

General fuzzy piecewise regression analysis with automatic change-point detection

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Abstract

Yu et al. (Fuzzy Sets and Systems 105 (1999) 429) performed general piecewise necessity regression analysis based on linear programming (LP) to obtain the necessity area. Their method is the same as that according to data distribution, even if the data are irregular, practitioners must specify the number and the positions of change-points. However, as the sample size increases, the number of change-points increases and the piecewise linear interval model also becomes complex. Therefore, this work devises general fuzzy piecewise regression analysis with automatic change-point detection to simultaneously obtain the fuzzy regression model and the positions of change-points. Fuzzy piecewise possibility and necessity regression models are employed when the function behaves differently in different parts of the range of crisp input variables. As stated, the above problem can be formulated as a mixed-integer programming problem. The proposed fuzzy piecewise regression method has three advantages: (a) Previously specifying the number of change-points, then the positions of change-points and the fuzzy piecewise regression model are obtained simultaneously. (b) It is more robust than conventional fuzzy regression. The conventional regression is sensitive to outliers. In contrast, utilizing piecewise concept, the proposed method can deal with outliers by automatically segmenting the data. (c) By employing the mixed integer programming, the solution is the global optimal rather than local optimal solution. For illustrating more detail, two numerical examples are shown in this paper. By using the proposed method, the fuzzy piecewise regression model with detecting change-points can be derived simultaneously. © 2001 Elsevier Science B.V. All rights reserved.

Keywords: Fuzzy regression; Piecewise regression; Change-point; Possibility; Necessity

1. Introduction

In the early 1980s, Tanaka et al. [8] introduced a linear programming (LP) based regression method using a linear fuzzy model with symmetrical triangular fuzzy parameters. Then the possibility and necessity analyses were clearly defined [5]. Recently, Sakawa and Yano [2,3] have generalized the minimization, maximization and conjunction formulation that were developed by Tanaka [5] and Tanaka et al. [6].

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However, two weaknesses involving the fuzzy regression model have arisen. First, in possibility analysis, Redden and Woodall [1,9] demonstrated that Tanaka's methodologies are extremely sensitive to outliers and ignore certain information contained in the data. Furthermore, the fuzzy predictive interval tends to become fuzzier as more data are collected and has no operational definition or interpretation. Second, in necessity analysis, the necessity area cannot be obtained owing to the large variation in data [2,5,7,10]. Tanaka et al. [5] suggested a polynomial or nonlinear model while dealing with the above-mentioned problem. Since the distribution of data are probably segmented, Yu et al. [10] proposed the piecewise model to address the necessity problem. They proposed a general piecewise necessity regression analysis based on linear programming (LP) to obtain the necessity area. The characteristic of their method is that, according to data distribution even if the data are irregular, practitioners must specify the number and the positions of change-points. However, as the sample size increases, the number of change-points increases and the piecewise linear interval model also become complex. Therefore, how to effectively control the number of change-intervals and obtain a parsimonious regression model are two serious problems. Hence, we adopt the piecewise concept to implement fuzzy piecewise regression. This work proposes a general fuzzy piecewise regression analysis with automatic change-point detection to simultaneously obtain the fuzzy regression model and change-points.

Fuzzy piecewise possibility and necessity regression models are employed when the function behaves differently in different parts of the range of crisp input variables. Namely, the above problem can be formulated as a mixed integer programming problem for solving fuzzy piecewise regression. The proposed fuzzy piecewise regression method has three merits: (a) Specifying previously the number of change-points, then the positions of change-points and the fuzzy piecewise regression model are obtained simultaneously. (b) It is more robust than conventional fuzzy regression. The conventional fuzzy regression is sensitive to outliers. In contrast, based on a piecewise concept, the proposed method can deal with outliers by automatically segmenting the data. (c) By employing the mixed integer programming, the solution is a global optimal rather than a local optimal solution.

The rest of this paper is organized as follows. Section 2 reviews conventional fuzzy regression. Based on the presentation in Section 2, Section 3 proposes fuzzy piecewise regression analysis with piecewise linear interval model including univariate and multivariate analyses. Finally, for more detail, univariate and multivariate examples illustrate our approach. These two examples indicate a rather useful approach to treat suspicious outliers.

