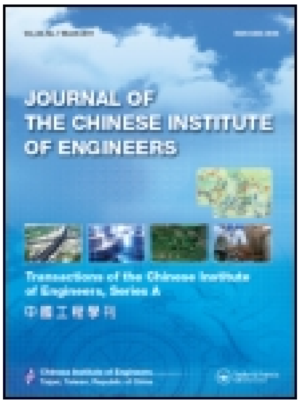

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Hui-Chen Su [a] & Wolfgang Kinzelbach [b]
[a] Department of Civil Engineering, National Chiao Tung University, Hsinchu, Taiwan 300, R.O.C.
[b] Institute of Hydromechanics and Water Resources Management, Swiss Federal Institute of Technology, Zurich, Switzerland

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APPLICATION OF 2-D RANDOM GENERATORS TO THE STUDY OF SOLUTE TRANSPORT IN FRACTURES

Hui-Chen Su*
Department of Civil Engineering
National Chiao Tung University
Hsinchu, Taiwan 300, R.O.C.

Wolfgang Kinzelbach
Institute of Hydromechanics and Water Resources Management
Swiss Federal Institute of Technology
Zurich, Switzerland

Key Words: fractured porous media, fractional Brownian motion, Hurst exponent, cubic law.

ABSTRACT

Two fields with random aperture distribution and different spatial structures are taken as models to study solute transport in fractures. One network has non-vanishing long range correlations and represents a fractal pattern. The other one has a finite correlation length and an exponential covariance function. Based on these fields, two physical fracture models were produced and used to record the movement of a coloured solute by means of a CCD camera. The pictures obtained were analyzed with image processing methods. A front tracking algorithm shows that the growth law of the frontal variance is a power law of time with the exponent depending on the Hurst coefficient of the aperture distribution in the case of the fractal pattern, while it is a linear function of time for the case of the finite correlation length.

I. INTRODUCTION

The complexity of natural fracture networks does not allow a deterministic microscopic description of the actual flow field. Even a single fracture constitutes a complex random medium. Here a single fracture is conceptualized as a two-dimensional porous medium with a local transmissivity which varies stochastically with the fracture aperture. Fractures may be filled with fault gouge or precipitates, further adding to the variability of the local transmissivity.

For media with a finite correlation length of a random property, such as transmissivity, an effective property description is possible if the size of the medium becomes large compared to the correlation length. Transport can then be described by an average flow field responsible for advective transport and an asymptotic dispersion tensor accounting for all mixing due to the disordered motion of the random medium. In this study, the random generator FGEN (Robin *et al*., 1993) is adopted to generate the corresponding transmissivity fields on which transport processes can be analyzed in detail.

If the medium considered is on a scale on which long range correlations still prevail and where a clear separation into a small and a large scale is not possible, the approximation of the medium by a fractal

*Correspondence addressee

may be useful. In a pragmatic approach, a fractal medium is a medium with a correlation function behaving as a power law in Laplace space. For such media a general description is still possible. The evolution of the width of a solute front through the spatial scales contained in the fractal will then follow a power law which is different from the ordinary diffusion law. In this study, a fractional Brownian motion process is used to generate a corresponding transmissivity distribution. A description by a dispersion-advection mechanism is possible. An average advective transport can be combined with a dispersion mechanism characterized by a dispersion coefficient which scales with the distance covered by the average front.

II. 2-DIMENSIONAL RANDOM GENERATORS

1. Fractal Geometry

Self-similar fractals are geometric objects which can be mapped into themselves by a scale transformation. This means a close-up view cannot be distinguished from a larger scale picture. Examples are the Koch curve or the Sierpinski carpet (Mandelbrot, 1982; Feder, 1988). These regular objects are, however, not suited to describe more random objects such as natural shapes of clouds, mountains, coastlines or porous media. For the characterization of those natural objects and especially the fractures considered here, self-affine random fractals are the more suited objects. They can be described by a stochastic process which is an extension of Brownian motion, i. e. fractional Brownian motion (fBm) (Barnsley, 1988; Voss, 1985).

