Optimal Synthesis of a Fractional Delay FIR Filter in a Reproducing Kernel Hilbert Space

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***Abstract*—Based on a bandlimited signal model, the optimal fractional delay finite impulse response (FIR) filter and corresponding interpolating error bound is derived in a reproducing kernel Hilbert space. The resulting optimal filtering is a projection onto a prescribed finite dimensional subspace. The connection of this filtering accuracy to the delay time and the filter order is investigated via error analysis.**

***Index Terms*—Bandlimited signal, fractional delay filter, reproducing kernel Hilbert space.**

I. Introduction

A FRACTIONAL delay (FD) filter is an interpolator used to interpolate between samples of a bandlimited signal. This filter is a digital signal processing (DSP) technique with applications to a wide range of physical problems such as beam steering of sensor arrays, modeling of musical instruments, and echo cancellation (see an excellent review paper [1]).

Although FD filters can be synthesized to approximate an ideal FD filter in either frequency or time domain, the frequency-domain methods appear to attract more attention. This fact is caused by various weighted least-squares fits in the frequency domain being comparatively easy to modify and solve. However, this letter focuses on a time-domain method based on a bandlimited signal model. The finite-energy bandlimited signals are notable for forming a reproducing kernel Hilbert space (RKHS) in which the sampled value of each signal can be evaluated using a kernel function (see [2], [3] for details). In the RKHS setting, the FD finite impulse response (FIR) filter is synthesized optimally by minimizing the worst-case magnitude error that may occur in the filter implementation.

The rest of this paper is organized as follows. Section II first recapitulates some important theorems of RKHS. Section III presents the solution of the optimal FD FIR filter. Finally, the properties of this optimal filter are investigated by error analysis in Section IV.

II. Bandlimited Signal Model

Without loss of generality, the signals are assumed herein to contain no frequency components above $\alpha\pi$ rad/sec and the sampling period is 1 s. In this way, signals are critically sampled when $\alpha = 1$, over-sampled when $\alpha < 1$. The Fourier transform (FT) of a bandlimited signal is a complex function with compact support. For the interval $[-\alpha\pi, \alpha\pi]$, consider the Hilbert space $\mathbf{L}^2[-\alpha\pi, \alpha\pi]$, of which the inner product is defined as

$$\langle F, G\rangle_{\mathbf{L}^2} = \frac{1}{2\pi}\int_{-\alpha\pi}^{\alpha\pi} F(\omega)\bar{G}(\omega)\, d\omega, \quad \text{for } F(\omega), G(\omega) \in \mathbf{L}^2[-\alpha\pi, \alpha\pi] \tag{1}$$

where $\bar{G}(\omega)$ denotes the complex conjugate of $G(\omega)$. Thus, any function $F(\omega)$ in $\mathbf{L}^2[-\alpha\pi, \alpha\pi]$ can be considered as the FT of a finite-energy $\alpha\pi$-bandlimited signal $f(t)$. Thus, the inverse FT of $f(t)$ can be written as

$$f(t) = \langle F(\omega), e^{-j\omega t}\rangle_{\mathbf{L}^2}. \tag{2}$$

All of these transforms $f(t)$ form a Hilbert space of $\alpha\pi$-bandlimited signals, represented by $\mathbf{H}(K)$. This Hilbert space is an RKHS with the kernel (see [3])

$$K(t, s) = \langle e^{-j\omega s}, e^{-j\omega t}\rangle_{\mathbf{L}^2} = \alpha\,\mathrm{sinc}[\alpha(t - s)] \tag{3}$$

and the inner product

$$\langle f, g\rangle_{\mathbf{H}(K)} = \langle F, G\rangle_{\mathbf{L}^2}. \tag{4}$$

Notably, the FT is unitary, and thus $\mathbf{H}(K)$ (time domain) is isomorphic with $\mathbf{L}^2[-\alpha\pi, \alpha\pi]$ (frequency domain). To derive the optimal FD FIR filter in $\mathbf{H}(K)$, the following theorems quoted without proof (see [2]) are used.