2. Interval arithmetic

A linear interval model with q independent variables is represented using interval parameters A_i as

$$Y(x_j) = A_0 + A_1x_{1j} + \cdots + A_qx_{qj}, \quad (1)$$

where $Y(x_j)$ is the predicted interval corresponding to the input vector x_j and j is the time j datum ($j = 1, 2, \dots, n$) and $x_j = (x_{1j}, x_{2j}, \dots, x_{qj})'$. In short, $\mathbf{x} = (x_1, x_2, \dots, x_q)'$ is a q -dimensional input vector. Throughout this work, closed intervals are denoted by upper case letters A, B . An interval is defined by an ordered pair in brackets as

$$A = [a_L, a_R] = \{a: a_L \leq a \leq a_R\}, \quad (2)$$

where a_L denotes the left limit and a_R denotes the right limit of A . Interval A is also denoted by its center and radius as

$$A = (a_c, a_w) = \{a: a_c - a_w \leq a \leq a_c + a_w\}, \quad (3)$$

where a_c denotes the center and a_w denotes the radius, i.e., half of the width of A . From Eqs. (2) and (3), the center and the radius of interval A can be calculated as

$$a_c = (a_R + a_L)/2, \tag{4}$$

$$a_w = (a_R - a_L)/2. \tag{5}$$

The following additions and multiplications are used herein

$$A + B = (a_c, a_w) + (b_c, b_w) = (a_c + b_c, a_w + b_w), \tag{6}$$

$$rA = r(a_c, a_w) = (ra_c, |r|a_w), \tag{7}$$

where r is the real number.

2.1. Linear interval model

The following linear model (1) is represented in detail:

$$Y(\mathbf{x}_j) = A_0 + A_1x_{1j} + \dots + A_qx_{qj} = (a_{0c}, a_{0w}) + (a_{1c}, a_{1w})x_{1j} + \dots + (a_{qc}, a_{qw})x_{qj} = (Y_c(\mathbf{x}_j), Y_w(\mathbf{x}_j)), \tag{8}$$

then

$$Y_c(\mathbf{x}_j) = a_{0c} + a_{1c}x_{1j} + \dots + a_{qc}x_{qj}, \tag{9}$$

$$Y_w(\mathbf{x}_j) = a_{0w} + a_{1w}|x_{1j}| + \dots + a_{qw}|x_{qj}|, \tag{10}$$

where $Y_c(\mathbf{x}_j)$ represents the center and $Y_w(\mathbf{x}_j)$ is the radius of the predicted interval $Y(\mathbf{x}_j)$.

2.2. Possibility regression analysis

$$\begin{aligned} Y^*(\mathbf{x}_j) &= A_0^* + A_1^*x_{1j} + \dots + A_q^*x_{qj} \\ &= (a_{0c}^*, a_{0w}^*) + (a_{1c}^*, a_{1w}^*)x_{1j} + \dots + (a_{qc}^*, a_{qw}^*)x_{qj} = (Y_c^*(x_q), Y_w^*(x_q)), \end{aligned} \tag{11}$$

which satisfies the following conditions:

$$Y^*(\mathbf{x}_j) \supseteq Y_j, \quad j = 1, 2, \dots, n, \tag{12}$$

where Y_j is the time j observation.

2.3. Minimization problem for interval-valued data

$$\text{Minimize} \quad Y_w^*(\mathbf{x}_1) + Y_w^*(\mathbf{x}_2) + \dots + Y_w^*(\mathbf{x}_n) \tag{13}$$

$$\text{Subject to} \quad Y^*(\mathbf{x}_j) \supseteq Y_j, \quad j = 1, 2, \dots, n, \tag{14}$$

$$a_{iw}^* \geq 0, \quad i = 0, 1, 2, \dots, q. \tag{15}$$

This LP problem is written as follows:

$$\text{Minimize } \sum_{j=1}^n (a_{0w}^* + a_{1w}^* |x_{1j}| + \dots + a_{qw}^* |x_{qj}|) \tag{16}$$

$$\text{Subject to } \left(a_{0c}^* + \sum_{i=1}^q a_{ic}^* x_{ij} \right) - \left(a_{0w}^* + \sum_{i=1}^q a_{iw}^* |x_{ij}| \right) \leq Y_{jL}, \quad j = 1, 2, \dots, n, \tag{17}$$

$$\left(a_{0c}^* + \sum_{i=1}^q a_{ic}^* x_{ij} \right) + \left(a_{0w}^* + \sum_{i=1}^q a_{iw}^* |x_{ij}| \right) \geq Y_{jR}, \quad j = 1, 2, \dots, n, \tag{18}$$

$$a_{iw}^* \geq 0, \quad i = 0, 1, 2, \dots, q. \tag{19}$$

The weakness in the above model is that it is sensitive to outliers. The model used to have a larger possibility than the system should have had and used to be warped and bent too much by various fluctuating data. The fuzzy predictive model tends to become fuzzier as more data are collected and has no operational definition or interpretation.