The mathematical description of fractional Brownian motion, $V_H(t)$, uses a single valued function of a variable t, which usually represents time. In this study, however, it is considered to be the spatial coordinate in two dimensions $X(x, y)$. The increments of this function $[V_H(X_2)-V_H(X_1)]$ have a Gaussian distribution with variance:

$$E\{[V_H(X_2)-V_H(X_1)]\}^2=(X_2-X_1)^{2H}\sigma^2 \qquad (1)$$

where H is the Hurst exponent ($0\leq H\leq 1$), and the relation between fractal dimension D and Hurst Exponent H is given by

$$D=d+1-H \qquad (2)$$

where d is the Euclidean dimension.

There are various methods for approximating a finite sample of fractional Brownian motion (Voss, 1985). The random midpoint displacement method (Barnsley, 1988; Voss, 1985) is applied in this study. In the two-dimensional approach of this method, the four corners of any square of a quadratic mesh, the midpoints of its sides, and the center point are assigned a randomized addition to the original (interpolated) value. While the process continues by subdividing the square into four smaller squares, the random perturbations are reduced according to a law involving the Hurst coefficient.

Here the equations are given for a one-dimensional process with coordinate t, between 0 and 1. One starts by setting $X(0)=0$ and selects $X(1)$ as a sample of a Gaussian random variable with mean 0 and variance σ^2. Then $Var(X(1)-X(0))=\sigma^2$ and

$$Var(X(t_2)-X(t_1))=|t_2-t_1|^{2H}\sigma^2 \qquad (3)$$

for $0\leq t_1\leq t_2\leq 1$. We then set the value at the midpoint $X(\frac{1}{2})$ to be the average of $X(0)$ and $X(1)$ plus some Gaussian random offset D_1 with mean 0 and variance Δ_1^2:

$$X(\frac{1}{2})-X(0)=\frac{1}{2}(X(1)-X(0))+D_1 \qquad (4)$$

and thus $X(\frac{1}{2})-X(0)$ has mean value 0 and the same holds for $X(1)-X(\frac{1}{2})$. The next step can be written as:

$$Var(X(\frac{1}{2})-X(0))=\frac{1}{4}Var(X(1)-X(0))+\Delta_1^2=(\frac{1}{2})^{2H}\sigma^2 \qquad (5)$$

therefore,

$$\Delta_1^2=\frac{\sigma^2}{2^{2H}}[1-2^{2H-2}] \qquad (6)$$

The same procedure is applied consecutively to finer resolutions, yielding

$$\Delta_n^2=\frac{\sigma^2}{(2^n)^{2H}}[1-2^{2H-2}] \qquad (7)$$

as the variance of displacement D_n.

2. Cross-correlated Random Field Generation (FGEN) (Robin *et al*., 1993)

This random generator was developed by Robin, *et al*. (1993). The purpose of FGEN is to generate two weakly stationary, real-valued processes that are cross correlated. It can generate 2- and 3-dimensional real random fields by applying an inverse Fast Fourier transform to randomized spectral fields obtained from the power spectral density functions and the cross-spectral density functions of the field parameters. Here we use it to generate 2D random fields without cross-correlations. The generated fields are on a regular grid with constant spacing between generation points in any given direction.

III. FLOW IN A FRACTURE

Laminar flow in an open parallel fracture can be described by Hagen-Poiseuille's law. A fracture is characterized by its geometry: one of its dimensions – the aperture – is much smaller than the other two. Furthermore the flow in individual fractures is assumed to be described by a low Reynolds number, laminar flow between parallel planar boundaries. When analyzing groundwater flow and solute transport in fractured rock, the transmissivity of the fracture is the decisive quantity.

The average velocity of flow between two plates can be written as

$$q = \frac{g b_{fr}^2 J}{12 v} \tag{8}$$

where b is the aperture of the fracture, J is the hydraulic gradient, g is the gravitational constant, and v is the kinematic viscosity.
Furthermore, the total volumetric flow rate per unit is

$$Q = q b_{fr} = \frac{g b_{fr}^3 J}{12 v} = T_{fr} J \tag{9}$$

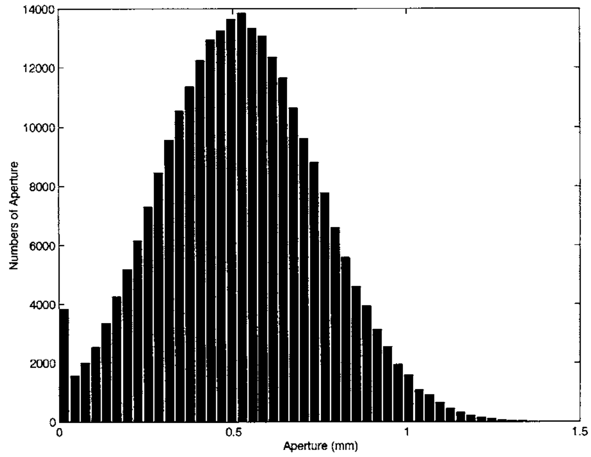
Thus, the fracture transmissivity T_{fr} can be written as:

