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European Journal of Operational Research 130 (2001) 653–664

EUROPEAN  
JOURNAL  
OF OPERATIONAL  
RESEARCH

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## Theory and Methodology

# Using fuzzy integral for evaluating subjectively perceived travel costs in a traffic assignment model

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Received 7 January 1999; accepted 4 November 1999

### Abstract

A good traffic assignment model can be a powerful tool to describe the characteristics of traffic behavior in a road network. The traffic assignment results often play an important role in transportation planning, e.g., an optimal and economical network design. Many traditional traffic assignment models rely heavily on the travel cost function established by Wardrop's principles; however, the Wardrop's travel cost function has been proven to be weak for explaining the uncertainty and interactivity of traffic among links. This study tries to construct a traffic assignment model that is different from Wardrop's in many aspects. First, it considers the cross-effect among the links. Second, a fuzzy travel cost function is established based on the possibility concept instead of precise calculation of traffic volumes. Third, the techniques of fuzzy measure and fuzzy integral are applied to calculate the subjectively perceived travel costs during traffic assignment. Furthermore, in order to validate our model, a detailed network with 22 nodes and 36 links is used to illustrate it. Study results show that our model explains more interactivity and uncertainty of traffic among links when compared with the traditional model of Wardrop's. © 2001 Elsevier Science B.V. All rights reserved.

*Keywords:* Network; Traffic assignment; Fuzzy integral; Fuzzy measure

### 1. Introduction

The traffic assignment skill, or the so-called network equilibrium technique, is popularly used

to analyze the traffic flows in equilibrium of a road-network (LeBlanc and Boyce, 1986; Nagurney, 1984, 1986, 1993). The traffic equilibrium is generally achieved by the following steps: first, the origin and destination (OD) of each driver for a desired trip in a road-network is identified; second, the optimal route (a combination of links) chosen by any driver for each trip is searched and recognized, where an optimal route is the route with the minimum perceived travel cost; and third, all the desired traffic flows (trips) are assigned to each link

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in a used route. The aforementioned three steps are iterated until the assigned traffic flow in each link converges to a stable state. The predicted convergent traffic flow in a road-network heavily affects the design of traffic management strategies, such as the road-network construction (Abdulaal and LeBlanc, 1979; LeBlanc, 1975; LeBlanc et al., 1975; Tzeng and Tsaur, 1997; Tzeng and Chen, 1998), traffic signal control, etc. Thus, it is very important to develop a good traffic assignment model in a transportation planning process (Roy, 1991; Chen, 1999).

Wardrop (1952) assumed that all drivers have the same perception of network costs and proposed Wardrop's first principle: each driver selects the route that minimizes his own trip cost. Traditionally, it is the basis of most equilibrium assignment procedures. Shortly after Wardrop, based on the assumption of separable link cost functions, Beckmann et al. (1956) constructed the initial equilibrium assignment model by mathematical programming. However, the strict assumption of separable link cost functions is far from the real traffic situation. Almost after 20 years, Dafermos (1980) proved that the equilibrium assignment model could also be resolved by mathematical programming under the condition that the Jacobian matrix of link cost functions is symmetric. Aashtiani (1979) used the nonlinear complementarity problem (NCP) to formulate the network equilibrium problem, whereas Smith (1979, 1983) and Dafermos (1980, 1984) introduced the concept of variational inequality problem (VIP) for the network equilibrium problem. The valuable work of Smith (1979, 1983) and Dafermos (1980, 1984) discarded the assumption suggested by Wardrop's so as to better describe the traffic characteristic in the real world. Nagurney (1984, 1986, 1993) later made much effort for solving the asymmetric Jacobian matrix of link cost functions. In Nagurney's work (1984), the link costs are not separable, and each link cost is related not only to the prevailing volume of the corresponding link, but is also affected by the volumes of other links, i.e., the cross-effects among links are considered. Thus, Nagurney's model obviously reflects greater reality compared

with other traditional models. But Nagurney (1984) did not provide a powerful and theoretic support for the construction of the link costs in his study.

The purpose of this paper is to construct the link costs by fuzzy approach in an effort to catch the vagueness in a real traffic situation. We also found that our formulation does provide a powerful and theoretic support for Nagurney's link costs with cross-effects among the links. Many link cost functions in related research are still established with a crisp concept, but the perceived cost of any link by a driver is obviously subjective and vague at all times. This also means the subjective vagueness inevitably compounds the recognition of travel cost by the drivers. Nevertheless, although sufficient traffic information is available to drivers, they are not able to perceive the exact and clear travel cost for any link. Such subjectively perceived costs can be appropriately formulated by fuzzy concept. Moreover, the fuzzy cost of each link is evaluated by fuzzy measure and fuzzy integral; the diagonalization method (Dafermos, 1980, 1984; Dafermos and Nagurney, 1984; Roy, 1991) is applied to find the convergent link flows. Finally, in order to validate our model, a detailed network with 22 nodes and 36 links is used as a numerical example. Study results show that our model explains more uncertainty of traffic among the links compared with the traditional models derived only by Wardrop's concept. Furthermore, our fuzzy model provides a general form for traffic assignment and can be expanded.