2.4. Necessity regression analysis

$$\begin{aligned} Y_*(x_j) &= A_{0*} + A_{1*}x_{1j} + \dots + A_{q*}x_{qj} \\ &= (a_{0c^*}, a_{0w^*}) + (a_{1c^*}, a_{1w^*})x_{1j} + \dots + (a_{qc^*}, a_{qw^*})x_{qj} = (Y_{c^*}(x_j), Y_{w^*}(x_j)), \end{aligned} \tag{20}$$

which satisfies the following conditions:

$$Y_*(x_j) \subseteq Y_j, \quad j = 1, 2, \dots, n. \tag{21}$$

2.5. Maximization problem for interval-valued data

$$\text{Maximize } Y_{w^*}(x_1) + Y_{w^*}(x_2) + \dots + Y_{w^*}(x_n) \tag{22}$$

$$\text{Subject to } Y_*(x_j) \subseteq Y_j, \quad j = 1, 2, \dots, n, \tag{23}$$

$$a_{iw^*} \geq 0, \quad i = 0, 1, 2, \dots, q. \tag{24}$$

This LP problem is written as follows:

$$\text{Maximize } \sum_{j=1}^n (a_{0w^*} + a_{1w^*} |x_{1j}| + \dots + a_{qw^*} |x_{qj}|) \tag{25}$$

$$\text{Subject to } \left(a_{0c^*} + \sum_{i=1}^q a_{ic^*} x_{ij} \right) - \left(a_{0w^*} + \sum_{i=1}^q a_{iw^*} |x_{ij}| \right) \geq Y_{jL}, \quad j = 1, 2, \dots, n, \tag{26}$$

$$\left(a_{0c^*} + \sum_{i=1}^q a_{ic^*} x_{ij} \right) + \left(a_{0w^*} + \sum_{i=1}^q a_{iw^*} |x_{ij}| \right) \leq Y_{jR}, \quad j = 1, 2, \dots, n, \tag{27}$$

$$a_{iw^*} \geq 0, \quad i = 0, 1, 2, \dots, q. \tag{28}$$

The above LP formulation of necessity has no feasible solution owing to large fluctuations of the given data. Therefore, a fuzzy piecewise regression was proposed by Yu et al. [10] for treating such problems. However,

the change-points must be given before employing the above method. For overcoming this restriction, we develop a general fuzzy piecewise regression analysis with automatic change-point detection.

3. Fuzzy piecewise regression analysis with automatic change-point detection

Change-points which are the joints of the pieces are quoted from conventional statistical piecewise regression [4]. This terminology is employed through the work. For an input variable x , $\{p_1, p_2, \dots, p_k\}$ are the values of variables x and are subject to an ordering constraint $p_1 < p_2 < \dots < p_k$, $k \leq n$. This work assumes that every datum is a change-point except p_k . Therefore, $k - 1$ change-point alternatives are in the initial possibility and necessity regression model. The suspected positions of change-points are $\{p_1, p_2, \dots, p_{k-1}\}$.

Definition 1. The point p_t is a change-point if $B(|b_c| > \delta, b_w)$. δ is a small positive value specified by the user. Otherwise, p_t is not a change-point if $B(|b_c| \leq \delta, b_w = 0)$.

If p_t is a change-point, then the operation of piecewise term is as follows:

$$(|x_j - p_t| + x_j - p_t)/2 = \begin{cases} x_j - p_t & \text{if } x_j > p_t, \\ 0 & \text{if } x_j \leq p_t, \end{cases} \tag{29}$$

where $t = 1, \dots, k - 1$.

An LP formulation is presented to determine the possibility area and necessity area of the piecewise linear interval model. For simplicity, we demonstrate the proposed method with the piecewise linear interval model by using an input variable x .

$$\begin{aligned} \hat{Y}(x_j) = & h(x_j) + B_1(|x_j - p_1| + x_j - p_1)/2 + B_2(|x_j - p_2| + x_j - p_2)/2 \\ & + B_3(|x_j - p_3| + x_j - p_3)/2 + \dots + B_{k-2}(|x_j - p_{k-2}| + x_j - p_{k-2})/2 \\ & + B_{k-1}(|x_j - p_{k-1}| + x_j - p_{k-1})/2, \end{aligned} \tag{30}$$

$$h(x_j) = A_0 + A_1 x_j.$$

Eq. (30) represents $Y^*(x_j)$ and $Y_*(x_j)$ in the initial possibility and necessity models, respectively. By prespecifying the number of change-points, the fuzzy regression model and the positions of the change-points are obtained simultaneously. The piecewise terms of LP formulation are as follows:

$$\begin{aligned} & B_1(|x_j - p_1| + x_j - p_1)/2 + B_2(|x_j - p_2| + x_j - p_2)/2 + B_3(|x_j - p_3| + x_j - p_3)/2 \\ & + \dots + B_{k-2}(|x_j - p_{k-2}| + x_j - p_{k-2})/2 + B_{k-1}(|x_j - p_{k-1}| + x_j - p_{k-1})/2, \quad j = 1, 2, \dots, n. \end{aligned} \tag{31}$$