$$T_{fr} = \frac{g b_{fr}^3}{12 v} K_{fr} b_{fr} \tag{10}$$

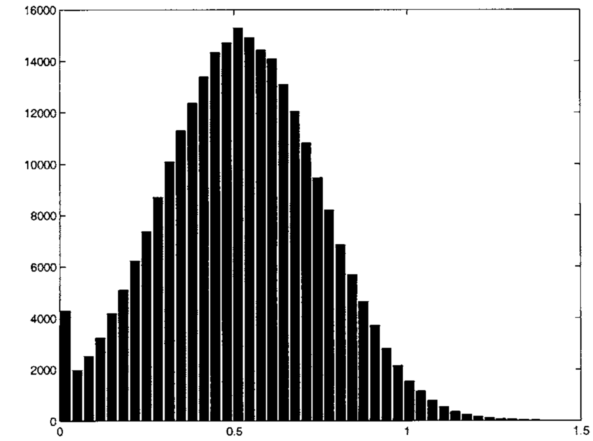
where K_{fr} is the hydraulic conductivity of the fracture. The above equation is known as the "cubic law" of the fracture. We assume this law to be also valid, locally, in a fracture with stochastically varying apertures.

IV. EXPERIMENT

To study solute transport in a fracture we built two physical fracture models. The models consist of two plates, one of which was etched with a pattern of varying depth while the other one was plane. We chose two different patterns. The first one (model A) was a fractal, the second one had a finite correlation length which was small compared to the model dimension (model B). The patterns were generated by the methods described in section II. The etching is illustrated by a sample shown in Figs. 1(a) and 1(b). From the histograms of both patterns, it can be seen that both models have the same fundamental statistical characteristics such as arithmetic mean 0.5204 mm and standard deviation 0.2267 mm. By the cubic law what is described in section III as the fracture transmissivity T_{fr} can be further evaluated. For example: the transmissivity is about 10^{-6} mm^2/sec (aperture = 0.001 mm), 1 mm^2/sec (aperture = 0.1 mm), and 10^3mm^2/sec(aperture = 1 mm).



(a)



(b)

Fig. 1 (a) Histogram of apertures obtained by random midpoint displacement; (b) Histogram of apertures used in FGEN

The two fracture patterns which we generated are shown in Figs. 2(a) and 2(b). The whole fracture model consists of a mesh 513×513 pixels with a total size of 400 mm by 400 mm. As the aperture cannot be smaller than zero the rather small number of negative apertures occurring in the generation process were set to zero. The pattern of Fig. 2(a) was obtained by the fractional random midpoint displacement method with a Hurst exponent of 0.79588. Here the Hurst exponent is a rough value that is estimated from the permeability data at Yunlin, Taiwan. This value is close to the averaged value, 0.73, what was investigated from conductivity by Hurst (Feder, 1988). The other pattern (Fig. 2(b)) was generated