This paper is organized as follows. In Section 2, the problem characteristics are described, and the basic concepts of fuzzy measure, fuzzy integral and the diagonalization method for traffic assignment are illustrated. In Section 3, the fuzzy link cost and the traffic assignment model are constructed, and the resolution steps of our model are discussed. In Section 4, an assumed network is used to validate our model, and our model is compared with the non-fuzzy traffic assignment model of Wardrop. Finally, the conclusions and recommendations are presented in Section 5.

**2. Problem characteristics and basic concepts for fuzzy measure, fuzzy integral and diagonalization method**

In the well-known traffic assignment or traffic equilibrium problem, a fixed travel demand is prescribed for every OD pair of nodes in the transportation network. The equilibrium property is that, once determined, no traveler can decrease his travel cost by making a unilateral decision to change his route (Roy, 1991). The travel cost depends in a prescribed way on the traffic pattern; and cost functions construct the core of traffic assignment.

Why the traffic assignment model should be modified by a fuzzy approach? There are two main reasons: first, a specified link cost is often uncertainly affected by the costs of other links, e.g., the low travel efficiency of the down-stream link always hampers the travel speed of the up-stream link. Thus, the travel cost on a link does not depend solely upon the certain flow on that link. This also means that the travel cost of a specified link should consider the uncertain cross-effects of other links. Second, a driver always perceives the travel cost subjectively, e.g., although a driver feels more congested than the normal traffic in a link, the driver cannot precisely tell how exactly the link cost increases. Thus, even if sufficient traffic information is available to drivers, they are still not able to perceive the exact and precise travel cost for any link – this also leads to the fuzzy route choice for drivers.

According to the aforementioned vagueness in driver decision, using the relevant techniques for evaluating a fuzzy link cost seems to be appropriate. The techniques implemented for evaluating a fuzzy link cost in this study are: fuzzy measure, fuzzy integral and the diagonalization method for traffic assignment. These basic concepts of aforementioned techniques are illustrated as follows.

*2.1. Fuzzy measure*

Fuzzy measure  $g$  is a set function defined on the power set  $\beta(X)$  of  $X$ , and satisfies the following properties (Lee and LeeKwang, 1995):

$$g : \beta(X) \rightarrow [0, 1],$$

- (a)  $g(\phi) = 0, g(X) = 1,$
- (b) if  $A, B \in \beta(X)$  and  $A \subset B,$  then  $g(A) \leq g(B),$
- (c) if  $F_k \in \beta(X)$  for  $1 \leq k \leq \infty,$  and a sequence  $\{F_k\}$  is monotone (in the sense of inclusion), then  $\lim_{k \rightarrow \infty} g(F_k) = g(\lim_{k \rightarrow \infty} F_k).$

A  $\lambda$ -fuzzy measure  $g_\lambda$  is a fuzzy measure with the following property:

$$\forall A, B \in \beta(X), \quad A \cap B = \phi,$$

$$g_\lambda(A \cup B) = g_\lambda(A) + g_\lambda(B) + \lambda g_\lambda(A)g_\lambda(B)$$

for  $-1 < \lambda < \infty.$

If  $X = \{x_1, x_2, \dots, x_n\},$  fuzzy density  $g_i = g_i(\{x_i\})$  will have the following form:

$$g_\lambda(\{x_1, x_2, \dots, x_n\}) = \sum_{i=1}^n g_i + \lambda \sum_{i=1}^{n-1} \sum_{i2=i1+1}^n g_{i1}g_{i2}$$

$$+ \dots + \lambda^{n-1}g_1g_2, \dots, g_n$$

$$= \frac{1}{\lambda} \left| \prod_{i=1}^n (1 + \lambda g_i) - 1 \right|$$

for  $-1 < \lambda < \infty.$  (1)

If  $\lambda > 0,$  then  $g_\lambda(A \cup B) > g_\lambda(A) + g_\lambda(B).$  This means that the evaluation of the set  $\{A, B\}$  is larger than the sum of evaluations for sets  $\{A\}$  and  $\{B\}.$  If  $\lambda = 0,$  then  $g_\lambda(A \cup B) = g_\lambda(A) + g_\lambda(B).$  This means that the evaluation of the set  $\{A, B\}$  equals the sum of evaluations for sets  $\{A\}$  and  $\{B\}.$  If  $\lambda < 0,$  then  $g_\lambda(A \cup B) < g_\lambda(A) + g_\lambda(B).$  This means that the evaluation of the set  $\{A, B\}$  is smaller than the sum of evaluations for sets  $\{A\}$  and  $\{B\},$  i.e., the substitutive effect exists in  $\{A, B\}.$  If  $\lambda > 0,$  then  $g_\lambda(A \cup B) > g_\lambda(A) + g_\lambda(B).$  This implies that the evaluation of the set  $\{A, B\}$  is larger than the sum of evaluations for sets  $\{A\}$  and  $\{B\},$  i.e., the multiplicative effect exists in  $\{A, B\}.$  The fuzzy measure is often used with fuzzy integral for the purpose of aggregating information evaluation.