The difference between Eqs. (30) and (11) is (31). This is the initial piecewise expression for the given data.

$$\begin{aligned} & B_1(|x_j - p_1| + x_j - p_1)/2 + B_2(|x_j - p_2| + x_j - p_2)/2 + B_3(|x_j - p_3| + x_j - p_3)/2 \\ & + \dots + B_{k-2}(|x_j - p_{k-2}| + x_j - p_{k-2})/2 + B_{k-1}(|x_j - p_{k-1}| + x_j - p_{k-1})/2 \\ & = \sum_{t=1}^{k-1} b_{lc}(|x_j - p_t| + x_j - p_t)/2 + \sum_{t=1}^{k-1} b_{lw}(|x_j - p_t| + x_j - p_t)/2, \quad j = 1, 2, \dots, n. \end{aligned}$$

The initial possibility formulation with q input variables is as follows:

$$Y^*(x_j) = A_0^* + \sum_{i=1}^q A_i^* x_{ij} + \sum_{i=1}^q \sum_{t=1}^{k_i-1} B_{it}^*(|x_{ij} - p_t| + x_{ij} - p_t)/2.$$

3.1. Possibility analysis with automatic change-point detection

$$\begin{aligned}
 \text{Minimize } \quad & zp = \sum_{j=1}^n \left[a_{0w}^* + \sum_{i=1}^q a_{iw}^* |x_{ij}| + \sum_{i=1}^q \sum_{t=1}^{k_i-1} b_{itw}^* (|x_{ij} - p_t| + x_{ij} - p_t) / 2 \right] \\
 \text{Subject to } \quad & \left[a_{0c}^* + \sum_{i=1}^q a_{ic}^* x_{ij} + \sum_{i=1}^q \sum_{t=1}^{k_i-1} b_{itc}^* (|x_{ij} - p_t| + x_{ij} - p_t) / 2 \right] \\
 & - \left[a_{0w}^* + \sum_{i=1}^q a_{iw}^* |x_{ij}| + \sum_{i=1}^q \sum_{t=1}^{k_i-1} b_{itw}^* (|x_{ij} - p_t| + x_{ij} - p_t) / 2 \right] \leq Y_{jL}, \\
 & \left[a_{0c}^* + \sum_{i=1}^q a_{ic}^* x_{ij} + \sum_{i=1}^q \sum_{t=1}^{k_i-1} b_{itc}^* (|x_{ij} - p_t| + x_{ij} - p_t) / 2 \right] \\
 & + \left[a_{0w}^* + \sum_{i=1}^q a_{iw}^* |x_{ij}| + \sum_{i=1}^q \sum_{t=1}^{k_i-1} b_{itw}^* (|x_{ij} - p_t| + x_{ij} - p_t) / 2 \right] \geq Y_{jR}, \quad j = 1, 2, \dots, n, \\
 & b_{itc}^* < Mu_{it} - 2\delta v_{it} + \delta + \varphi v_{it}, \tag{32} \\
 & b_{itc}^* > 2\delta u_{it} - Mv_{it} - \delta - \varphi u_{it}, \tag{33} \\
 & u_{it} + v_{it} \leq I_{it}, \tag{34} \\
 & \sum_{i=1}^q \sum_{t=1}^{k_i-1} I_{it} \leq C, \tag{35} \\
 & b_{itw}^* \leq MI_{it}, \quad i = 1, 2, \dots, q \text{ and } t = 1, 2, \dots, k_i - 1. \tag{36}
 \end{aligned}$$

The initial necessity formulation with input variables is as follows:

$$Y_*(x_j) = A_{0*} + \sum_{i=1}^q A_{i*} x_{ij} + \sum_{i=1}^q \sum_{t=1}^{k_i-1} B_{it*} (|x_{ij} - p_t| + x_{ij} - p_t) / 2, \quad j = 1, 2, \dots, n. \tag{37}$$

