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# $H^\infty$ control for nonlinear affine systems: a chain-scattering matrix description approach

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SUMMARY

This paper combines an alternative chain-scattering matrix description with $(J, J')$-lossless and a class of conjugate $(-J, -J')$-lossless systems to design a family of nonlinear $H^\infty$ output feedback controllers. The present systems introduce a new chain-scattering setting, which not only offers a clearer expression for the solving process of the nonlinear $H^\infty$ control problem but also removes the fictitious signals introduced by the traditional chain-scattering approach. The intricate nonlinear affine control problem thus can be transformed into a simple lossless network and is easy to deal with in a network-theory context. The relationship among these $(J, J')$ systems, $L_2$-gain, and Hamilton–Jacobi equations is also given. Block diagrams are used to illustrate the central theme. Copyright © 2001 John Wiley & Sons, Ltd.

KEY WORDS: nonlinear systems; $L_2$-gain; hamilton–Jacobi equations; state-space method

## 1. INTRODUCTION

Since Zames [1] proposed the concept of sensitivity minimization in the $H^\infty$ domain, many researchers have made valuable contributions to the study of the $H^\infty$ domain. Parameterization of all linear $H^\infty$-(sub) optimal output feedback controllers were given by Glover *et al.* [2]. Green *et al.* [3] and Kimura [4] then offered an alternative method by using $J$-spectral or $(J, J')$-lossless factorization. Also, Kimura [5] and Ball *et al.* [6] developed a fictitious signals method to solve the linear 4-block control problem. Hong and Teng [7] then developed a new method which both matched the famous results of Glover *et al.* [2] and removed the fictitious signals.

As in the extension of linear $H^\infty$ control theory to nonlinear settings, the local disturbance attenuation with internal stability was first studied by Ball and Helton [8], Başar and Bernhard [9], and Van der Schaft [10, 11]. Van der Schaft used the notion of dissipativity in a nonlinear

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system to show that the Hamilton–Jacobi equation is the nonlinear version of the Riccati equation considered in linear systems which yields the solution of a nonlinear $H^\infty$-state feedback control problem. As for measurement feedback, Ball *et al.* [12] established the necessary conditions for the existence of a solution. Moreover, Isidori [4, 13] summarized the notion of the dissipative system and the theory of differential games to define sufficient conditions based on two Hamilton–Jacobi equations.

An alternative approach using $(J, J')$-inner-outer factorization or the chain-scattering approach, Helton and James [14, 26], Baramov and Kimura [15], and Ball *et al.* [16, 17] solved the so-called 2-block case. Following this approach, Pavel and Fairman [18] introduced a nonlinear version of the fictitious signals method to solve the general 4-block case, which reduced the 4-block case to a simple 2-block case. However, one must then be careful to ignore the fictitious signals when seeking the solution for the original problem.

The present paper aims to reformulate the earlier results by combining the traditional $(J, J')$-lossless system with a class of nonlinear conjugate $(-J, -J')$-lossless system to solve the 4-block nonlinear $H^\infty$-output feedback control problem. This new chain-scattering matrix description extends the concept of Hong and Teng's [7] to the nonlinear setting and discrads the fictitious signals proposed recently by Pavel and Fairman [18]. Therefore, the controller thus obtained is quite straight-forward and provides deeper insight into the synthesis of the controllers.

In Section 2, we briefly state the standard nonlinear affine $H^\infty$ control problem. Section 3 proposes the relationships among the Hamiltonian system, $(J, J')$-lossless, conjugate $(J, J')$-lossless, and conjugate $(-J, -J')$-lossless matrices. The main results and the relation between the nonlinear $H^\infty$ control problem and the chain-scattering matrix description are presented in Section 4.

## 2. NOTATIONS AND PRELIMINARY INFORMATION

$\mathscr{R}$ denotes a real number
$\mathbb{R}^n$ denotes $n$-dimensional Euclidean space
$\mathrm{RL}^\infty$ the set of proper real rational function matrices with no poles on the $jw$ axis.
dom(Ric) denotes the Hamiltonian matrix with no eigenvalues on the $jw$-axis.
$G^\sim(s)$ denotes $G^{\mathrm{T}}(-s)$ and $G^*(s)$ denotes $G^{\mathrm{T}}(\bar{s})$.
The chain-scattering matrix description is abbreviated as CSMD.

Consider a smooth nonlinear affine system $\hat{P}$ given by

$$\hat{P} := \begin{cases} \dot{x} = A(x) + B(x)u \\ y = C(x) + D(x)u \end{cases}$$

with $y \in \mathbb{R}^{(p_1+p_2)}$, $u \in \mathbb{R}^{(m_1+m_2)}$, and $x = (x_1, x_2, \ldots, x_n)$ are local co-ordinates for a smooth state-space manifold $M$ defined in a neighbourhood $\Omega$ of the origin. Also, we assume $x = 0$ is an equilibrium point and $C(0) = 0$.