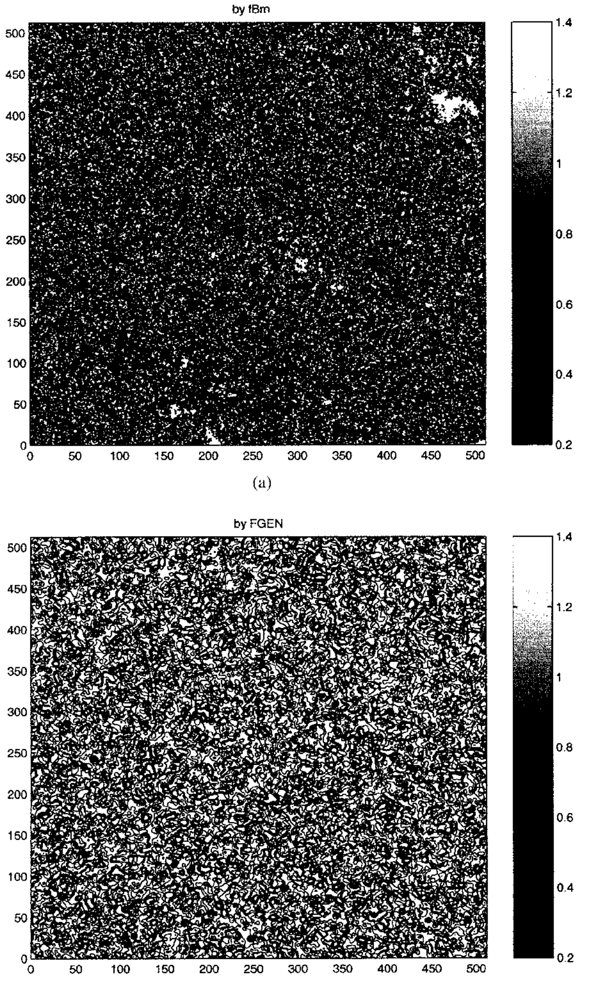
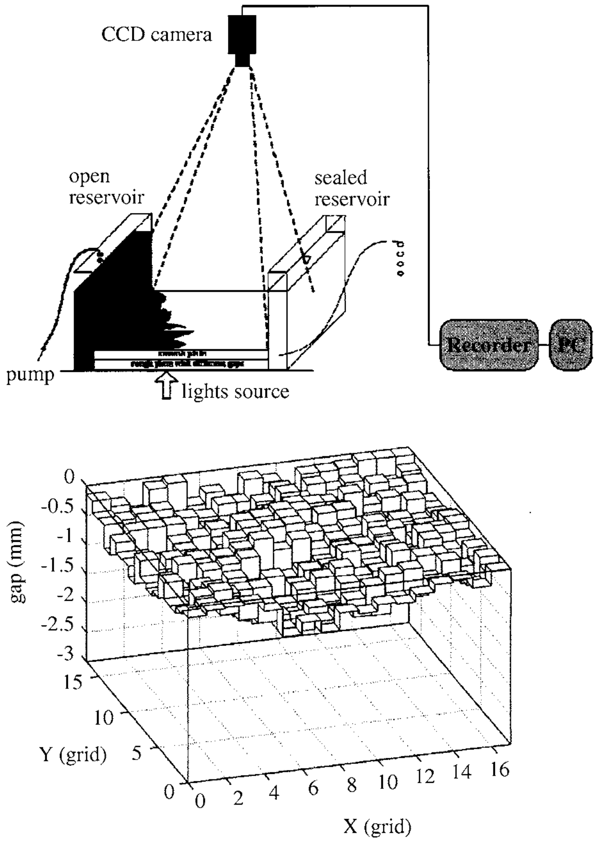


Fig. 2 (a) Apertures pattern of fracture obtained by random midpoint displacement; (b) Apertures pattern of fracture obtained by FGEN

by FGEN using a correlation length of 10 mm ($=\frac{1}{10}\times$*model's length*).

By connecting the square shaped fracture on two opposite sides with reservoirs, and sealing the other two sides, a saturated flow in the fracture could be produced (Refer to Fig. 3). When saturating the empty model with water, care must be taken as trapped air may take a long time to dissolve. A fast saturation could be achieved by first filling the fracture with CO_2–*gas* and then flushing with degassed water. This method is much faster than methods where degassed water is used to flush out trapped air. When the model was successfully degassed the upstream reservoir was coloured by a powder dye and the experiment was started.

The model was mounted on a cold light source. The evolution of the dye front was filmed with a CCD-camera. The observed dye movement is shown in Fig. 4.



Demonstration of Rough Plate (the lower one)

Fig. 3 Setup of the experiment

V. DATA ANALYSIS AND EXPERIMENTAL RESULTS

The front at different times was analyzed by image processing methods. A front tracking algorithm is developed using MATLAB. It allows computing the variance of the dye front at any time from the digitized image data. The frequency of digitized pictures is 1 image/sec.