3.2. Necessity analysis with automatic change-point detection

$$\begin{aligned}
 \text{Minimize } \quad & zn = \sum_{j=1}^n \left[a_{0w^*} + \sum_{i=1}^q a_{iw^*} |x_{ij}| + \sum_{i=1}^q \sum_{t=1}^{k_i-1} b_{itw^*} (|x_{ij} - p_t| + x_{ij} - p_t) / 2 \right] \\
 \text{Subject to } \quad & \left[a_{0c^*} + \sum_{i=1}^q a_{ic^*} x_{ij} + \sum_{i=1}^q \sum_{t=1}^{k_i-1} b_{itc^*} (|x_{ij} - p_t| + x_{ij} - p_t) / 2 \right] \\
 & - \left[a_{0w^*} + \sum_{i=1}^q a_{iw^*} |x_{ij}| + \sum_{i=1}^q \sum_{t=1}^{k_i-1} b_{itw^*} (|x_{ij} - p_t| + x_{ij} - p_t) / 2 \right] \geq Y_{jL}, \\
 & \left[a_{0c^*} + \sum_{i=1}^q a_{ic^*} x_{ij} + \sum_{i=1}^q \sum_{t=1}^{k_i-1} b_{itc^*} (|x_{ij} - p_t| + x_{ij} - p_t) / 2 \right] \\
 & + \left[a_{0w^*} + \sum_{i=1}^q a_{iw^*} |x_{ij}| + \sum_{i=1}^q \sum_{t=1}^{k_i-1} b_{itw^*} (|x_{ij} - p_t| + x_{ij} - p_t) / 2 \right] \leq Y_{jR},
 \end{aligned}$$

$$b_{itc}^* < Mu_{it} - 2\delta v_{it} + \delta + \varphi v_{it}, \tag{38}$$

$$b_{itc}^* > 2\delta u_{it} - Mv_{it} - \delta - \varphi u_{it}, \tag{39}$$

$$u_{it} + v_{it} \leq I_{it}, \tag{40}$$

$$\sum_{i=1}^q \sum_{t=1}^{k_i-1} I_{it} \leq C, \tag{41}$$

$$b_{itw}^* \leq MI_{it}, \quad i = 1, 2, \dots, q \text{ and } t = 1, 2, \dots, k_i - 1, \tag{42}$$

where $u_{it}, v_{it}, I_{it} \in \{0, 1\}$, δ denotes a small positive value and $\delta > \varphi$; b_{itc}^* and b_{itw}^* are unconstrained in sign. M is a large positive constant and can be specified as $M \geq \max\{|b_{itc}^*| \text{ and } |b_{itw}^*| \mid i = 1, \dots, q \text{ and } t = 1, \dots, k_i - 1\}$. Eqs. (32)–(36), the same as (38)–(42) are checked to see whether they will satisfy the definition of change-point in Definition 1. For instance, in possibility regression analysis:

- (i) If $b_{itc}^* > \delta$, then $u_{it} = 1$ and $v_{it} = 0$ (from Eqs. (33) and (34));
 - (ii) If $b_{itc}^* < -\delta$, then $u_{it} = 0$ and $v_{it} = 1$ (from Eqs. (32) and (34));
 - (iii) If $|b_{itc}^*| \leq \delta$, then $u_{it} = v_{it} = 0$ (from Eqs. (32)–(34));
 - (iv) $\sum_{i=1}^q \sum_{t=1}^{k_i-1} I_{it} \leq C$ limits the number of change-points. C is a reasonable measurement of the number of change-points that depends on data distribution (from Eq. (35));
 - (v) $b_{itw}^* \leq MI_{it}$ means if $b_{itc}^* = 0$ then $b_{itw}^* = 0$ (from Eq. (36)).
- (i)–(iii) demonstrate that if $|b_{itc}^*| > \delta$, i.e., a change-point occurs in the time it position, then $u_{it} + v_{it} = I_{it} = 1$, and otherwise $u_{it} + v_{it} = I_{it} = 0$.

Practitioners previously specify the number of change-points (C). Then, he can obtain the position of change interval and the fuzzy piecewise regression model simultaneously. Based on piecewise characteristics, in possibility analysis, our methodology is insensitive to outliers. Furthermore, the fuzzy predicted interval does not tend to become fuzzier as more data are collected. In necessity analysis, regardless of regular or irregular data, the case that probably does not obtain necessity area need not be considered by practitioners. All that the practitioners have to do is to specify the conceivable number of change-points (C).

The detected change-points or their neighborhood can be viewed as outliers under some circumstances. The proposed piecewise method is more robust than conventional fuzzy regression. By employing the mixed integer programming, this method can give the global optimal solution.

4. Numerical example

The data used herein provide an example that makes a linear necessity model of Tanaka–Ishibuchi’s method [7] not obtainable. By applying the piecewise concept, the possibility and necessity area represented by a piecewise linear interval model can be obtained rather than a non-linear interval regression one. By adjusting the under terms of $h(x)$, the proposed method can also be presented as the non-linear interval model. In the following example, for simplicity, $h(x)$ is a linear interval function. If the property of data is non-linear, we can adjust the order term of $h(x)$ and, then, $Y^*(x)$ and $Y_*(x)$ will turn into a non-linear piecewise interval regression model.

