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Uniqueness of positive radial solutions for semilinear elliptic equations on annular domains☆

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1. Introduction

In this paper we shall study the uniqueness problem of positive radial solutions for semilinear elliptic equations on annular domains. Indeed, we consider the following equations:

$$\Delta u + f(u) = 0 \quad \text{in } \Omega, \tag{1.1}$$

$$u = 0 \quad \text{on } \partial\Omega, \tag{1.2}$$

where $\Omega = \Omega_{a,b} = \{x \in \mathbf{R}^n,\ 0 < a < |x| < b\}$ is an annular domain in $\mathbf{R}^n$, $n \geq 3$.

It is well known that the uniqueness problems of solutions of (1.1) and (1.2) are both fundamental and often difficult. During the last decade, there has been tremendous progress in studying these problems when Ω is a ball in $\mathbf{R}^n$ or entire $\mathbf{R}^n$ $n \geq 3$, see, e.g., [1,2,5,10,11]. However, very little was known concerning the annular domain until the work of Ni and Nussbaum [12]. Until very recently, Coffman [3] showed that when $n = 3$, $f(u) = -u + u^p$, $1 < p \leq 3$, (1.1) and (1.2) have unique positive radial solution for any annular domain, see also [13]. Then, it is of interest to extend Coffman's result to a more general situation. For example, we may ask the following question.

Question. Do (1.1) and (1.2) have unique positive radial solution when $f(u) = -u + u^p$, $1 < p \leq n/(n-2)$, and $n \geq 4$?

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In this paper, after carefully investigating the auxiliary functions which have been used in [3,11] and some techniques developed in dealing with the problem on ball [1], we can prove the uniqueness result in the case $n=4$ and partial result when $n=5$.

Indeed, we can prove the following general result.

Theorem 1. *Assume* $f \in C^1([0,\infty))$ *and satisfies the following conditions*:
(A-1) $f(u)(u-1) > 0$ *for* $u > 0$ *and* $u \neq 1$,
(A-2) $f'(u)u > f(u)$ *for* $u > 0$,
(A-3) $f'(u)(u-1) > f(u)$ *for* $u > 1$,
(A-4) $I(u) < 0$ *for* $u > 1$,
where

$$I(u) = (n-2)f'(u)(u-1) - nf(u). \tag{1.3}$$

Then (1.1) *and* (1.2) *have a unique positive radial solution on any annular domain in* $\mathbf{R}^n$, $n \geq 3$.

One consequence of Theorem 1 is the following result.

Theorem 2. *For* $3 \leq n \leq 5$, $f(u) = -u + u^p$, *and* $1 < p \leq \min(4/(n-2), n/(n-2))$. *Then* (1.1) *and* (1.2) *have a unique positive radial solution on any annular domain in* $\mathbf{R}^n$.

Higher dimensional cases are a subject of further study in future.

This paper is organized as follows: In Section 2, we present some preliminary results concerning the positive radial solutions on annular domains. In Section 3, we prove the main results.

2. Preliminaries

In this section we shall recall and prove some useful properties of positive radial solution of (1.1). We always assume (A-1) holds.

The radial symmetric solution of (1.1) satisfies the following equation:

$$u''(r) + \frac{n-1}{r}u'(r) + f(u(r)) = 0, \quad r > a > 0. \tag{2.1}$$

Denote by $u(\cdot,\alpha)$ the unique solution of Eq. (2.1) and initial conditions

$$u(a,\alpha) = 0 \quad \text{and} \quad u'(a,\alpha) = \alpha > 0. \tag{2.2}$$

If $u(\cdot,\alpha)$ has a zero, then denote its first zero by $R(\alpha)(<\infty)$; otherwise, $R(\alpha) = \infty$. It is known that the solutions can be classified into the following three types:

$$N = \{\alpha > 0\colon u(r,\alpha) > 0 \text{ in } (a, R(\alpha)) \text{ and } u(R(\alpha),\alpha) = 0,\ R(\alpha) < \infty\},$$