*2.2. Fuzzy integral*

Let  $h$  be a measurable function from  $X$  to  $[0, 1].$  Assuming that  $h(x_1) \geq h(x_2) \geq \dots \geq h(x_n),$  then

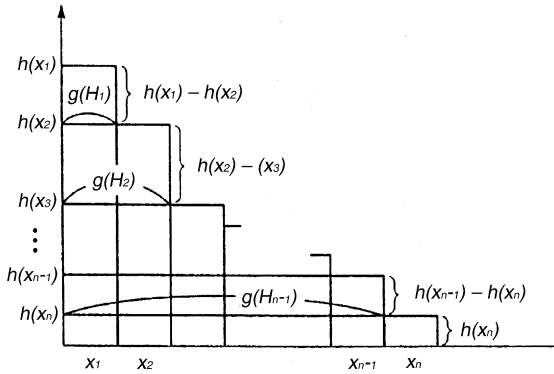


Fig. 1. The basic concept for fuzzy integral.

the fuzzy integral is defined as follows (Sugeno, 1974; Ishii and Sugeno, 1985):

$$\begin{aligned}
 (C) \int h dg &= h(x_n)g(H_n) + [h(x_{n-1}) - h(x_n)]g(H_{n-1}) \\
 &+ \dots + [h(x_1) - h(x_2)]g(H_1) \\
 &= h(x_n)[g(H_n) - g(H_{n-1})] + h(x_{n-1})[g(H_{n-1}) \\
 &- g(H_{n-2})] + \dots + h(x_1)g(H_1), \quad (2)
 \end{aligned}$$

where  $H_1 = \{x_1\}$ ,  $H_2 = \{x_1, x_2\}, \dots, H_n = \{x_1, x_2, \dots, x_n\} = X$ .

The basic concept of Eq. (2) can be illustrated as shown in Fig. 1.

Furthermore, if  $\lambda = 0$  and  $g_1 = g_2 = \dots = g_n$ , then  $h(x_1) \geq h(x_2) \geq \dots \geq h(x_n)$  is not a necessary condition.

### 2.3. Diagonalization method

An asymmetric traffic assignment problem can be rewritten as a VIP as follows (Dafermos, 1980, 1984; Dafermos and Nagurney, 1984; Roy, 1991):

Find  $f^* \in \Omega$ , such that

$$\begin{aligned}
 \sum_a C_a(f^*)(f_a - f_a^*) &\geq 0 \quad \forall f \in \Omega, \\
 \Omega &= \left\{ f : f_a - \sum_a \delta_{ar} h_r = 0 \quad \forall r; \right. \\
 &\left. \sum_r h_r - T_{xy} = 0 \quad \forall x, y; \quad h_r \geq 0 \quad \forall r \right\}, \\
 C_a(f) &= C_a(f_1, f_2, \dots, f_a, \dots) \quad \forall f \in \Omega, \quad (3)
 \end{aligned}$$

where  $f_a$  is the flow in link  $a$  of the detailed network during asymmetric traffic assignment;  $f_a^*$  the convergent flow in link  $a$ ;  $f^*$  the convergent flow in network;  $C_a$  the travel-cost of link  $a$  in the detailed network during asymmetric traffic assignment,  $C_a$  is only related to  $f \forall f \in \Omega$ ;  $h_r$  the traffic volume of route  $r$  in a detailed network during traffic assignment;  $\delta_{ar}$  is the dummy variable, if route  $r$  is used and link  $a$  is passed, then  $\delta_{ar} = 1$ ; otherwise  $\delta_{ar} = 0$  and  $T_{xy}$  is the traffic demand between node  $x$  and  $y$  in a physical network.

To solve Eq. (3) for convergent flows, the diagonalization method can be employed by the following steps (Roy, 1991):

- (a) Using all or nothing (AON) traffic assignment skill to find an initial flow set  $f_{a,it}$ , let  $it$  (iteration) = 0.
- (b) Let  $C_{a,it}(f) = C_{a,it}(f_{1,it}, f_{2,it}, \dots, f_{a,it}, \dots) \forall f \in \Omega$ .
- (c) Find an optimal  $\alpha_{it}$  by the bi-section method, such that

$$\text{Min}_{0 \leq \alpha_{it} \leq 1} \sum_a \int_0^{f_{a,it} + \alpha_{it}(f_{a,it+1} - f_{a,it})} C_{a,it}(f) df \quad \forall f \in \Omega.$$

- (d) If  $\text{Max}_a |f_{a,it+1} - f_{a,it}| / f_{a,it} \leq \epsilon, \forall f \in \Omega$ , then the convergent flows are obtained and the iteration stops; otherwise, go to (b) and  $it = it + 1$ .  $\epsilon$  is an acceptable error, which is given by the planner.

It is clear that if the flow set

$$f = \{f_{1,it}, f_{2,it}, \dots, f_{a,it}, \dots\} \quad \forall f \in \Omega,$$

is a fuzzy set, then  $f$  will be similar to the set  $X$  in Eq. (1). Nevertheless, the fuzzy measure and integral technique in Eqs. (1) and (2) will be applied as to evaluate such a fuzzy link cost:

$$C_{a,it}(f) = C_{a,it}(f_{1,it}, f_{2,it}, \dots, f_{a,it}, \dots) \quad \forall f \in \Omega.$$

In the next section, the fuzzy link cost function, the fuzzy measure, fuzzy integral and diagonalization method will be combined to form our asymmetric traffic assignment model based on subjectively perceived travel costs.