Theorem 1 (The Reproducing Property): In an RKHS $\mathbf{H}(K)$, a uniquely determined kernel function $K(\cdot, s) \in \mathbf{H}(K)$ exists such that

$$f(s) = \langle f(\cdot), K(\cdot, s)\rangle_{\mathbf{H}(K)} \quad f \in \mathbf{H}(K). \tag{5}$$

Theorem 2: Assume that M is a closed subspace of an RKHS $\mathbf{H}(K)$, then M is also an RKHS. Additionally, if

$$K(\cdot, s) = K_1(\cdot, s) + K_2(\cdot, s) \quad K_1 \in M, \quad K_2 \in M^{\perp} \tag{6}$$

then K_1 is the kernel of M, and K_2 is the kernel of $M^{\perp}$.

Theorem 3: Assume that if H is a Hilbert space, and M is an RKHS subspace of H, then for every $f \in H$

$$(P_M f)(t) = \langle f(\cdot), K_M(\cdot, t)\rangle$$

where P_M is the projection operator onto M, and K_M is the reproducing kernel of M.

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III. Optimal FD FIR Filter

In this section, the optimal solution for a general FD FIR filter is derived by minimizing the worst-case interpolating error. In the following, the inner product and the norm are defined in $\mathbf{H}(k)$ and thus the subscript $\mathbf{H}(k)$ is omitted.

Proposition 1 (Optimal FD FIR Filter): The best estimate $f(d)$ of an arbitrary $f \in \mathbf{H}(K)$ through linear combination of the sample values $f(0), f(1), \ldots, f(N)$ is

$$\hat{f}(d) = \sum_{n=0}^{N} h_n f(n) \tag{7}$$

and

$$\begin{bmatrix} h_0 \\ h_1 \\ h_2 \\ \vdots \\ h_N \end{bmatrix} = \begin{bmatrix} K(0,0) & K(0,1) & \cdots & K(0,N) \\ K(1,0) & K(1,1) & & K(1,N) \\ K(2,0) & K(2,1) & \cdots & K(2,N) \\ \vdots & \vdots & & \vdots \\ K(N,0) & K(N-1,1) & \cdots & K(N,N) \end{bmatrix}^{-1} \times \begin{bmatrix} K(0,d) \\ K(1,d) \\ K(2,d) \\ \vdots \\ K(N,d) \end{bmatrix} \tag{8}$$

where

$$K(t,s) = \alpha \operatorname{sinc}[\alpha(t-s)]. \tag{9}$$

Proof: According to the reproducing property (Theorem 1) and Cauchy-Schwarz inequality

$$\begin{aligned} |f(d) - \hat{f}(d)|^2 &= \left| f(d) - \sum_{n=0}^{N} h_n f(n) \right|^2 \\ &= \left| \left\langle f(\cdot), K(\cdot,d) - \sum_{n=0}^{N} h_n K(\cdot,n) \right\rangle \right|^2 \\ &\le \|f\|^2 \left\| K(\cdot,d) - \sum_{n=0}^{N} h_n K(\cdot,n) \right\|^2. \end{aligned} \tag{10}$$

To minimize the absolute value of the estimate error for arbitrary $f \in \mathbf{H}(K)$, the following problem is equivalently solved:

$$\min_{h_n \in \mathbf{R}, n=0,1,\ldots,N} \left\| K(\cdot,d) - \sum_{n=0}^{N} h_n K(\cdot,n). \right\| \tag{11}$$

According to the projection principle, the aforementioned error norm is minimized when

$$\left\langle K(\cdot,d) - \sum_{n=0}^{N} h_n K(\cdot,n), K(\cdot,m) \right\rangle = 0, \quad m = 0,1,\ldots,N. \tag{12}$$

Further applying the reproducing property to (12) yields

$$\sum_{n=0}^{N} h_n K(m,n) = K(m,d), \quad m = 0,1,\ldots,N. \tag{13}$$

Next, the optimal coefficients h_n can be determined for any specific d by solving the aforementioned N linear equations. ■