Basic properties for this system which will be used in the present paper are stated in the following definitions (see Pavel *et al.* [18]).

*Definition 1*

The system $\hat{P}$ is said to be *stabilizable* if there exists a continuous function $F(x)$ with $F(0) = 0$ such that $A(x) + B(x)F(x)$ is asymptotically stable.

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*Definition 2*
The system $\hat{P}$ is said to be *zero-state detectable* if for all $x \in \mathbb{R}^n$, $u = 0$, and $y = 0$, $\forall t \geqslant 0$, implies $x \to 0$ as $t \to \infty$.

*Definition 3*
The *Zero dynamics* of system $\hat{P}$ are defined as the set of state trajectories $\{(x(t)\}$ generated by the set of input $\mathscr{U}$ and initial conditions $\mathscr{X}_0$ such that the output is identically null, i.e., $\dot{x} = A(x) + B(x)u$, with $0 = C(x) + D(x)u$, $\forall t \geqslant 0$, $u \in \mathscr{U}$, $x(0) \in \mathscr{X}_0$.

*Definition 4*
The system $\hat{P}$ is said to have $L_2$-gain less than or equal to $\gamma$ if its zero state response satisfies

$$\int_0^{\mathrm{T}} \|y(t)\|^2 \,\mathrm{d}t \leqslant \gamma^2 \int_0^{\mathrm{T}} \|u(t)\|^2 \,\mathrm{d}t \quad \textit{with } T > 0$$

The following definition of the right-coprime factorization can be found in Pavel *et al.* [18] or Scherpen *et al.* [19]. Which is well known in linear case. As this is one of the key ideas of this paper, for completeness, we rewrite it below.

*Definition 5*
A right-coprime factorization of system $\hat{P}$, with $(A(x), B(x))$ being stabilizable, is given by two systems

$$N := \begin{cases} \dot{x} = A(x) + B(x)F(x) + B(x)U_a(x)\zeta \\ y = C(x) + D(x)F(x) + D(x)U_a(x)\zeta \end{cases} \quad M := \begin{cases} \dot{x} = A(x) + B(x)F(x) + B(x)U_a(x)\zeta \\ y = F(x) + U_a(x)\zeta \end{cases}$$

with $U_a(x)$ is invertible and $M^{-1}$ is the inverse system of $M$ such that:

(i) for every initial condition of $\hat{P}$ there exist initial conditions for $N$ and for $M^{-1}$ such that the input–output behaviour of $\hat{P}$ equals the input–output behaviour of $N \circ M^{-1}$, where $N \circ M^{-1}$ denotes the system obtained by the series interconnection of $M^{-1}$ followed by $N$,
(ii) $A(x) + B(x)F(x)$ is asymptotically stable;
(iii) $N$ and $M$ are right coprime, i.e. the zero dynamics of the system $\left[\begin{smallmatrix} N \\ M \end{smallmatrix}\right]$ is asymptotically stable.

*The standard nonlinear affine $H^{\infty}$ control problem*

Consider the following smooth ($C^{\infty}$) nonlinear affine $H^{\infty}$ framework

$$P := \begin{cases} \dot{x} = A(x) + B_1(x)w + B_2(x)u \\ z = C_1(x) + D_{12}(x)u \\ y = C_2(x) + D_{21}(x)w \end{cases} \tag{1}$$

where $z(t) \in \mathbb{R}^{p_1}$, $y(t) \in \mathbb{R}^{p_2}$, $w(t) \in \mathbb{R}^{m_1}$, and $u(t) \in \mathbb{R}^{m_2}$ are the error, observation, disturbance, and control input, respectively. The states $x = (x_1, x_2, \ldots, x_n)$ are local co-ordinates for a state-space manifold $M$ defined in a neighbourhood $\Omega$ of the origin in $\mathbb{R}^n$. Assume $x = 0$, an equilibrium point, also $A(0) = 0$, $C_1(0) = 0$, and $C_2(0) = 0$. Furthermore, as in the general 4-block nonlinear $H^{\infty}$-control problem, the inequalities $m_1 > p_2$ and $p_1 > m_2$ must hold.

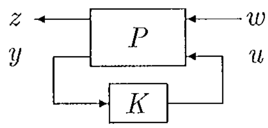



Figure 1.

The suboptimal nonlinear affine $H^\infty$ control problem is then modelled so as to choose a controller $K$ which connects the observation vector $y$ to $u$ so that $K$ locally, asymptotically stabilizes the closed-loop system in a neighbourhood $\Omega$ of the origin with internal stability. Furthermore, the closed-loop system with a local $L_2$-gain is less than or equal to a prescibed number $\gamma$.