### 3. Model construction and resolution

To formulate the mathematical model for our traffic assignment with fuzzy travel costs, which

are subjectively perceived by drivers, a detailed network shown in Fig. 2 from Nagurney (1984) is used for illustration purposes. Assume a specified link cost, for example, the travel cost of link 1; it is affected by the volumes of link 1 and link 14. From the definition of fuzzy measure in Section 2.1, let  $ff_a$  denote the fuzzy set of travel costs in link  $a$  ( $a = 1, 2, \dots, 36$ ), and  $fc_a$  represent the free flow cost (a constant) for link  $a$  ( $a = 1, 2, \dots, 36$ ), our fuzzy link costs can be defined.

Let  $ff_1 = \{0.01f_1^2 + fc_1, f_{14}\}$ , then the fuzzy density,  $g_i = g_\lambda(\{f_i\})$  in Eq. (1) can be rewritten in the following form:

$$g_\lambda(\{0.01f_1^2 + fc_1, f_{14}\}) = \frac{1}{\lambda} \left| \prod_{i=1}^2 (1 + \lambda g_i) - 1 \right| \tag{4}$$

Furthermore, let  $g_\lambda(\{0.01f_1^2 + fc_1, f_{14}\}) = 1, \lambda = 0, g_1 = g_2 = \dots = g_5$  and  $h$  be a measurable function from  $f$  to  $[0,1]$ ; thus, the fuzzy travel cost evaluated by the fuzzy integral can be defined as follows:

$$\begin{aligned} C_1(f) &= (C) \int h dg \\ &= f_{14}[g(H_2) - g(H_1)] + (0.01f_1^2 + fc_1)g(H_1) \\ &= f_{14} \times (1 - 0.5) + (0.01f_1^2 + fc_1) \times 0.5 \\ &= 0.005f_1^2 + 0.5f_{14} + 0.5fc_1, \end{aligned} \tag{5}$$

where

$$H_1 = \{0.01f_1^2 + fc_1\}, H_2 = \{0.01f_1^2 + fc_1, f_{14}\} = ff_1.$$

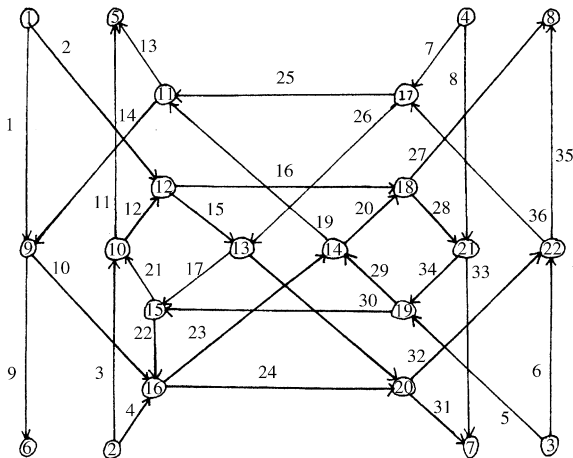


Fig. 2. Nagurney’s detailed network. (Source: Nagurney, 1984).

Thus, following similar steps in Eqs. (4) and (5), the fuzzy travel costs of all detailed links in Fig. 2 can be evaluated. Of course, each fuzzy set  $ff_a$  ( $a = 1, 2, \dots, 36$ ) is subjectively determined by the experienced planner. When  $\lambda$  changes, Eq. (5) forms a basis for the diagonalization method of Section 2.3, and each detailed link cost is continuously evaluated by the fuzzy integral for traffic assignment. Three cases:  $\lambda = 0, \lambda = 1$  and  $\lambda = -0.9999$  ( $\lambda \rightarrow -1$ ) are used in this study for comparison. The meaning of  $\lambda$  values will be discussed in the next section.

After inputting the available data, e.g., OD pattern of traffic demand of this model, which will determine convergent flows, the convergent flows are obtained through an asymmetric traffic assignment model. The fuzzy measure and integral are used to evaluate fuzzy travel costs of the detailed links. In the next section, a numerical example from Fig. 2 will be used to illustrate Eqs. (1)–(5). Furthermore, our model with two different  $\lambda$  values and a Wardrop’s model are compared and discussed.

#### 4. Numerical example and discussions

An assumed network in Fig. 2 is used to validate our model, and the traffic demand of Fig. 2 expressed in an OD matrix, which is shown in Table 1.

First, each fuzzy set of each detailed link cost in Fig. 2 is subjectively defined by the experienced planner. The fuzzy set of all detailed link costs with different  $\lambda$  values are given in Appendix A. Second, the diagonalization method of Section 2.3 is applied to find a set of convergent flows. Finally, the results of traffic assignments from our models

Table 1  
The assumed traffic demand of network in Fig. 2

From	To			
	5	6	7	8
1	0	235	230	220
2	240	0	235	225
3	230	220	0	235
4	235	225	240	0

and Wardrop's model ( $\lambda = -0.9999$ ) are compared in Table 2.