$$G = \left\{\alpha > 0\colon u(r,\alpha) > 0 \text{ in } (a,\infty) \text{ and } \lim_{r\to\infty} u(r,\alpha) = 0\right\}$$

and

$$P = (0, \infty) \setminus (N \cup G).$$

It is also proved that $N \neq \emptyset$ when (A-1) and (A-2) holds, see, e.g., [6,8,9]. Indeed, $N \supset (\alpha^*, \infty)$ for some $\alpha^* > 0$. In this case, there is a unique $R_0(\alpha) \in (\alpha, R(\alpha))$ such that

$$u'(r, \alpha) > 0 \quad \text{in } (a, R_0(\alpha))$$

and

$$u'(r, \alpha) < 0 \quad \text{in } (R_0(\alpha), R(\alpha)),$$

see, e.g., [4,9].

To study the uniqueness problem, it is useful to study the variation φ of u with respect to α, i.e.,

$$\varphi(r, \alpha) = \frac{\partial u(r, \alpha)}{\partial \alpha}.$$

It is clear that φ satisfies

$$\varphi''(r) + \frac{n-1}{r}\varphi'(r) + f'(u(r))\varphi(r) = 0, \quad r > a, \tag{2.3}$$

$$\varphi(a, \alpha) = 0 \quad \text{and} \quad \varphi'(a, \alpha) = 1. \tag{2.4}$$

It is also known that φ has a zero in $(a, R(\alpha))$ when (A-1) and (A-2) hold, see e.g., [7]. The first zero of φ in $(a, R(\alpha)]$ is denoted by $r_1(\alpha)$. Denote the maximum $\varphi(\xi)$ of φ in (a, r_1); then it is easy to verify

$$\varphi'(r, \alpha) > 0 \text{ in } (a, \xi(\alpha)) \quad \text{and} \quad \varphi'(r, \alpha) > 0 \text{ in } (\xi(\alpha), r_1(\alpha)), \tag{2.5}$$

see, e.g., [4,9].

The following lemma due to Coffman [3] is critical.

Lemma 2.1. (Coffman [2,3]). *If* (A-1)–(A-3) *hold, then*

$$u(\xi, \alpha) > 1. \tag{2.6}$$

Proof. We first claim

$$\varphi' < \frac{1}{\alpha}u' \quad \text{in } (a, \xi). \tag{2.7}$$

Indeed, let $V = (1/\alpha)u - \varphi$. Then V satisfies

$$V'' + \frac{n-1}{r}V' + f'(u)V = \frac{1}{\alpha}(f'(u)u - f(u)). \tag{2.8}$$

From (2.3) and (2.8), we have

$$r^{n-1}\{V'(r)\varphi(r) - V(r)\varphi'(r)\} = \frac{1}{\alpha}\int_a^r s^{n-1}(f'(u(s))u(s) - f(u(s)))\varphi(s)\,\mathrm{d}s.$$

By (A-2), the right-hand side is positive in (a, r_1), which implies

$$V'(r)\varphi(r) - V(r)\varphi'(r) > 0 \tag{2.9}$$

in (a, r_1). Hence,

$$\frac{\mathrm{d}}{\mathrm{d}r}\left\{\frac{V(r)}{\varphi(r)}\right\} = \frac{1}{\varphi^2(r)}\{V'(r)\varphi(r) - V(r)\varphi'(r)\} > 0 \quad \text{in } (a, r_1).$$

Now,

$$\frac{V(a)}{\varphi(a)} = \lim_{r\to a^+}\left\{\frac{1}{\alpha}\frac{u(r,\alpha)}{\varphi(r,\alpha)} - 1\right\} = 0.$$

Therefore, we have

$$V(r) > 0 \quad \text{in } (a, r_1).$$

By (2.9) and the last inequality, we have

$$V'(r) > 0 \quad \text{in } (a, \xi),$$

and thus (2.7) follows.