Figure 1 shows a general set-up for the nonlinear affine $H^\infty$ control system.

For simplicity and yet without any loss of generality of the derivations in subsequent sections, let $\gamma = 1$ and take the following assumptions for the 4-block nonlinear affine $H^\infty$ control problem.

*Assumptions:*

A1. $(A(x), B_2(x))$ is locally stabilizable and $(C_2(x), A(x))$ is locally detectable in a neighbourhood $\Omega$ of the origin.

A2. $D_{12}^{\mathrm{T}}(x)D_{12}(x) = I_{m_2}$ and $D_{21}(x)D_{21}^{\mathrm{T}}(x) = I_{p_2}$.

$D_{12}^{\mathrm{T}}(x)C_1(x) = 0$ and $B_1(x)D_{21}^{\mathrm{T}}(x) = 0$

A3. Any bounded trajectory $x(t)$ of system $\dot{x}(t) = A(x(t)) + B_2(x(t))u(t)$, satisfying $C_1(x(t)) + D_{12}(x(t))u(t) = 0$, for all $t \geqslant 0$, in such that $\lim_{t\to\infty} x(t) = 0$.

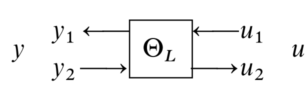
A4. $$\operatorname{rank}\begin{bmatrix} \dfrac{\partial A}{\partial x}(0) - \mathrm{j}wI & B_1(0) \\ \dfrac{\partial C_2}{\partial x}(0) & D_{21}(0) \end{bmatrix} = n + p_2, \quad \forall \omega \in \mathscr{R}$$

Assumption A1 is necessary for the existence of stabilizing controllers. Assumptions A3 and A4 imply that the pair $\{A(x), C_1(x)\}$ is locally zero state detectable and $\{A(x), B_1(x)\}$ is locally stabilizable at the origin. These assumptions are the nonlinear version of standard assumptions usually considered in linear case (see References [2, 3, 7, 20]).

## 3. $(J, J')$-LOSSLESS SYSTEMS AND HAMILTONIAN SYSTEMS

### 3.1. *The $(J, J')$-lossless system* $(\Theta^* J \Theta = J')$

Before discussing the $(J, J')$-lossless property in nonlinear system, let's consider this property in the following linear chain-scattering setting.




$$\Theta_L := \begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases} \tag{2}$$

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with $y_1 \in \mathbb{R}^p$, $y_2 \in \mathbb{R}^q$, $u_1 \in \mathbb{R}^m$, and $u_2 \in \mathbb{R}^n$. It is well known (see references [5, 7, 21, 22]) that, matrix $\Theta_L(s) \in \mathrm{RL}^\infty_{(p+q)\times(m+n)}$ is said to be a $(J, J')$-lossless matrix if $p \geqslant m$, $q \geqslant n$ and

$$\Theta_L(s)^{\sim} J \Theta_L(s) = J' \quad \text{for each } s \in jw$$

$$\Theta_L(s)^* J \Theta_L(s) \leqslant J' \quad \text{for each } Re[s] \geqslant 0$$

where $J = \mathrm{diag}\{I_p, -I_q\}$ and $J' = \mathrm{diag}\{I_m, -I_n\}$. Its relevant state-space properties are stated below.

*Lemma 1*

Let $\Theta_L(s) \in \mathrm{RL}^\infty_{(p+q)\times(m+n)}$ with $(A, B)$ controllable, $(C, A)$ detectable.
Then $\Theta_L$ is $(J, J')$-lossless if:

(i) $A^\mathrm{T} X + XA + C^\mathrm{T} JC = 0$;
(ii) $XB + C^\mathrm{T} JD = 0$;
(iii) $D$ is $(J, J')$-unitary (i.e. $D^\mathrm{T} JD = J'$ and $DJ'D^\mathrm{T} = J$);
(iv) $X \geqslant 0$.

Obviously, from the above lemma, one has

$$\begin{aligned}\Theta_L^* J \Theta_L &= (D + C(sI-A)^{-1}B)^* J (D + C(sI-A)^{-1}B) \\ &= D^\mathrm{T} JD + B^\mathrm{T}(s^*I - A^\mathrm{T})^{-1} C^\mathrm{T} JD + D^\mathrm{T} JC(sI-A)^{-1}B \\ &\quad + B^\mathrm{T}(s^*I - A^\mathrm{T})^{-1} C^\mathrm{T} JC(sI-A)^{-1}B \\ &= J' - B^\mathrm{T}(s^*I - A^\mathrm{T})^{-1}(s^*X + sX)(sI-A)^{-1}B \end{aligned} \tag{3}$$