From Table 2 and our model of Section 3, we found that our modeling concept is very similar to Nagurney's (1984) for the asymmetric assignment, but our model explains more interactivity and uncertainty because of varying the  $\lambda$  values. The  $\lambda$  can be regarded as an adjustment factor in our model, and changing of  $\lambda$  value dominates the

characteristics of our traffic assignment model. In Section 3, if  $\lambda \rightarrow -1$ , then  $g(H_1) \rightarrow 1$ , and Eq. (5) equals  $0.005f_1^2 + 0.5fc_1$ . Thus, the link cost function is only related to the volume of link 1 when  $\lambda \rightarrow -1$ , this is a Wardrop's model. On the other hand, if  $\lambda \rightarrow \infty$ , then  $g(H_1) \rightarrow 0$ , and Eq. (5) equals  $0.5f_{14}$ . The link cost function is only related to the volume of link 14 when  $\lambda \rightarrow \infty$ , this case is an ultimate model of the cross-effects among de-

Table 2  
The traffic assignment results of our models<sup>a</sup>

Link		Assigned traffic volume		
To node	From node	Our model		Wardrop's model
		$\lambda = 1$	$\lambda = 0$	$\lambda = -0.9999 \rightarrow -1$
5	10	408	373	340
5	11	297	332	365
6	9	286	308	330
6	15	393	372	350
7	20	297	353	358
7	21	408	352	347
8	18	465	438	349
8	22	215	242	331
9	1	289	237	330
9	11	187	78	104
10	2	406	335	339
10	15	196	133	109
11	14	188	149	188
11	17	296	261	281
12	1	396	448	355
12	10	194	95	108
13	12	231	135	187
13	17	297	123	188
14	16	294	142	189
14	19	296	149	188
15	13	305	128	188
15	19	285	377	270
16	2	294	365	361
16	9	190	7	104
17	4	404	361	362
17	22	190	23	107
18	12	359	408	275
18	14	402	142	189
19	3	396	427	351
19	21	184	99	106
20	13	239	130	187
20	16	190	229	275
21	4	296	339	338
21	18	296	112	115
22	3	289	258	334
22	20	116	7	104

<sup>a</sup> The assigned traffic is rounded to its nearest integer.

tailed links. For reflecting greater reality,  $\lambda$  should be chosen within the interval  $(-1, \infty)$ ; therefore, the fuzzy link cost function will be established with the cross-effects among links and the main volume effect in a link.

Furthermore, it is clear that the output of our model will change as the  $\lambda$  value varies; thus, our model includes interactivity and explains the uncertainty. This also implies that our fuzzy formulation in this study expands the scope of traditional traffic assignment models, and  $\lambda$  can be regarded as an adjustment operator for developing different traffic assignment models in order to meet the special need of a transportation planner. The varying  $\lambda$  values reflect the different congestion degree of a road-network perceived by the transportation planner – this interactive approach is seldom considered in traditional traffic assignment models. For example, if a planner predicts a less-congested traffic situation, e.g., designing a new network, then a lower  $\lambda$  is suggested, e.g.,  $\lambda = -0.9999$  in Table 2. On the other hand, if a planner forecasts an over-congested traffic situation, e.g., simulating the reconstruction plan of a post-disaster network (Tzeng and Chen, 1998), then a higher  $\lambda$  is recommended because the cross-effects are notable, e.g.,  $\lambda = 1$  in Table 2. Thus, the transportation planner can use the same traffic demand and network to draw many traffic control plans by different levels of traffic congestion. This also leads to a more elastic and interactive strategy in traffic control. Moreover, we suggest that the planner may use genetic algorithms (Goldberg, 1989; Michalewicz, 1996) to find an optimal  $\lambda$  value to minimize the difference between the assigned traffic and the observed traffic.

## 5. Conclusions and recommendations

In the well-known traffic assignment or traffic equilibrium problem, the cost functions define the core of the problem. The traffic assignment technique is widely used in the resolution of many problems, e.g., the optimal network design, the optimal plan of traffic-signal control, the optimal schedule of network restoration, the

network economics, the optimal pricing strategy of network users, the dynamic network of intelligent transportation systems (ITS), . . . , etc. (Chen, 1999). The traditional cost functions proposed by Wardrop are too precise and too simple to catch the vagueness of driver's decision, because a specified link cost is often uncertainly affected by the changing costs of other links and subjectively perceived by a driver. Nevertheless, this study successfully combines the fuzzy measure and fuzzy integral to expand the traffic assignment model so as to describe the vagueness in the decision of drivers. The traffic assignment technique implemented by our model is very interactive and elastic. Our traffic assignment model evaluated by the fuzzy measure and integral theoretically not only supports Nagurney's idea but is also more appropriate for predicting the uncertain traffic in a real situation. Furthermore, the technique reasonably explains the fuzzy traffic with different  $\lambda$  values. The transportation planner may use our model to develop many scenarios of traffic prediction so as to promote the elasticity of transportation planning, e.g., a fuzzy network design.

With more precise data for our model in the near future, this study can be regarded as a basis of decision support system (DSS) on a geographic information system (GIS) for fuzzy transportation planning, e.g., the network investment. Furthermore, more topics may be taken into consideration as to make our model reflect greater reality, such as, how to construct an appropriate fuzzy set for each link cost, using genetic algorithms to find the optimal  $\lambda$  values of link costs (Goldberg, 1989; Michalewicz, 1996), and so on.

## Acknowledgements

The authors would like to deeply thank the anonymous referees for their valuable comments and Prof. Subhash C. Narula in Department of Management Science & Statistics, Virginia Commonwealth University (VCU) for correcting this paper.