Example 1. Let us consider the following interval of valued data

$$\{(x; y)\} = \{(3; [12, 17]), (6; [10, 13]), (9; [13, 18]), (12; [14, 18]), (15; [19, 24]), (18; [16, 19])\}.$$

Sample size is 6. The distribution of x range from 3 to 18. We use $p_1 = 3, p_2 = 6, p_3 = 9, p_4 = 12, p_5 = 15$ as change-points. The initial models are as follows:

The initial possibility model is as follows:

$$\begin{aligned}
 Y^*(\mathbf{x}_j) &= (a_{0c}^*, a_{0w}^*) + (a_{1c}^*, a_{1w}^*)x_j + (b_{1c}^*, b_{1w}^*)(|x_j - 3| + x_j - 3)/2 \\
 &\quad + (b_{2c}^*, b_{2w}^*)(|x_j - 6| + x_j - 6)/2 + (b_{3c}^*, b_{3w}^*)(|x_j - 9| + x_j - 9)/2 \\
 &\quad + (b_{4c}^*, b_{4w}^*)(|x_j - 12| + x_j - 12)/2 + (b_{5c}^*, b_{5w}^*)(|x_j - 15| + x_j - 15)/2.
 \end{aligned}$$

The initial necessity model is as follows:

$$\begin{aligned}
 Y_*(\mathbf{x}_j) &= (a_{0c}^*, a_{0w}^*) + (a_{1c}^*, a_{1w}^*)x_j + (b_{1c}^*, b_{1w}^*)(|x_j - 3| + x_j - 3)/2 \\
 &\quad + (b_{2c}^*, b_{2w}^*)(|x_j - 6| + x_j - 6)/2 + (b_{3c}^*, b_{3w}^*)(|x_j - 9| + x_j - 9)/2 \\
 &\quad + (b_{4c}^*, b_{4w}^*)(|x_j - 12| + x_j - 12)/2 + (b_{5c}^*, b_{5w}^*)(|x_j - 15| + x_j - 15)/2.
 \end{aligned}$$

Let $M = 1000$, $\delta = 0.00000001$, $\varphi = 0.000000001$, $C = 2$, solving this program by LINDO. The following piecewise linear models are subsequently obtained,

$$\begin{aligned}
 Y^*(\mathbf{x}_j) &= (16.5, 2.5) + (-0.667, 0)x_j + (1.5, 0.167)(|x_j - 6| + x_j - 6)/2 \\
 &\quad + (-0.667, 0)(|x_j - 15| + x_j - 15)/2,
 \end{aligned}$$

$$\begin{aligned}
 Y_*(\mathbf{x}_j) &= (15, 1) + (-0.66, 0)x_j + (1.667, 0)(|x_j - 6| + x_j - 6)/2 \\
 &\quad + (-1.833, 0.167)(|x_j - 15| + x_j - 15)/2
 \end{aligned}$$

which satisfies the conditions:

$$Y_*(\mathbf{x}_j) \subseteq Y_j \subseteq Y^*(\mathbf{x}_j), \quad j = 1, 2, \dots, 6.$$

The comparison of observed output with predicted output in linear piecewise model is shown in Table 1, Figs. 1 and 2. Figs. 1 and 2 depict that change intervals of the possibility and necessity model appear at the same positions $x_j = 6$ and 15. Fig. 2 illustrates that the necessity model can be found without regard to the data distribution. According to the data physical phenomenon, practitioners tune the number of change

Table 1
The comparison of observed output with predicted output

Observed output		Predicted output	
x	y	Possibility	Necessity
3	[12, 17]	[12, 17]	[12, 14]
6	[10, 13]	[10, 15]	[10, 12]
9	[13, 18]	[12, 18]	[13, 15]
12	[14, 18]	[14, 21]	[16, 18]
15	[19, 24]	[16, 24]	[19, 21]
18	[16, 19]	[16, 25]	[16, 19]