That is $\Theta_L(s)^{\sim} J\Theta_L(s) = J'$, $\forall s \in jw$ and $\Theta_L(s)^* J \Theta_L(s) \leqslant J'$, $\forall Re\,[s] \geqslant 0$.
Since $y = \Theta_L u$ and $\Theta_L^* J \Theta_L \leqslant J'$, one has

$$\begin{aligned} u^* \Theta_L^* J \Theta_L u \leqslant u^* J' u \Rightarrow y^* Jy \leqslant u^* J' u &\Rightarrow \|y_1\|_2^2 - \|y_2\|_2^2 \leqslant \|u_1\|_2^2 - \|u_2\|_2^2 \\ &\Rightarrow \|y_1\|_2^2 + \|u_2\|_2^2 \leqslant \|u_1\|_2^2 + \|y_2\|_2^2 \end{aligned} \tag{4}$$

where

$$\|u\|_2 := \left( \int_{-\infty}^{\infty} \|u(t)\|^2 \, \mathrm{d}t \right)^{1/2} = \left( \int_{-\infty}^{\infty} u(t)^* u(t) \, \mathrm{d}t \right)^{1/2}$$

This implies that the output energy is less then or equal to the input energy.

Furthermore, from Lemma 3 in Hong *et al.* [7], one has the following lemma for a conjugate $(J, J')$-lossless system. For simplifying the mathematical narration, we use 'if' instead of 'if and only if' in these lemmas for linear case.

*Lemma 2*

Let $\Theta_{cL}(s) \in \mathrm{RL}^\infty_{(m+n)\times(p+q)}$ with $(C, A)$ observable, $(A, B)$ stabilizable. Then $\Theta_{cL}$ is conjugate $(J, J')$-lossless if

(i) $AY + YA^\mathrm{T} + BJ'B^\mathrm{T} = 0$;
(ii) $DJ'B^\mathrm{T} + CY = 0$;
(iii) $D$ is $(J, J')$-unitary;
(iv) $Y \geqslant 0$.

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*Remark 1*

As a similar computation as in Equation (3), one obtains that

$$\Theta_{cL} J' \Theta_{cL}^{\sim} = J \quad \text{for each } s \in \mathrm{j}w$$

$$\Theta_{cL} J' \Theta_{cL}^{*} \leqslant J \quad \text{for each } Re[s] \geqslant 0$$

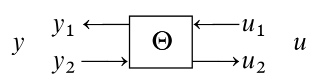
Besides, if $X$ in Lemma 1 is invertible, then (i), (ii) in Lemma 1 and (i), (ii) in Lemma 2 are related by reciprocity: $Y = X^{-1}$. Furthermore, from the $(J, J')$-unitary of $D$, this indicates that $\Theta_L$ in Lemma 1 also is a conjugate, $(J, J')$-lossless system (i.e. $\Theta_L(s) J' \Theta_L(s)^{\sim} = J, \forall s \in \mathrm{j}w$ and $\Theta_L(s) J' \Theta_L(s)^* \leqslant J$, $\forall\, Re[s] \geqslant 0$).

It immediately shows that, if $Y$ in Lemma 2 is invertible, then $\Theta_{cL}$ is conjugate $(J, J')$-lossless implies that $\Theta_{cL}$ also in $(J, J')$-lossless. By a similar computation as in Equation (4), one obtains that $\Theta_{cL}$ also has the property of output energy is less then or equal to the input energy.

Now, consider the following definition of $(J, J')$-losslessness for the nonlinear system. This definition is a modified version of the well-known results of the *dissipative* system while applied to the chain-scattering setting (see Willems [23] and Pavel *et al.* [18]).

*Definition 6*

A nonlinear $C^{\infty}$ chain-scattering system $\Theta$ given by




$$\Theta := \begin{cases} \dot{x} = a(x) + b(x)u \\ y = c(x) + d(x)u \end{cases} \tag{5}$$

with $y_1 \in \mathbb{R}^p$, $y_2 \in \mathbb{R}^q$, $u_1 \in \mathbb{R}^m$, and $u_2 \in \mathbb{R}^n$, is called a $(J, J')$-lossless system, if $\Theta$ has an equilibrium point at $x = 0$ with $c(0) = 0$ and if there exists a storage function $V(x) \geqslant 0$, such that:

$$V(x(T)) - V(x(0)) = \frac{1}{2} \int_0^{\mathrm{T}} (u^{\mathrm{T}}(t) J' u(t) - y^{\mathrm{T}}(t) J y(t))\, \mathrm{d}t \geqslant 0 \tag{6}$$

with $x(0) = 0$, $V(0) = 0$, and $T \geqslant 0$, where $J = \mathrm{diag}\{I_p, -I_q\}$ and $J' = \mathrm{diag}\{I_m, -I_n\}$

Obviously, from Equation (6), one can see that this $(J, J')$-lossless system has a same property as it is in linear case, (i.e., the output energy is less than or equal to the input energy). Furthermore, if $V(x)$ is differentiable then Equation (6) becomes

$$V_x(x)[a(x) + b(x)u] = \tfrac{1}{2} u^{\mathrm{T}} J' u - \tfrac{1}{2}[c(x) + d(x)u]^{\mathrm{T}} J [c(x) + d(x)u]$$

Direct computation yields the following lemma for the $(J, J')$-lossless system (see Ball *et al.* [17] or Pavel *et al.* [18]).