**Appendix A**

1. The fuzzy flow set and equal individual fuzzy density for each link with  $\lambda = 0$ .

$$f_1 = \{0.01f_1^2 + 100, f_{14}\}, \quad g(H_2) = 1, \\ g(H_1) = 0.5,$$

$$f_2 = \{0.01f_2^2 + 50, f_{12}, f_{13}, f_{14}\}, \\ g(H_4) = 1, \quad g(H_3) = 0.75, \quad g(H_2) = 0.5, \\ g(H_1) = 0.25,$$

$$f_3 = \{0.01f_3^2 + 150, f_{10}, f_{21}, f_{22}\}, \\ g(H_4) = 1, \quad g(H_3) = 0.75, \quad g(H_2) = 0.5, \\ g(H_1) = 0.25,$$

$$f_4 = \{0.01f_4^2 + 50, f_{10}\}, \quad g(H_2) = 1, \\ g(H_1) = 0.5,$$

$$f_5 = \{0.01f_5^2 + 100, f_6, f_{29}, f_{30}, f_{34}\}, \\ g(H_5) = 1, \quad g(H_4) = 0.8, \quad g(H_3) = 0.6, \\ g(H_2) = 0.4, \quad g(H_1) = 0.2,$$

$$f_6 = \{0.01f_6^2 + 75, f_{32}, f_{33}, f_{34}\}, \\ g(H_4) = 1, \quad g(H_3) = 0.75, \quad g(H_2) = 0.5, \\ g(H_1) = 0.25,$$

$$f_7 = \{0.01f_7^2 + 75, f_{36}\}, \quad g(H_2) = 1, \\ g(H_1) = 0.5,$$

$$f_8 = \{0.01f_8^2 + 100, f_{27}, f_{28}, f_{36}\}, \\ g(H_4) = 1, \quad g(H_3) = 0.75, \quad g(H_2) = 0.5, \\ g(H_1) = 0.25,$$

$$f_9 = \{0.01f_9^2 + 100, f_{21}\}, \quad g(H_2) = 1, \\ g(H_1) = 0.5,$$

$$f_{10} = \{0.01f_{10}^2 + 50, f_3, f_4, f_{21}\}, \\ g(H_4) = 1, \quad g(H_3) = 0.75, \quad g(H_2) = 0.5, \\ g(H_1) = 0.25,$$

$$f_{11} = \{0.01f_{11}^2 + 50, f_2, f_{13}, f_{14}\}, \\ g(H_4) = 1, \quad g(H_3) = 0.75, \quad g(H_2) = 0.5, \\ g(H_1) = 0.25,$$

$$f_{12} = \{0.01f_{12}^2 + 50, f_2\}, \quad g(H_2) = 1, \\ g(H_1) = 0.5,$$

$$f_{13} = \{0.01f_{13}^2 + 50, f_{11}\}, \quad g(H_2) = 1, \\ g(H_1) = 0.5,$$

$$f_{14} = \{0.01f_{14}^2 + 50, f_1, f_2, f_{13}\}, \\ g(H_4) = 1, \quad g(H_3) = 0.75, \quad g(H_2) = 0.5, \\ g(H_1) = 0.25,$$

$$f_{15} = \{0.01f_{15}^2 + 50, f_{26}\}, \quad g(H_2) = 1, \\ g(H_1) = 0.5,$$

$$f_{16} = \{0.01f_{16}^2 + 50, f_{19}, f_{20}, f_{26}\}, \\ g(H_4) = 1, \quad g(H_3) = 0.75, \quad g(H_2) = 0.5, \\ g(H_1) = 0.25,$$

$$f_{17} = \{0.01f_{17}^2 + 50, f_{30}\}, \quad g(H_2) = 1, \\ g(H_1) = 0.5,$$

$$f_{18} = \{0.01f_{18}^2 + 50, f_{23}, f_{24}, f_{30}\}, \\ g(H_4) = 1, \quad g(H_3) = 0.75, \quad g(H_2) = 0.5, \\ g(H_1) = 0.25,$$

$$f_{19} = \{0.01f_{19}^2 + 50, f_{16}, f_{25}, f_{26}\}, \\ g(H_4) = 1, \quad g(H_3) = 0.75, \quad g(H_2) = 0.5, \\ g(H_1) = 0.25,$$



$$f_{20} = \{0.01f_{20}^2 + 50, f_{16}\}, \quad g(H_2) = 1, \\ g(H_1) = 0.5,$$

$$f_{21} = \{0.01f_{21}^2 + 50, f_3, f_9, f_{10}\}, \\ g(H_4) = 1, \quad g(H_3) = 0.75, \quad g(H_2) = 0.5, \\ g(H_1) = 0.25,$$

$$f_{22} = \{0.01f_{22}^2 + 50, f_3\}, \quad g(H_2) = 1, \\ g(H_1) = 0.5,$$

$$f_{23} = \{0.01f_{23}^2 + 100, f_{17}, f_{18}, f_{29}, f_{30}\}, \\ g(H_5) = 1, \quad g(H_4) = 0.8, \quad g(H_3) = 0.6, \\ g(H_2) = 0.4, \quad g(H_1) = 0.2,$$