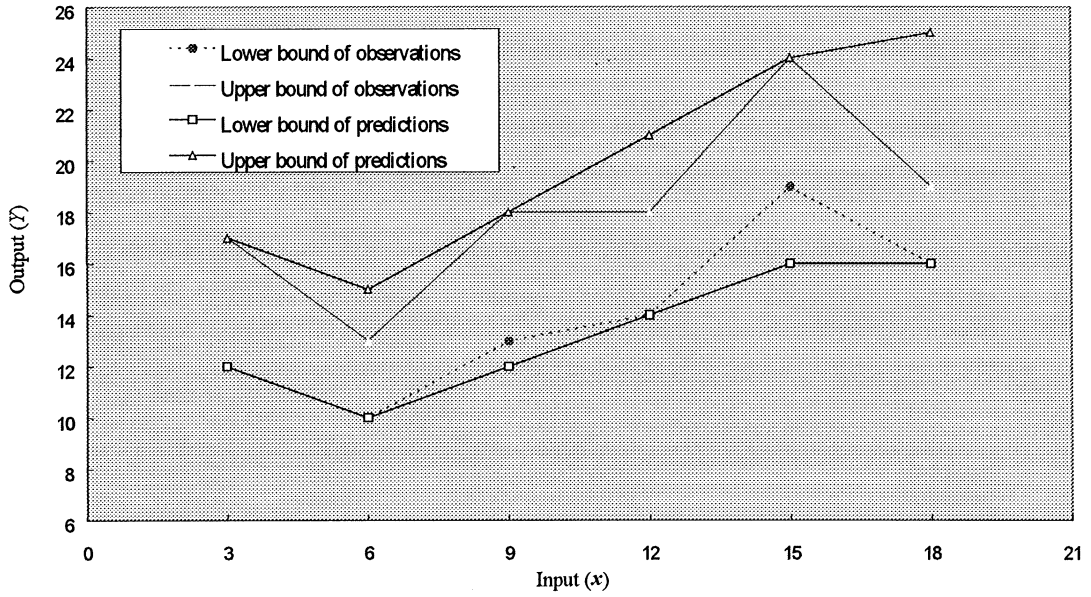


Fig. 1. Possibility analysis compared with raw data.

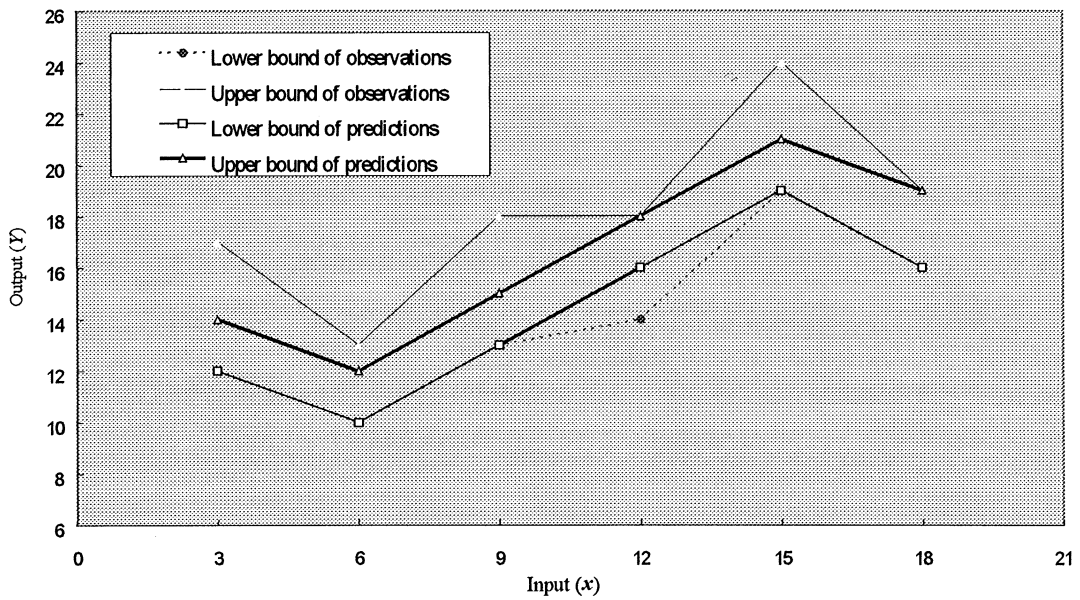


Fig. 2. Necessity analysis compared with raw data.

intervals. If the data distribution is nonlinear, practitioners can modify $h(\mathbf{x})$ as a nonlinear function to satisfy the practical requirements.

Example 2. Multivariate fuzzy data are from Tanaka [5]. In this example, we merely demonstrate how to obtain the necessity model.

$$\begin{aligned} \{(x_1; x_2; x_3; y)\} = & \{(3; 5; 9; [96, 42]), (14; 8; 3; [120, 47]), (7; 1; 4; [52, 33]) \\ & (11; 7; 3; [106, 45]), (7; 12; 15; [189, 79]), (8; 15; 10; [194, 65]), \\ & (3; 9; 6; [107, 42]), (12; 15; 11; [216, 78]), (10; 5; 8; [108, 52]), \\ & (9; 7; 4; [103, 44])\}, \quad n = 10. \end{aligned}$$

The distributions of variable x_1 , x_2 and x_3 have 8 different crisp values $\{3, 7, 8, 9, 10, 11, 12, 14\}$, 7 crisp values $\{1, 5, 7, 8, 9, 12, 15\}$ and 8 crisp values $\{3, 4, 6, 8, 9, 10, 11, 15\}$, respectively. In the extreme case, variable x_1 , x_2 and x_3 have 7 change-points $\{3, 7, 8, 9, 10, 11, 12\}$, 6 change-points $\{1, 5, 7, 8, 9, 12\}$ and 7 change-points $\{3, 4, 6, 8, 9, 10, 11\}$, respectively. There are 20 change-points in the beginning. If we cannot control the number of change-point, the multivariate necessity model with 10 change-points is obtained by [10].