*Lemma 3*

System $\Theta$ is $(J, J')$-lossless with respect to a smooth function $V(x)$, if:

(i) $V_x(x) a(x) + \frac{1}{2} c^{\mathrm{T}}(x) J c(x) = 0$;
(ii) $V_x(x) b(x) + c^{\mathrm{T}}(x) J d(x) = 0$;
(iii) $d^{\mathrm{T}}(x) J d(x) = J'$;
(iv) $V(x) \geqslant 0$, $V(0) = 0$.

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*Remark 2*

If one further defines the input vector $u$ in $\Theta$ as $u = \left[\begin{smallmatrix} z' \\ w' \end{smallmatrix}\right]$ and the output $y$ as $y = \left[\begin{smallmatrix} z \\ w \end{smallmatrix}\right]$, then from Equation (6), it is obvious that:

$$\int_0^{\mathrm{T}} [(\|z'(t)\|^2 - \|w'(t)\|^2) - (\|z(t)\|^2 - \|w(t)\|^2)]\,\mathrm{d}t \geqslant 0$$

That is

$$\int_0^{\mathrm{T}} (\|z'(t)\|^2 - \|w'(t)\|^2)\,\mathrm{d}t \leqslant 0 \Rightarrow \int_0^{\mathrm{T}} (\|z(t)\|^2 - \|w(t)\|^2)\,\mathrm{d}t \leqslant 0$$

Furthermore, as proposed by Crouch *et al.* [24], the Hamiltonian extension of system $\Theta$ is

$$\begin{aligned} \dot{x} &= a(x) + b(x)u \\ \dot{p} &= -\left(\frac{\partial a(x)}{\partial x} + \frac{\partial b(x)}{\partial x}u\right)^{\mathrm{T}} p - \left(\frac{\partial c(x)}{\partial x} + \frac{\partial d(x)}{\partial x}u\right)^{\mathrm{T}} u_a \\ y &= c(x) + d(x)u \\ y_a &= b^{\mathrm{T}}(x)p + d^{\mathrm{T}}(x)u_a \end{aligned} \tag{7}$$

Imposing $u_a = Jy$ in Equation (7) leads to the following Hamiltonian system for $\Theta^* J\Theta$ (with input $u$ and output $y_a$).

$$\Theta^* J\Theta: \begin{cases} \dot{x} = a(x) + b(x)u \\ \dot{p} = -\left(\dfrac{\partial a(x)}{\partial x} + \dfrac{\partial b(x)}{\partial x}u\right)^{\mathrm{T}} p - \left(\dfrac{\partial c(x)}{\partial x} + \dfrac{\partial d(x)}{\partial x}u\right)^{\mathrm{T}} J(c(x) + d(x)u) \\ y_a = b^{\mathrm{T}}(x)p + d^{\mathrm{T}}(x)Jc(x) + d^{\mathrm{T}}(x)Jd(x)u \end{cases} \tag{8}$$

This Hamiltonian system can also be denoted by

$$\Theta^* J\Theta: \begin{cases} \dot{x} = \left[\dfrac{\partial \hat{H}}{\partial p}(x, p, u)\right]^{\mathrm{T}} \\ \dot{p} = -\left[\dfrac{\partial \hat{H}}{\partial x}(x, p, u)\right]^{\mathrm{T}} \\ y_a = \left[\dfrac{\partial \hat{H}}{\partial u}(x, p, u)\right]^{\mathrm{T}} \end{cases}$$

with Hamiltonian function $\hat{H}(x, p, u) = p^{\mathrm{T}}(a(x) + b(x)u) + \frac{1}{2}(c(x) + d(x)u)^{\mathrm{T}} J(c(x) + d(x)u)$.

Recalling from Proposition 7.1.3 in Van der Schaft [25] that, $\{(x, p): p = V_x^{\mathrm{T}}\}$ being an invariant mainfold for $\Theta^* J\Theta$ (with $u = 0$) if and only if the smooth function $V(x)$ is such that the Hamilton–Jacobi equation $\hat{H}(x, V_x^{\mathrm{T}}(x), 0) = 0$, which is equal to condition (i) in Lemma 3. It immediately follows that, if $d^{\mathrm{T}}(x)Jd(x) = J'$ and the smooth function $V(x)$ satisfies conditions (ii) and (iv) in Lemma 3 then the system $\Theta$ is $(J, J')$-lossless. Also, from the local properties in Ball and

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Van der Schaft [17, 11] such an invariant manifold exists if the Jacobian matrix of the Hamiltonian flow associated with $\Theta^* J\Theta$ (with $u=0$) at equilibrium belongs to dom(Ric).