The optimal filter in Proposition 1 has numerous interpretations. First, this optimal filter can be interpreted in a geometrical way. Suppose the topic of interest is the projection of an arbitrary $f \in \mathbf{H}(K)$ onto the subspace M generated by all finite linear combinations of $\{K(\cdot,n); n = 0,1,\ldots,N\}$. Since every closed subspace of an RKHS is also an RKHS, M is an RKHS, and its reproducing kernel $K_M(t,d) \in M$ is assumed to be some linear combination of $\{K(\cdot,n); n = 0,1,\ldots,N\}$. Then Theorem 2 can be applied to achieve

$$K_M(t,d) = h_0 K(t,0) + h_1 K(t,1) + \cdots + h_N K(t,N) \tag{14}$$

where h_n has the same expression as (8). Finally, applying Theorems 1 and 3 yields the projection

$$\begin{aligned} P_M f(d) = \langle f(\cdot), K_M(\cdot,d) \rangle &= \sum_{n=0}^{N} h_n \langle f(\cdot), K(\cdot,n) \rangle \\ &= \sum_{n=0}^{N} h_n f(n) \end{aligned} \tag{15}$$

which is identical to the best estimate obtained in Proposition 1. Thus, the proposed optimal interpolation is actually a projection operation onto the subspace M.

An alternative interpretation uses a frequency-domain viewpoint. When attempting to solve the following problem (see [1]):

$$\min_{G(z)=\sum_{n=0}^{N} h_n z^{-n}} \int_0^{\alpha\pi} |e^{-j\omega d} - G(e^{j\omega})|^2 \, d\omega \tag{16}$$

the same solution as in Proposition 1 is obtained. Therefore, the proposed filter approximates an ideal FD response in the signal bandwidth while leaving the response in the remainder of the frequency band as "don't care." This behavior is caused by the $\alpha\pi$-bandlimited signal model, and the optimization process assumes no signal frequency content above $\alpha\pi$.

IV. Error Analysis

The relationship between the interpolating performance and three design parameters, namely, α, N, and d, is interesting. To clarify this relationship, this section estimates the magnitude error, which may arise when implementing the optimal FD FIR filter.

Proposition 2 (Magnitude Error Bound): When interpolating with the optimal FD FIR linear filter presented in Proposition 1, the magnitude error bound of the signal can be expressed as follows:

$$|f(d) - \hat{f}(d)| \le \|f\| \sqrt{\alpha \left(1 - \sum_{n=0}^{N} h_n \operatorname{sinc}[\alpha(d-n)]\right)} \tag{17}$$

Proof: According to (15) and the Cauchy-Schwarz inequality

$$\begin{aligned} |f(d) - \hat{f}(d)|^2 &= |f(d) - P_M f(d)|^2 \\ &= |\langle f(\cdot), K(\cdot,d) - K_M(\cdot,d) \rangle|^2 \\ &\le \|f\|^2 \|K(\cdot,d) - K_M(\cdot,d)\|^2. \end{aligned} \tag{18}$$

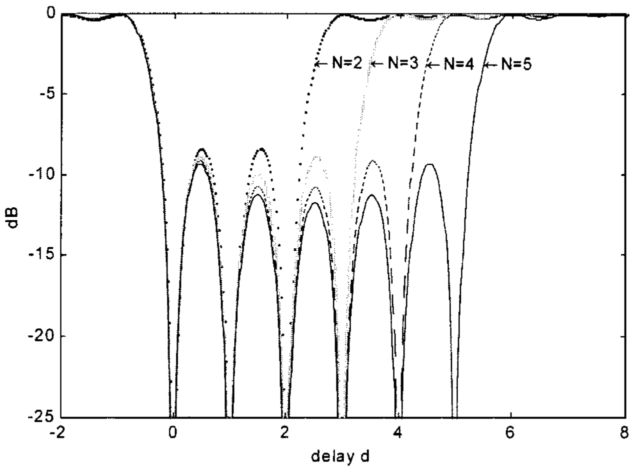

Fig. 1. Normalized error bounds for the critically sampled case ($\alpha = 1$).