Figure 4 shows the dye front movement in models A and B at different times. Due to varying thickness, the image of the analyzed model without dye inside should not have a constant color index matrix (This image is called the "blank matrix" in the following) under the unsaturated lighting used. The values of the blank matrix depend on the aperture depth. While the dye flows into the model, the color index matrix of the images (called the "front matrix" in the following) should change due to the color contrast. Comparing the difference between the front matrix of time t and the blank matrix, the color index difference matrix at each time t (defined as ΔX_t here) could be determined. In the mixing zone between water and dye lighter color index differences should be observed. Furthermore, the threshold value

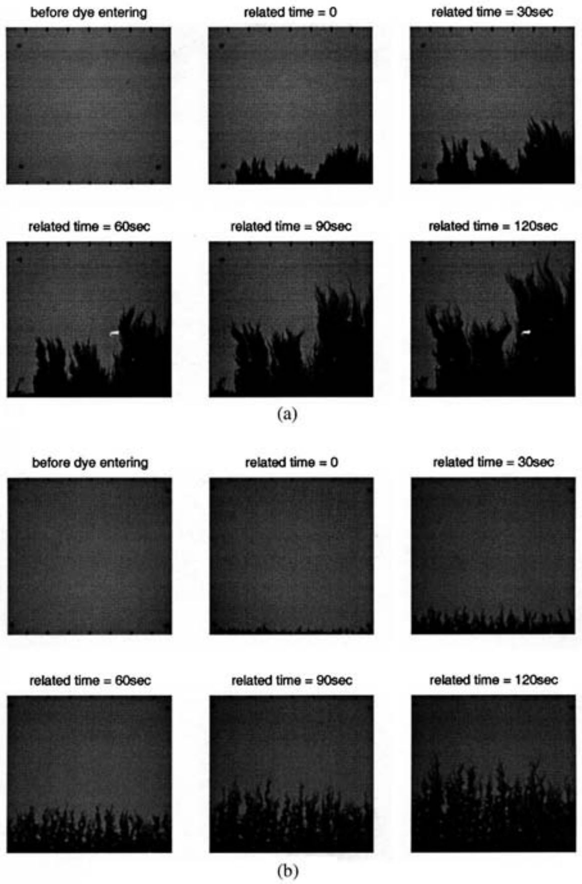


Fig. 4 (a) Dye front at different times: model A; (b) Dye front at different times: model B