For discussing the ((J, J′)-lossless)-(minimal-phase) factorization for a nonlinear affine system $G$ so that $G$ can be factorized as $G=\Theta\Pi$ (with $\Pi$ minimal phase and $\Theta$ being $(J, J')$-lossless), let's consider the following nonlinear affine system $G$:

$$G:\begin{cases}\dot{\bar{x}} = a(\bar{x}) + b(\bar{x})u\\ y = c(\bar{x}) + d(\bar{x})u\end{cases}$$

where $\bar{x}=0$ is an equilibrium point and $c(0)=0$. As shown in Ball and Van der Schaft [17, 11] that, for such a nonlinear affine system, while the Hamiltonian system of $(G^*JG)^{-1}$ is given by

$$(G^*JG)^{-1}:\begin{cases}\dot{\bar{x}} = \left[\dfrac{\partial \hat{H}^\times}{\partial p}(\bar{x}, p, y_a)\right]^{\mathrm{T}}\\ \dot{p} = -\left[\dfrac{\partial \hat{H}^\times}{\partial \bar{x}}(\bar{x}, p, y_a)\right]^{\mathrm{T}}\\ u = \left[\dfrac{\partial \hat{H}^\times}{\partial y_a}(\bar{x}, p, y_a)\right]^{\mathrm{T}}\end{cases}$$

then $G$ has a ((J, J′)-lossless)-(minimal-phase) factorization, suppose there exists an invariant manifold $\{(x, p): p = V_x^{\mathrm{T}}\}$ for $(G^*JG)^{-1}$ (with $y_a=0$) so that the Hamilton–Jacobi equation $\hat{H}^\times(\bar{x}, V_{\bar{x}}^{\mathrm{T}}(\bar{x}), 0)=0$ with the stability side condition $\partial\hat{H}^\times/\partial\bar{x}\,(\bar{x}, V_{\bar{x}}^{\mathrm{T}}(\bar{x}), 0)$ is Lyapunov stable. Furthermore, such an invariant manifold does exist if the Jacobian matrix of the Hamiltonian flow associated with $(G^*JG)^{-1}$ (with $y_a=0$) at equilibrium belongs to dom(Ric).

The idea behind this local result is easy to understand when one considers the following characteristics for linear systems. One can further compare these characteristics with the related nonlinear Hamiltonian system.

If the linear chain-scattering system $G_L$ is denoted by

$$G_L = C(sI-A)^{-1}B + D = \left[\begin{array}{c|c} A & B\\ \hline C & D\end{array}\right] = \begin{cases}\dot{\bar{x}} = A\bar{x} + Bu\\ y = C\bar{x} + Du\end{cases}$$

with $R = D^{\mathrm{T}}JD$ being invertible, then the linear Hamiltonian system $G_L{}^*JG_L$ given by

$$G_L{}^*JG_L = \left[\begin{array}{cc|c} A & 0 & B\\ -C^{\mathrm{T}}JC & -A^{\mathrm{T}} & -C^{\mathrm{T}}JD\\ \hline D^{\mathrm{T}}JC & B^{\mathrm{T}} & D^{\mathrm{T}}JD\end{array}\right] \text{ is contrasted with } G^*JG:\begin{cases}\dot{\bar{x}} = \left[\dfrac{\partial \hat{H}}{\partial p}(\bar{x}, p, u)\right]^{\mathrm{T}}\\ \dot{p} = -\left[\dfrac{\partial \hat{H}}{\partial \bar{x}}(\bar{x}, p, u)\right]^{\mathrm{T}}\\ y_a = \left[\dfrac{\partial \hat{H}}{\partial u}(\bar{x}, p, u)\right]^{\mathrm{T}}\end{cases}$$

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Furthermore, the $(G_L{}^*JG_L)^{-1}$ denoted by

$$(G_L{}^*JG_L)^{-1} = \left[\begin{array}{c|c} A_{G^\times} & B_{G^\times} \\ \hline C_{G^\times} & D_{G^\times} \end{array}\right] \quad \text{is contrasted with } (G^*JG)^{-1}: \begin{cases} \dot{\bar{x}} = \left[\dfrac{\partial \hat{H}^\times}{\partial p}(\bar{x}, p, y_a)\right]^{\mathrm{T}} \\ \dot{p} = -\left[\dfrac{\partial \hat{H}^\times}{\partial \bar{x}}(\bar{x}, p, y_a)\right]^{\mathrm{T}} \\ y_a = \left[\dfrac{\partial \hat{H}^\times}{\partial y_a}(x, p, y_a)\right]^{\mathrm{T}} \end{cases}$$

where $\hat{H}^\times(\bar{x}, p, y_a)$ is the Hamiltonian function for $(G^*JG)^{-1}$.

