wide-sense nonblocking* and *rearrangeably nonblocking* (RNB). The focus of this paper is to establish lower bounds for the number of center switches needed to guarantee wide-sense nonblocking. A network is wide-sense nonblocking (WSNB) if a new call is always routable as long as all previous requests were routed according to a given routing algorithm.

In the multirate switching environment, a request is a triple (u, v, w) , where u is an inlet, v an outlet and w a weight which can be thought of as the bandwidth requirement (rate) of that request. We normalize the weights so that $1 \geq w > 0$, and each link has capacity one; i.e., it can carry any number of calls as long as the sum of weights of these calls does not exceed one.

Clos [3] proved that for the classical model, $2n - 1$ center switches are necessary and sufficient to guarantee SNB for $C(n, m, r)$. Benes [1, 2] proved that $C(n, m, 2)$ is WSNB (using the packing routing) if and only if $m \geq 3n/2$, thus giving hope that WSNB can be achieved with fewer center switches than SNB in general. However recently, Du et al. [4] gave the surprising result that $C(n, m, r)$ for $r \geq 3$ is WSNB under the packing routing if and only if $m \geq 2n - 1$; namely, it requires the same number of center switches as SNB. In this paper, we further dash the hope by showing that for large r , $C(n, m, r)$ is WSNB (under any routing algorithm) if and only if $m \geq 2n - 1$.

While WSNB, as commented above, plays a very restrictive role in the classical model, the multirate environment provides a fertile playground. This is because we have a new dimension, the *rate*, to design routing algorithm. For example, Gao and Hwang [5] gave a routing algorithm such that $C(n, m, r)$ is WSNB if $m \geq 5.75n$. If there are only two different rates, then the requirement reduced to $4n$. In this paper we show that $3n - 2$ is a lower bound of WSNB under any routing algorithm, and this bound is obtained by using only two different rates. This lower bound provides a gauge to measure how good the algorithm of Gao and Hwang is and how much room is left for improvement. We will also talk about some impacts on repackable algorithms, a relatively new type of nonblocking.

2. Main results

We first study the classical model where every link can serve only one connection request.

Theorem 1. *For*

$$r \geq (n-1) \binom{2n-2}{n-1} + 1, C(n, m, r)$$

is WSNB if and only if $m \geq 2n - 1$.

Proof. Since SNB implies WSNB, it suffices to prove the “only if” part. Suppose the number of center switches, m , is $2n - 2$. We shall prove that if

$$r = (n-1) \binom{2n-2}{n-1} + 1,$$

then the network is not WSNB. In order to prove it, let the network carry a set of initial requests. Then we show that there must exist a set of new requests such that at least one of them is not routable.

Suppose there are $r(n-1)$ initial requests, which involve $(n-1)$ inlets from each input switch but not to a particular output switch O . Since the number of total outputs excluding output switch O is $(r-1)n$ and $r > n$. This is a feasible set of requests. The network then routes these initial requests (according to any routing algorithm). Notice that no two requests from an input switch can share a common center switch. Hence, the $n-1$ connections from an input switch are routed by a set of $n-1$ center switches. We say that the set of these center switches is the *routing set* of that input switch. Since there are $2n-2$ center switches, and the size of each routing set is $n-1$, the number of distinct routing set is

$$\binom{2n-2}{n-1}.$$

If the number of input switches is large enough, i.e.,

$$r = (n-1) \binom{2n-2}{n-1} + 1,$$

then by the pigeon-hole principle, there must exist X , a set of n input switches, which has the same routing set, Y .

Consider a new set of n requests $\{(x, o): x \in X, o \in O\}$. Each of the n requests must be routed through a distinct center switch, which is not in Y . Hence, at least $|X| + |Y| = 2n - 1$ center switches are needed (See Fig. 2). \square

Next, we consider the multirate model where each request has a rate requirement.

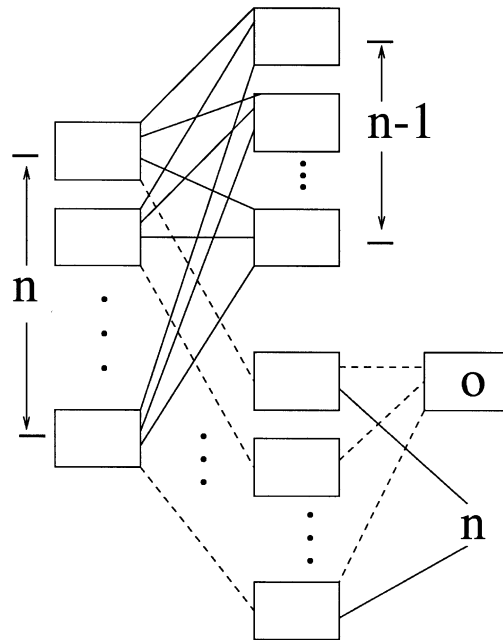


Fig. 2. Figure for Theorem 1.

Theorem 2. $C(n, m, r)$ is not WSNB for any two rates B, b , $B > b$, satisfying $B + b > 1$ if $n \geq 2$, $m \leq \min(k + 2n - 3, 3n - 3)$ and r is large enough, where $k = \lfloor 1/b \rfloor$.