To prove (2.6), we need to introduce more auxiliary functions as follows: Denote

$$F(u) = \int_0^u f(v)\,\mathrm{d}v, \tag{2.10}$$

$$\Phi(r) = u'^2(r) + 2F(u(r)), \tag{2.11}$$

and

$$\Psi(r) = u'(r)\varphi'(r) + f(u(r))\varphi(r). \tag{2.12}$$

It is easy to verify

$$\Phi'(r) = -\frac{2(n-1)}{r}u'^2(r) \tag{2.13}$$

and

$$\Psi'(r) = -\frac{2(n-1)}{r}u'(r)\varphi'(r). \tag{2.14}$$

Therefore, by (2.11) and (2.13), we have

$$\Phi(r) = \alpha^2 - 2(n-1)\int_a^r \frac{u'^2(s)}{s}\,\mathrm{d}s. \tag{2.15}$$

Since $u(R(\alpha), \alpha) = 0$, we have

$$\Phi(R(\alpha)) = u'^2(R(\alpha)) > 0.$$

Hence (2.13) implies

$$\Phi(r) > 0 \quad \text{in } [a, R(\alpha)].$$

Eq. (2.15) implies

$$2(n-1)\int_a^r \frac{u'^2(s)}{s}\,\mathrm{d}s < \alpha^2. \tag{2.16}$$

Similarly, by (2.12) and (2.14) we have

$$\Psi(r) = \alpha - 2(n-1)\int_a^r \frac{1}{s}\varphi'(s)u'(s)\,\mathrm{d}s. \tag{2.17}$$

By Schwartz inequality, (2.7) and (2.16), we have

$$\begin{aligned}
&2(n-1)\left|\int_a^r \frac{1}{s}\varphi' s(s)u'(s)\,\mathrm{d}s\right| \\
&\quad\le 2(n-1)\left(\int_a^r \frac{1}{s}\varphi'^2(s)\,\mathrm{d}s\right)^{1/2}\left(\int_a^r \frac{1}{s}u'^2(s)\,\mathrm{d}s\right)^{1/2} \\
&\quad< 2(n-1)\cdot\frac{1}{\alpha}\cdot\int_a^r \frac{1}{s}u'^2(s)\,\mathrm{d}s \\
&\quad< \alpha.
\end{aligned}$$

By (2.17) and the last inequality, we have

$$\Psi(\xi) > 0.$$

Since $\varphi(\xi) > 0$ and $\varphi'(\xi) = 0$, by (2.12) we have

$$f(u(\xi,\alpha)) > 0.$$

By (A-1), we have

$$u(\xi,\alpha) > 1.$$

The proof is complete. □

Next, we shall prove the following lemma.

Lemma 2.2. *If* (A-1)–(A-3) *hold*, *then*

$$u(r_1(\alpha),\alpha) > 1. \tag{2.18}$$

Proof. Let $W = u - 1$, then W satisfies

$$W'' + \frac{n-1}{r}W' + f'(u)W = f'(u)(u-1) - f(u). \tag{2.19}$$

By (2.3) and (2.19), we obtain

$$\begin{aligned}
r^{n-1}(W'\varphi - W\varphi')|_\xi^r = \int_\xi^r s^{n-1}\{&f'(u(s))(u(s)-1) \\
&- f(u(s))\}\varphi(s)\,\mathrm{d}s > 0
\end{aligned} \tag{2.20}$$

for any $r \in (\xi, r_1]$.

We now claim $W(r_1) > 0$. Otherwise, if there were $\hat{r} \in (\xi, r_1]$ such that $W(\hat{r}) = 0$, then, by (2.20) and(2.7) we have

$$\hat{r}^{n-1}u'(\hat{r})\varphi(\hat{r}) > \xi^{n-1}u'(\xi)\varphi(\xi) > 0$$

which implies

$$u'(\hat{r}) > 0.$$

Now (2.6) and $u(\hat{r}) = 1$ imply $u'(\hat{r}) < 0$, a contradiction. This implies

$$W(r_1) > 0.$$

The proof is complete. □

3. Proofs of main results

In this section, we shall prove Theorems 1 and 2.