It immediately follows that

$$A_{G^\times} = \begin{bmatrix} A - BR^{-1}D^{\mathrm{T}}JC & -BR^{-1}B^{\mathrm{T}} \\ -C^{\mathrm{T}}(J - JDR^{-1}D^{\mathrm{T}}J)C & -(A - BR^{-1}D^{\mathrm{T}}JC)^{\mathrm{T}} \end{bmatrix}$$

is contrasted with the Hamiltonian flow induced by $\hat{H}^\times(\bar{x}, p, y_a)$ (with $y_a = 0$) or the Hamiltonian flow associated with $(G^*JG)^{-1}$ (with $y_a = 0$). Now, introducing a similarity transformation matrix $T = \left[\begin{smallmatrix} I & 0 \\ X & I \end{smallmatrix}\right]$ into $A_{G^\times}$, one has

$$T^{-1}A_{G^\times}T = \begin{bmatrix} I & O \\ -X & I \end{bmatrix} A_{G^\times} \begin{bmatrix} I & 0 \\ X & I \end{bmatrix} = \begin{bmatrix} \mathscr{P} & \mathscr{R} \\ \mathscr{Q} & -\mathscr{P}^{\mathrm{T}} \end{bmatrix}$$

where

$$\mathscr{P} = A - BR^{-1}(D^{\mathrm{T}}JC + B^{\mathrm{T}}X)$$

$$\mathscr{R} = -BR^{-1}B^{\mathrm{T}}$$

$$\mathscr{Q} = X(A - BR^{-1}D^{\mathrm{T}}JC) + (A - BR^{-1}D^{\mathrm{T}}JC)^{\mathrm{T}}X - XBR^{-1}B^{\mathrm{T}}X + C^{\mathrm{T}}(J - JDR^{-1}D^{\mathrm{T}}J)C$$

Obviously, the stability side condition $\partial\hat{H}^\times/\partial\bar{x}(\bar{x}, V_{\bar{x}}^{\mathrm{T}}(\bar{x}), 0)$ is contrasted with $\mathscr{P} = A + BF$ (with the state-feedback gain $F = -R^{-1}(D^{\mathrm{T}}JC + B^{\mathrm{T}}X)$).

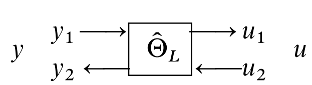
As we know, in linear system, the above Hamiltonian matrix $A_{G^\times}$ is related to the algebraic Riccati equation:

$$X(A - BR^{-1}D^{\mathrm{T}}JC) + (A - BR^{-1}D^{\mathrm{T}}JC)^{\mathrm{T}}X - XBR^{-1}B^{\mathrm{T}}X + C^{\mathrm{T}}(J - JDR^{-1}D^{\mathrm{T}}J)C = 0$$

Furthermore, this algebraic Riccati equation can be solved if the eigenvalues of the related Hamiltonian matrix $A_{G^\times}$ are not on the j$w$-axis (i.e. the Hamiltonian matrix belongs to dom(Ric)).

### *3.2. Conjugate $(-J, -J')$-lossless system $(-\hat{\Theta}J'\hat{\Theta}^* = -J)$*

Consider the following linear chain-scattering system $\hat{\Theta}_L$ given as




$$\hat{\Theta}_L := \begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases} \tag{9}$$

Carefully comparing the chain-scattering structure of $\hat{\Theta}_L$ with $\Theta_L$'s are shown in Equation (2) reveals that the directions of the arrow signals in $\hat{\Theta}_L$ are contrary to $\Theta_L$'s. One thus has the following lemma for conjugate $(-J, -J')$-lossless or conjugate $(J, J')$-expansive system (see Hong *et al.* [7]).

*Lemma 4*

Let $\hat{\Theta}_L(s) \in \mathrm{RL}^{\infty}_{(m+n)\times(p+q)}$ with $m \leqslant p$, $n \leqslant q$, $(C, A)$ observable, and $(A, B)$ stabilizable. Then $\hat{\Theta}_L$ is conjugate $(-J, -J')$-lossless if

(i) $-AY - YA^{\mathrm{T}} + BJ'B^{\mathrm{T}} = 0$;
(ii) $DJ'B^{\mathrm{T}} - CY = 0$;
(iii) $D$ *is* $(J, J')$-unitary;
(iv) $Y \geqslant 0$, where $J = \mathrm{diag}\{I_m, -I_n\}$ and $J' = \mathrm{diag}\{I_p, -I_q\}$.