$$f_{24} = \{0.01f_{24}^2 + 100, f_{18}\}, \quad g(H_2) = 1, \\ g(H_1) = 0.5,$$

$$f_{25} = \{0.01f_{25}^2 + 100, f_{19}\}, \quad g(H_2) = 1, \\ g(H_1) = 0.5,$$

$$f_{26} = \{0.01f_{26}^2 + 75, f_{15}, f_{16}, f_{19}, f_{20}\}, \\ g(H_5) = 1, \quad g(H_4) = 0.8, \quad g(H_3) = 0.6, \\ g(H_2) = 0.4, \quad g(H_1) = 0.2,$$

$$f_{27} = \{0.01f_{27}^2 + 75, f_8, f_{35}, f_{36}\}, \\ g(H_4) = 1, \quad g(H_3) = 0.75, \quad g(H_2) = 0.5, \\ g(H_1) = 0.25,$$

$$f_{28} = \{0.01f_{28}^2 + 50, f_8\}, \quad g(H_2) = 1, \\ g(H_1) = 0.5,$$

$$f_{29} = \{0.01f_{29}^2 + 50, f_{23}\}, \quad g(H_2) = 1, \\ g(H_1) = 0.5,$$

$$f_{30} = \{0.01 + 50, f_{17}, f_{18}, f_{23}\}, \\ g(H_4) = 1, \quad g(H_3) = 0.75, \quad g(H_2) = 0.5, \\ g(H_1) = 0.25,$$

$$f_{31} = \{0.01f_3^2 + 75, f_{33}\}, \quad g(H_2) = 1, \\ g(H_1) = 0.5,$$

$$f_{32} = \{0.01f_{32}^2 + 75, f_5, f_6, f_{33}\}, \\ g(H_4) = 1, \quad g(H_3) = 0.75, \quad g(H_2) = 0.5, \\ g(H_1) = 0.25,$$

$$f_{33} = \{0.01f_{33}^2 + 50, f_5, f_{31}, f_{32}\}, \\ g(H_4) = 1, \quad g(H_3) = 0.75, \quad g(H_2) = 0.5, \\ g(H_1) = 0.25,$$

$$f_{34} = \{0.01f_{34}^2 + 50, f_5\}, \quad g(H_2) = 1, \\ g(H_1) = 0.5,$$

$$f_{35} = \{0.01f_{35}^2 + 40, f_{27}\}, \quad g(H_2) = 1, \\ g(H_1) = 0.5,$$

$$f_{36} = \{0.01f_{36}^2 + 75, f_7, f_8, f_{27}\}, \\ g(H_4) = 1, \quad g(H_3) = 0.75, \quad g(H_2) = 0.5, \\ g(H_1) = 0.25.$$

2. The fuzzy flow set and equal individual fuzzy density for each link with  $\lambda = 1$ .

$$f_1 = \{0.01f_1^2 + 100, f_{14}\}, \\ g(H_2) = 1, \quad g(H_1) = 0.41,$$

$$f_2 = \{0.01f_2^2 + 50, f_{12}, f_{13}, f_{14}\}, \\ g(H_4) = 1, \quad g(H_3) = 0.68, \\ g(H_2) = 0.41, \quad g(H_1) = 0.19,$$

$$f_3 = \{0.01f_3^2 + 150, f_{10}, f_{21}, f_{22}\}, \\ g(H_4) = 1, \quad g(H_3) = 0.68, \\ g(H_2) = 0.41, \quad g(H_1) = 0.19,$$