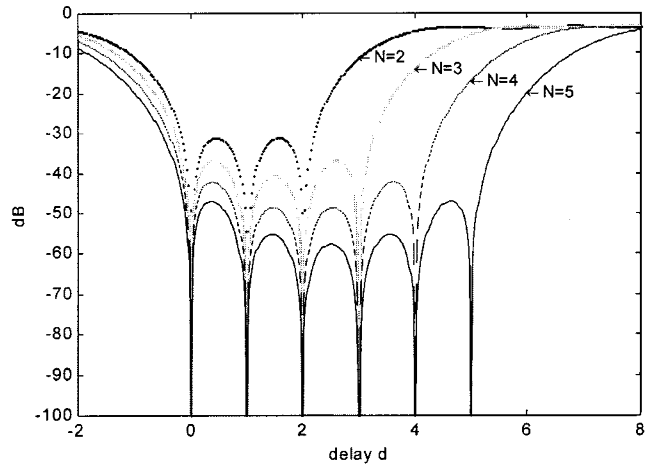

Fig. 2. Normalized error bounds for the oversampled case ($\alpha = 0.5$).

Since $K(\cdot, d) - K_M(\cdot, d)$ is orthogonal to $K_M(\cdot, d)$ by Theorem 2, then applying the Pythagorean Theorem produces

$$\begin{aligned} &\|K(\cdot, d) - K_M(\cdot, d)\|^2 \\ &= \|K(\cdot, d)\|^2 - \|K_M(\cdot, d)\|^2 \\ &= \langle K(\cdot, d), K(\cdot, d)\rangle - \langle K_M(\cdot, d), K_M(\cdot, d)\rangle \\ &= K(d, d) - K_M(d, d). \end{aligned} \tag{19}$$

Substitution of (9) and (14) into (19) yields

$$\|K(\cdot, d) - K_M(\cdot, d)\|^2 = \alpha \left(1 - \sum_{n=0}^{N} h_n \operatorname{sinc}[\alpha(d - n)]\right). \tag{20}$$

Finally, substitution of (20) into (18) gives (17). ■

As an example, two cases can be considered: one critically sampled ($\alpha = 1$) and the other with a sampling frequency four times the signal bandlimit ($\alpha = 0.5$). By Proposition 1, the optimal filter for any specific delay d and filter order N can be obtained. Then, as Proposition 2 suggests, the normalized error bound $\sqrt{\alpha(1 - \sum_{n=0}^{N} \operatorname{sinc}[\alpha(d - n)])}$ is calculated. Figs. 1 and 2 show some results.

Several features of Figs. 1 and 2 are notable.

1) Signal bandlimit α. As the figures illustrate, the oversampled case performs much better than the critically sampled case. Closely examining (17) reveals that the interpolating error would be exactly zero for an extreme case ($\alpha = 0$). In this case, the system (13) is underdetermined and thus has infinite possible solutions. Consequently, infinite filters can interpolate a constant signal perfectly. Notably, however, (8) may have the numerical stability problem with a small α. With a very small α, the columns of the matrix in (8) become only "slightly" linearly independent, making the matrix almost singular. Therefore, to avoid an absurd solution from (8), α should not be allowed to be too small, even when the sampling rate is actually much higher than the signal bandlimit.
2) The filter order N. Errors decrease with increasing filter order N. However, for the critically sampled case, the improvements are only slight.
3) The delay time d. Like a Lagrange interpolation, these optimal filters interpolate the signal at the sample instants. Thus, the filtering errors are zero for an integer d within the filter length. Notably, for a fixed N, the intersampled interpolations are generally more accurate when the delay time d is close to half of the filter length. Therefore, to optimally realize a long delay with a short-length FIR filter, the desired delay can be split into an integer and a short delay approximately half of the filter length. A cascade of unit delay elements can then realize the integer delay, while the optimal FD FIR filter realizes the remaining short delay.

V. Conclusion

In this letter, the optimal FD FIR filter is synthesized by minimizing the worst-case interpolating error in a Hilbert space of bandlimited signals. The resulting optimal filter interpolates the sampled signal by projecting the original signal onto a prescribed finite dimensional subspace. The underlying RKHS geometry not only helps explain the optimal interpolating process, but also helps estimate the interpolating error, thereby benefiting investigation of the properties of the proposed optimal filter.

References

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