$$\begin{aligned} Y_*(\mathbf{x}_j) = & (0, 2.35) + (1.87, 0.6)x_{1j} + (4.08, 0)x_{2j} + (6.9, 0.7)x_{3j} \\ & + (0, 0.21)(|x_{1j} - 7| + x_{1j} - 7) + (1.75, 0)(|x_{1j} - 8| + x_{1j} - 8) + (-1.36, 0)(|x_{1j} - 9| + x_{1j} - 9) \\ & + (-3.57, 0.3)(|x_{1j} - 12| + x_{1j} - 12) + (0.21, 0)(|x_{2j} - 5| + x_{2j} - 5) \\ & + (2.61, 2.02)(|x_{2j} - 12| + x_{2j} - 12) + (-2.28, 0)(|x_{3j} - 3| + x_{3j} - 3) \\ & + (-1.59, 0.87)(|x_{3j} - 4| + x_{3j} - 4) + (0, 0.38)(|x_{3j} - 6| + x_{3j} - 6) \\ & + (2.99, 0)(|x_{3j} - 11| + x_{3j} - 11). \end{aligned} \tag{43}$$

The total vagueness of Eq. (43) $zn_1 = 378.7$ with 10 change-points.

From the raw data distribution and Eq. (43), specify $C = 3$ i.e. the number of change-points is less than or equal to 3. Employing the proposed method, the necessity model with 3 change-points is as following:

$$\begin{aligned} Y_*(\mathbf{x}_j) = & (10.9, 0) + (2.26, 0.66)x_{1j} + (4.4, 3.1)x_{2j} + (2.82, 0.44)x_{3j} \\ & + (-1.51, 0.56)(|x_{1j} - 12| + x_{1j} - 12) + (0.81, 1.2)(|x_{2j} - 12| + x_{2j} - 12) \\ & + (1.31, 0.73)(|x_{3j} - 6| + x_{3j} - 6). \end{aligned} \tag{44}$$

The total vagueness of Eq. (44) $zn_2 = 365.8$ with three change-points.

The concept is assuming every datum to be a change-point, except for the last one which is like interpolation. However, C in Eq. (41) is to control the number of change-points and to reduce the redundant change-points. From Table 2, Eq. (44) with seven less change-points than Eq. (43) achieves a promising result which approximates the result of Eq. (44), and the total vagueness zn_2 is only slightly smaller than zn_1 . Hence, for an effective and parsimonious form, considering that the number of change-points must depend on the raw data distribution and regarding the number of change-points, fewer is better [10]. Besides, the proposed method brings about future research in [10] and searches the change-points automatically as well.

Table 2
The comparison of possibility models with 3 and 10 change-points, respectively

Observed output	Prediction of (43) with 10 change-points [10] $zn_1 = 378.7$	Prediction of (44) with 3 change-points $zn_2 = 373.12$
1. [96, 42]	[90.2, 48]	[92.8, 45.6]
2. [120, 47]	[120.0, 47]	[120.0, 47.0]
3. [52, 33]	[52.0, 33]	[52.0, 33.0]
4. [106, 45]	[106.0, 45]	[105.6, 45.0]
5. [189, 79]	[189.0, 79]	[189.0, 79.0]
6. [194, 64]	[194.0, 64]	[194.0, 68.6]
7. [107, 42]	[107.0, 42]	[107.0, 42.0]
8. [216, 78]	[216.0, 78]	[211.0, 78.0]
9. [108, 52]	[108.0, 52]	[108.0, 52.8]
10. [103, 44]	[101.8, 44]	[103.0, 44.2]

5. Conclusions and remarks

This work proposes general fuzzy piecewise regression analysis with automatic change-point detection. Fuzzy piecewise possibility and necessity regression models are utilized when the function behaves differently in different parts of the range of crisp input variables. By employing this method, users can obtain the positions of change-points and the fuzzy regression model simultaneously. Based on piecewise characteristics, practitioners need not consider the cases where the fuzzy predictive interval tends to become fuzzier as more data are collected in possibility model and the necessity area is not obtained. In Section 4, Example 1 demonstrates how to solve a simple fuzzy piecewise regression problem. In addition, the fewer number of change-points the better, as shown in Example 2. As the sample size increases, users still manipulate the number of change-points that depend on structure change of data distribution. Moreover, outliers generally occur at the positions or in the neighborhood of the change-points.

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