$$f_4 = \{0.01f_4^2 + 50, f_{10}\}, \quad g(H_2) = 1, \\ g(H_1) = 0.41,$$

$$f_5 = \{0.01f_5^2 + 100, f_6, f_{29}, f_{30}, f_{34}\},$$

$$g(H_5) = 1, \quad g(H_4) = 0.74, \quad g(H_3) = 0.51,$$

$$g(H_2) = 0.32, \quad g(H_1) = 0.15,$$

$$f_6 = \{0.01f_6^2 + 75, f_{32}, f_{33}, f_{34}\},$$

$$g(H_4) = 1, \quad g(H_3) = 0.68,$$

$$g(H_2) = 0.41, \quad g(H_1) = 0.19,$$

$$f_7 = \{0.01f_7^2 + 75, f_{36}\}, \quad g(H_2) = 1,$$

$$g(H_1) = 0.41,$$

$$f_8 = \{0.01f_8^2 + 100, f_{27}, f_{28}, f_{36}\},$$

$$g(H_4) = 1, \quad g(H_3) = 0.68, \quad g(H_2) = 0.41,$$

$$g(H_1) = 0.19,$$

$$f_9 = \{0.01f_9^2 + 100, f_{21}\}, \quad g(H_2) = 1,$$

$$g(H_1) = 0.41,$$

$$f_{10} = \{0.01f_{10}^2 + 50, f_3, f_4, f_{21}\},$$

$$g(H_4) = 1, \quad g(H_3) = 0.68,$$

$$g(H_2) = 0.41, \quad g(H_1) = 0.19,$$

$$f_{11} = \{0.01f_{11}^2 + 50, f_2, f_{13}, f_{14}\},$$

$$g(H_4) = 1, \quad g(H_3) = 0.68,$$

$$g(H_2) = 0.41, \quad g(H_1) = 0.19,$$

$$f_{12} = \{0.01f_{12}^2 + 50, f_2\}, \quad g(H_2) = 1,$$

$$g(H_1) = 0.41,$$

$$f_{13} = \{0.01f_{13}^2 + 50, f_{11}\}, \quad g(H_2) = 1,$$

$$g(H_1) = 0.41,$$

$$f_{14} = \{0.01f_{14}^2 + 50, f_1, f_2, f_{13}\},$$

$$g(H_4) = 1, \quad g(H_3) = 0.68,$$

$$g(H_2) = 0.41, \quad g(H_1) = 0.19,$$

$$f_{15} = \{0.01f_{15}^2 + 50, f_{26}\}, \quad g(H_2) = 1,$$

$$g(H_1) = 0.41,$$

$$f_{16} = \{0.01f_{16}^2 + 50, f_{19}, f_{20}, f_{26}\},$$

$$g(H_4) = 1, \quad g(H_3) = 0.68,$$

$$g(H_2) = 0.41, \quad g(H_1) = 0.19,$$

$$f_{17} = \{0.01f_{17}^2 + 50, f_{30}\}, \quad g(H_2) = 1,$$

$$g(H_1) = 0.41,$$

$$f_{18} = \{0.01f_{18}^2 + 50, f_{23}, f_{24}, f_{30}\},$$

$$g(H_4) = 1, \quad g(H_3) = 0.68,$$

$$g(H_2) = 0.41, \quad g(H_1) = 0.19,$$

$$f_{19} = \{0.01f_{19}^2 + 50, f_{16}, f_{25}, f_{26}\},$$

$$g(H_4) = 1, \quad g(H_3) = 0.68,$$

$$g(H_2) = 0.41, \quad g(H_1) = 0.19,$$

$$f_{20} = \{0.01f_{20}^2 + 50, f_{16}\}, \quad g(H_2) = 1,$$

$$g(H_1) = 0.41,$$

$$f_{21} = \{0.01f_{21}^2 + 50, f_3, f_9, f_{10}\},$$

$$g(H_4) = 1, \quad g(H_3) = 0.68,$$

$$g(H_2) = 0.41, \quad g(H_1) = 0.19,$$

$$f_{22} = \{0.01f_{22}^2 + 50, f_3\}, \quad g(H_2) = 1,$$

$$g(H_1) = 0.41,$$

$$f_{23} = \{0.01f_{23}^2 + 100, f_{17}, f_{18}, f_{29}, f_{30}\},$$

$$g(H_5) = 1, \quad g(H_4) = 0.74g(H_3) = 0.51,$$

$$g(H_2) = 0.32, \quad g(H_1) = 0.15,$$

$$f_{24} = \{0.01f_{24}^2 + 100, f_{18}\}, \quad g(H_2) = 1,$$

$$g(H_1) = 0.41,$$

$$f_{25} = \{0.01f_{25}^2 + 100, f_{19}\}, \quad g(H_2) = 1,$$

$$g(H_1) = 0.41,$$

$$f_{26} = \{0.01f_{26}^2 + 75, f_{15}, f_{16}, f_{19}, f_{20}\},$$

$$g(H_5) = 1, \quad g(H_4) = 0.74, \quad g(H_3) = 0.51,$$

$$g(H_2) = 0.32, \quad g(H_1) = 0.15,$$

$$f_{27} = \{0.01f_{27}^2 + 75, f_8, f_{35}, f_{36}\},$$

$$g(H_4) = 1, \quad g(H_3) = 0.68,$$

$$g(H_2) = 0.41, \quad g(H_1) = 0.19,$$

$$f_{28} = \{0.01f_{28}^2 + 50, f_8\}, \quad g(H_2) = 1,$$

$$g(H_1) = 0.41,$$

$$f_{29} = \{0.01f_{29}^2 + 50, f_{23}\}, \quad g(H_2) = 1,$$

$$g(H_1) = 0.41,$$

$$f_{30} = \{0.01 + 50, f_{17}, f_{18}, f_{23}\},$$

$$g(H_4) = 1, \quad g(H_3) = 0.68, \quad g(H_2) = 0.41,$$

$$g(H_1) = 0.19,$$

$$f_{31} = \{0.01f_3^2 + 75, f_{33}\}, \quad g(H_2) = 1,$$

$$g(H_1) = 0.41,$$

$$f_{32} = \{0.01f_{32}^2 + 75, f_5, f_6, f_{33}\},$$

$$g(H_4) = 1, \quad g(H_3) = 0.68,$$

$$g(H_2) = 0.41, \quad g(H_1) = 0.19,$$

$$f_{33} = \{0.01f_{33}^2 + 50, f_5, f_{31}, f_{32}\},$$

$$g(H_4) = 1, \quad g(H_3) = 0.68,$$

$$g(H_2) = 0.41, \quad g(H_1) = 0.19,$$

$$f_{34} = \{0.01f_{34}^2 + 50, f_5\}, \quad g(H_2) = 1,$$

$$g(H_1) = 0.41,$$

$$f_{35} = \{0.01f_{35}^2 + 40, f_{27}\}, \quad g(H_2) = 1,$$

$$g(H_1) = 0.41,$$

$$f_{36} = \{0.01f_{36}^2 + 75, f_7, f_8, f_{27}\},$$

$$g(H_4) = 1, \quad g(H_3) = 0.68,$$

$$g(H_2) = 0.41, \quad g(H_1) = 0.19.$$

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