Proof. The condition $B + b > 1$ forces $B > \frac{1}{2}$ and also the consequence that any link carrying a B -request cannot carry a second request, therefore we may assume $B = 1$. Without loss of generality, assume that $b = 1/k$ for some integer $k \geq 2$. We mention that B is too large to share a link with other requests. Another fact is that the number of B -requests is limited as each inlet may generate at most one B -request. On the contrary, b is small, and an inlet may generate up to k b -requests. To employ these special properties of two rate B, b , we devise a two-phase construction of requests. In Phase 1, only b -requests are generated, and B -requests appear in Phase 2. Since a B -request cannot share a link with a b -request, any link used in Phase 1 is no longer available in Phase 2.

The object of Phase 1 is to spread out some b -requests to some common center switches. On the other hand, we also hope that the input switch maintain the capability to generate the maximum number of B -requests in Phase 2. Hence, in our construction, only the first inlet of each input switch can generate requests in Phase 1. The construction in Phase 1 can be further divided into several steps. In each step, one b -request is generated in the first inlet of the input switches. The exact number of steps is bounded by two inequalities, which depend on the relation between k and n , and will be examined later.

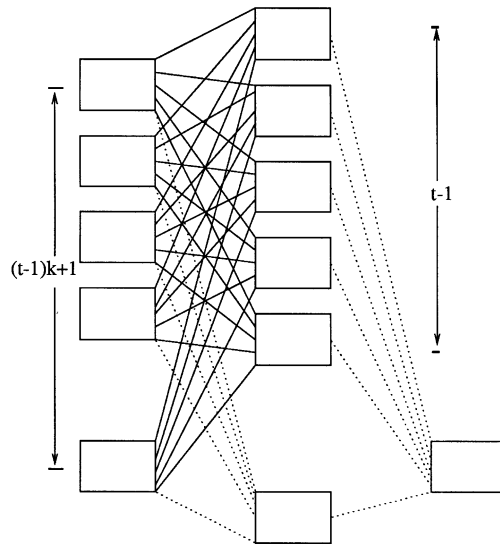


Fig. 3. Figure for step t of Theorem 2.

Step 1: Each input generates one request. Since r is large enough, there exists at least a large set I_1 of input switches whose requests are all routed through the same center switch M_1 .

Step 2: Each input in I_1 generates a second request. Partition I_1 into size $k + 1$ groups such that inputs in the same group all have their second requests going to the same fresh output switch O . This is possible since $b(k + 1) \leq n$, the capacity of a fresh output switch. But the total weights of each group, $b(k + 1)$, require at least a new center switch M_2 to carry since the link between M_1 and O can carry only bk . Therefore in each group at least one input switch whose second request is carried by some center switch other than M_1 . Since I_1 is large there exists a large set I_2 of input switches whose requests are routed through the same set of center switches M_1, M_2 .

Step t : Each input in I_{t-1} generates a t th request. Partition I_{t-1} into size $(t - 1)k + 1$ groups such that inputs in the same group all make the t th request to the same fresh output, this is possible if $(t - 1)k + 1 \leq nk$. But the total weights of each group, $(t - 1)bk + b$, is greater than $t - 1$, the total capacity between the specified output switch and M_1, M_2, \dots, M_{t-1} . Therefore, in each group at least one input switch whose t th request is carried by some center switch other than M_1, M_2, \dots, M_{t-1} . Since I_{t-1} is large there exists at least a large set I_t of input switches whose requests are routed through the same set of center switches (see Fig. 3).

The number of steps is controlled by the following two inequalities. Firstly, there is at most k iterations, as the first inlet can generate not more than k b -request. Secondly, these requests coming from the same group are directed to an output switch. Hence, the size of the group as well as the number of steps are bounded. Suppose that Step t is the last step. Then, $t \leq k$ and $(t - 1)k + 1 \leq nk$. It can be verified that $t = \min(k, n)$.

In Phase 2, Consider only the input switches in I_t . Since each switch is able to generate $n - 1$ B -requests going to a set of fresh output switches, which are able to receive n B -requests. By applying the same argument in Theorem 1, we may argue that $2n - 2$ center switches are needed. Notice that M_1, M_2, \dots, M_{t-1} are not available in this phase as the links between them and I_t already carried a b -request, which was generated in Phase 1. Hence, a total of $t + 2n - 2$ or $\min(k, n) + 2n - 2$ center switches are necessary. \square

Corollary 1. *In a multirate environment, $C(n, m, r)$ is not WSNB if $m \leq 3n - 3$ and r is large enough.*

3. Some concluding remarks

A network is *repackable* if existing calls can be rearranged at any moment when a connection is deleted (e.g., a call hangs up). Repacking algorithms have been studied for both the classical model [6] and the multirate model [7, 8] to show that they can help to reduce the number of center switches needed for $C(n, m, r)$ to be nonblocking. Theorems 1 and 2 imply that no repacking algorithm can be effective when r is large. Since the request sequences we construct in these theorems have no deletion, these results apply to all repacking algorithms. Thus, our results solidly confirm a major difference between $\{\text{WSNB, Repackable}\}$ and $\{\text{SNB, RNB}\}$, namely, the numbers of center switches required for $C(n, m, r)$ are dependent on r for the former pair, but independent from r for the latter pair. In other words, the hope of finding an effective WSNB or repackable algorithm is restricted to small r .

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