For any $c \in \mathbf{R}^1$, denote

$$v_c(r) = ru'(r) + c(u(r) - 1). \tag{3.1}$$

Then,

$$v_c'(r) = (c + 2 - n)u'(r) - rf(u(r)) \tag{3.2}$$

and v_c satisfies

$$v_c''(r) + \frac{n-1}{r}v_c'(r) + f'(u(r))v_c(r) = I_c(u(r)), \tag{3.3}$$

where

$$I_c(u) = cf'(u)(u - 1) - (c + 2)f(u). \tag{3.4}$$

For notational simplicity, when $c = n - 2$, we denote

$$v = v_{n-2} = ru'(r) + (n - 2)(u(r) - 1)$$

and

$$I = I_{n-2} = (n - 2)f'(u)(u - 1) - nf(u).$$

It is easy to verify

$$I'(u) = (n - 2)f''(u)(u - 1) - 2f'(u)$$

and

$$I''(u) = (n - 2)f'''(u)(u - 1) + (n - 4)f''(u).$$

The following lemma is crucial in studying the uniqueness problem.

Lemma 3.1. *If* (A-1)–(A-4) *hold, then*

$$v(r_1(\alpha), \alpha) < 0. \tag{3.5}$$

Proof. By (2.3) and (3.3), we have

$$r^{n-1}(v'\varphi - v\varphi')|_{\xi}^{r_1} = \int_{\xi}^{r_1} r^{n-1} I(u(r))\varphi(r)\,\mathrm{d}r.$$

Now, (A-4) implies $I(u(r)) < 0$ in (ξ, r_1). Therefore, we have

$$-r_1^{n-1} v(r_1)\varphi'(r_1) - \xi^{n-1} v'(\xi)\varphi(\xi) < 0.$$

By (3.2), we have

$$-r_1^{n-1} v(r_1)\varphi'(r_1) + \xi^{n} f(u(\xi))\varphi(\xi) < 0.$$

Now (2.6) implies $f(u(\xi)) > 0$, therefore

$$v(r_1) < 0.$$

The proof is complete. □

To prove the main result, we need to introduce the following auxiliary function Z which also has been used in [3]. Denote

$$Z(r) = v(r)\cdot\{r\varphi'(r) + (n-2)\varphi(r)\} + r^2 f(u(r))\varphi(r). \tag{3.6}$$

Then it is easy to verify that

$$Z'(r) = r\varphi(r)J(u(r)), \tag{3.7}$$

where

$$J(u) = (4-n)f(u) - (n-2)f'(u)(u-1). \tag{3.8}$$

It is easy to verify that

$$J(u) \le 0 \quad \text{for } u > 1 \tag{3.9}$$

when (A-3) holds.

After these preparations, we can now prove the following lemma which leads to the main result.

Lemma 3.2. *If* (A-1)–(A-4) *hold*, *then* φ *has exactly one zero in* $(a, R(\alpha)]$.

Proof. Suppose that $\varphi(r,\alpha)$ has a second zero $r_2 \in (a, R(\alpha)]$. Then there exists $\eta \in (r_1, r_2)$ such that

$$\eta\varphi'(\eta) + (n-2)\varphi(\eta) = 0. \tag{3.10}$$

If (3.10) holds, then we claim that

$$u(\eta) < 1. \tag{3.11}$$

Indeed, by (3.5), we have

$$Z(r_1) = r_1 v(r_1)\varphi'(r_1) > 0. \tag{3.12}$$

By (3.7) and (3.9) we have

$$Z'(r) \ge 0 \quad \text{in } (r_1, \hat{r}), \tag{3.13}$$

whenever $u(\hat{r}) \geq 1$. Otherwise, if (3.11) were false, i.e., $u(\eta) \geq 1$. Then (3.12) and(3.13) imply

$$Z(\eta) > 0. \tag{3.14}$$

On the other hand, by (3.6), (3.10), and (3.14), we have

$$\eta^2 f(u(\eta))\varphi(\eta) > 0$$

which implies $f(u(\eta)) < 0$ or $u(\eta) < 1$, a contradiction. Hence (3.11) follows.