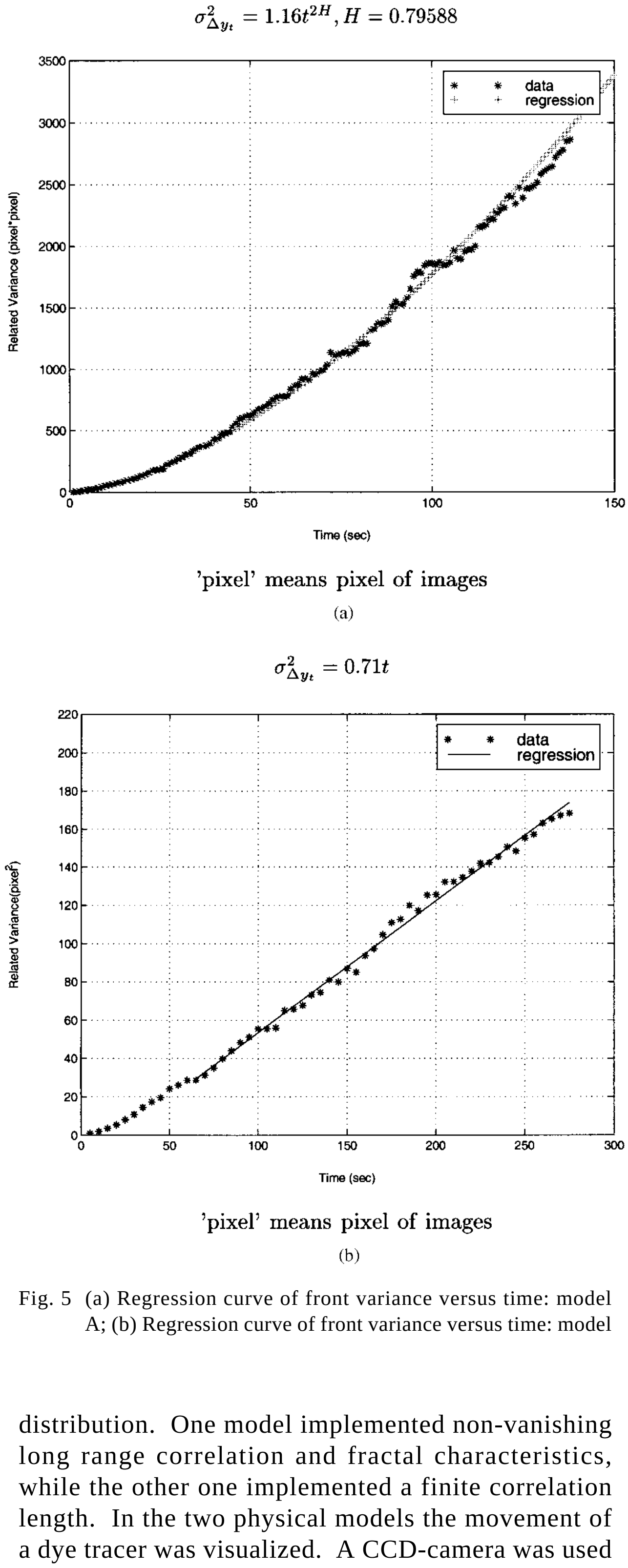


Fig. 5 (a) Regression curve of front variance versus time: model A; (b) Regression curve of front variance versus time: model

ΔX_t of the dye front was defined as 30. The locations on the y axis could be found when the points on the x axis were specified at different times.

The quantity analyzed is the frontal width characterized by the variance of the fingers at the front. Its behaviour differs in the two media. If drawn against time the frontal variance of the fractal model grows in time proportionally to a power of time which is larger than 1. The exponent found from the experiment is exactly 2×0.79588 (0.79588 is the Hurst coefficient here). This process is called anomalous diffusion. The growth law of the variance in the finite correlation length medium is a linear function of time after the front has passed a number of correlation lengths. This is the expected normal diffusion result. As reference time zero, a time is taken at which the first fingers become visible in the window of the video recording. The resulting laws are shown as Fig. 5.

VI. CONCLUSIONS

On the basis of the cubic law, aperture meshes were generated corresponding to a transmissivity distribution. One model implemented non-vanishing long range correlation and fractal characteristics, while the other one implemented a finite correlation length. In the two physical models the movement of a dye tracer was visualized. A CCD-camera was used to record the tracer front movement. The images were digitized and analyzed with an algorithm for front tracking. The variance of the front line as a function of time was determined. When the velocity is slow enough and molecular diffusion is given sufficient time to mix the solute over the aperture, simple laws for the frontal behaviour can be found. Transport behaves quite differently in the two media. In model A, a power law expression was obtained, the power being twice the Hurst exponent. In model B, a linear function was found when the front had proceeded to about half the model's length. The behaviour in model B is the one expected, asymptotically, for length scales large compared to the correlation length of the medium.

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NOMENCLATURE

H	Hurst exponent
D	fractal dimension
D	Euclidean dimension
b	aperture of the fracture
J	hydraulic gradient
g	gravitational constant
v	kinematic viscosity
Q	total volumetric flow rate per unit
T_{fr}	transmissivity of single fracture
K_{fr}	hydraulic conductivity of single fracture

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應用碎形隨機布朗運動於裂隙傳輸之研究

蘇惠珍

國立交通大學土木工程學系

Wolfgang Kinzelbach

蘇黎士聯邦理工學院水利暨水資源研究所

摘　要

地下水於裂隙中之傳輸行爲往往因裂隙分佈的不易掌握使相關研究變得相當棘手且不確定，例如當有污染物洩漏時，瞭解其傳輸特性方能有效進行整治計劃。本研究應用具有碎形特性的碎形隨機布朗運動及應用地質統計所發展的兩種不同的亂數產生器產生兩組具有相同資料點及相同統計特性的裂隙寬度分佈，進而將這兩組裂隙分佈刻製於玻璃上並進行定水頭試驗，以數位式攝影機錄下染色液體在兩組模型中的傳輸分佈後進行影像分析。影像分析的結果映證了液體於不同裂隙分佈中亦有不同的傳輸特性：在具有碎形特性的裂隙中其流動軌跡之變異量爲傳輸時間的次方關係（次方法則；power law），且其冪次爲代表碎形特性之赫斯特指數(Hurst exponent)的兩倍；而在具有一定相關性長度的裂隙分佈中，其傳輸軌跡的變異量則在數倍相關性長度後呈現爲時間的線性關係。

關鍵詞：裂隙介質，碎形布朗運動，赫斯特指數，次方法則。