Now, let $v_0 = ru'(r)$ as in (3.1). Then $I_0 = -2f(u)$, i.e., v_0 satisfies

$$v_0''(r) + \frac{n-1}{r} v_0'(r) + f'(u(r))v_0 = -2f(u(r)). \tag{3.15}$$

By (2.3) and (3.15), we have

$$r^{n-1}(v_0'\varphi - v_0\varphi')|_\eta^{r_2} = -2\int_\eta^{r_2} r^{n-1} f(r(r))\varphi(r)\,\mathrm{d}r < 0.$$

However, by (3.11), the left-hand side of the last equality is

$$-r_2^n u'(r_2)\varphi'(r_2) + \eta^n f(u(\eta))\varphi(\eta) > 0,$$

a contradiction. Therefore, φ has no second zero in $(a, R(\alpha))$. The proof is complete. □

Remark 3.3. In the proof of Lemma 3.2, it is critical to establish (3.11) which is immediate when (3.5) holds. Therefore, to prove (3.5), more careful study is needed for $v(r,\alpha)$ when (A-4) does not hold.

Proof of Theorem 1. Since

$$u(R(\alpha),\alpha) = 0 \tag{3.16}$$

after differentiating (3.16) with respect to $\alpha \in N$, we have

$$u'(r(\alpha),\alpha)\frac{\mathrm{d}R(\alpha)}{\mathrm{d}\alpha} + \varphi(R(\alpha),\alpha) = 0. \tag{3.17}$$

Therefore, Lemma 3.2 implies

$$\frac{\mathrm{d}R(\alpha)}{\mathrm{d}\alpha} < 0 \tag{3.18}$$

for any $\alpha \in N$. Now, it is easy to see $N = (\alpha^*, \infty)$, for some $\alpha^* > 0$. Otherwise, if there were $(\alpha_1,\alpha_2) \subset N$, $0 < \alpha_1 < \alpha_2 < \infty$ and $\alpha_1 \notin N$, $\alpha_2 \notin N$, then α_1 and α_2 will be in G, a contradiction to (3.18). The proof is complete. □

Now, we can apply Theorem 1 to prove Theorem 2.

Proof of Theorem 2. We will verify that $f(u)$ satisfies all assumptions of Theorem 1. It is easy to see that $f(u)$ satisfies (A-1)–(A-3). To prove (A-4), by (1.3) we have

$I(u) = (n-2)(pu^{p-1}-1)(u-1) - n(u^p - u)$, then

$$I'(u) = (n-2)(p(p-1)u^{p-2}-1)(u-1) - n(pu^{p-1}-1),$$

$$I''(u) = p(p-1)[(n-4)u^{p-2} + (u-2)(p-2)u^{p-3}(u-1)].$$

For $3 \le n \le 4$, $p-2 \le n/(n-2) - 2 = (4-n)/(n-2)$. Since $I(1)=0$, $I'(1) = -2(p-1) < 0$, and

$$\begin{aligned} I''(u) &\le p(p-1)[(n-4)u^{p-2} + (4-n)u^{p-3}(u-1)] \\ &= p(p-1)(n-4)u^{p-3} \le 0, \end{aligned}$$

we have $I(u)$ negative for $u > 1$.

For $4 < n < 6$, if $I(u)$ is nonnegative somewhere in $u > 1$, then there is a $u_0 > 1$ such that $I'(u_0) = 0$ and $I''(u_0) \ge 0$. $I'(u_0) = 0$ is equivalent to

$$2(pu_0^{p-1} - 1) = (n-2)p(p-1)u_0^{p-2}(u_0 - 1).$$

Hence,

$$\begin{aligned} 0 \le u_0 I''(u_0) &= p(p-1)[(n-4)u_0^{p-1} + (n-2)(p-2)u_0^{p-2}(u_0-1)] \\ &= p(p-1)(n-4)u_0^{p-1} + 2(p-2)(pu_0^{p-1} - 1) \\ &= p[p(n-2)-n]u_0^{p-1} - 2(p-2) \\ &< p[p(n-2)-n] - 2(p-2) \\ &\le 0 \quad \text{for } 1 < p \le \frac{4}{n-2}, \end{aligned}$$

a contradiction. The proof is complete. □

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