The above lemma indicates that

$$-\hat{\Theta}_L J' \hat{\Theta}_L^* = -(D + C(sI - A)^{-1}B)J'(D + C(sI - A)^{-1}B)^*$$

$$= -J - C(sI - A)^{-1}(sY + s^*Y)(s^*I - A^{\mathrm{T}})^{-1}C^{\mathrm{T}}$$

That is $-\hat{\Theta}_L(\mathrm{s})J'\hat{\Theta}_L(\mathrm{s})^{\sim} = -J$, $\forall s \in jw$ and $-\hat{\Theta}_L(\mathrm{s})J'\hat{\Theta}_L(\mathrm{s})^* \leqslant -J$, $\forall\, Re[s] \geqslant 0$ or $\hat{\Theta}_L(\mathrm{s})J'\hat{\Theta}_L(\mathrm{s})^* \geqslant J$, $\forall\, Re[s] \geqslant 0$.

*Remark 3*

This conjugate $(-J, -J')$-lossless system also have the same property as the statement in Remark 1. That is, if $Y$ is invertible, then $\hat{\Theta}_L$ is conjugate $(-J, -J')$-lossless implies that $\hat{\Theta}_L$ also is $(-J, -J')$-lossless (i.e., $\hat{\Theta}_L^* J \hat{\Theta}_L \geqslant J'$).

Since $y = \hat{\Theta}_L u$ and $\hat{\Theta}_L^* J \hat{\Theta}_L \geqslant J'$, one has

$$y^*Jy = u^*\hat{\Theta}_L^* J \hat{\Theta}_L u \geqslant u^*J'u \Rightarrow y^*Jy \geqslant u^*J'u$$

$$\Rightarrow \|y_1\|_2^2 - \|y_2\|_2^2 \geqslant \|u_1\|_2^2 - \|u_2\|_2^2$$

$$\Rightarrow \|y_1\|_2^2 + \|u_2\|_2^2 \geqslant \|\mathrm{u}_1\|_2^2 + \|\mathrm{y}_2\|_2^2$$

From Equation (9), this implies that the output energy is less than or equal to the input energy.

Now, supposing the interconnection law $u$ in Equation (7) is as $u = -J'y_a$, and substituting $\hat{\Theta}$ for $\Theta$, one has the following Hamiltonian system for $-\hat{\Theta}J'\hat{\Theta}^*$ (with input $u_a$ and output $y$).

$$-\hat{\Theta}J'\hat{\Theta}^*: \begin{cases} \dot{x} = a(x) - b(x)J'b^{\mathrm{T}}(x)p - b(x)J'd^{\mathrm{T}}(x)u_a \\ \dot{p} = -\left(\dfrac{\partial a(x)}{\partial x} - \dfrac{\partial b(x)}{\partial x} J'b^{\mathrm{T}}(x)p - \dfrac{\partial b(x)}{\partial x} J'd^{\mathrm{T}}(x)u_a\right)^{\mathrm{T}} p \\ \qquad -\left(\dfrac{\partial c(x)}{\partial x} - \dfrac{\partial d(x)}{\partial x} J'b^{\mathrm{T}}(x)p - \dfrac{\partial d(x)}{\partial x} J'd^{\mathrm{T}}(x)u_a\right)^{\mathrm{T}} u_a \\ y = c(x) - d(x)J'b^{\mathrm{T}}(x)p - d(x)J'd^{\mathrm{T}}(x)u_a \end{cases} \tag{10}$$

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The Hamiltonian function $\bar{H}$ is such that

$$-\hat{\Theta}J'\hat{\Theta}^{*}:\begin{cases}\dot{x}= & \left[\dfrac{\partial\bar{H}}{\partial p}(x,p,u_a)\right]^{\mathrm{T}}\\ \dot{p}=- & \left[\dfrac{\partial\bar{H}}{\partial x}(x,p,u_a)\right]^{\mathrm{T}}\\ y= & \left[\dfrac{\partial\bar{H}}{\partial u_a}(x,p,u_a)\right]^{\mathrm{T}}\end{cases}$$

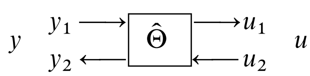
is thus given as

$$\bar{H}(x,p,u_a)=p^{\mathrm{T}}a(x)-\tfrac{1}{2}p^{\mathrm{T}}b(x)J'b^{\mathrm{T}}(x)p-p^{\mathrm{T}}b(x)J'd^{\mathrm{T}}(x)u_a+c^{\mathrm{T}}(x)u_a-\tfrac{1}{2}u_a^{\mathrm{T}}d(x)J'd^{\mathrm{T}}(x)u_a \quad (11)$$

The following definition gives the property for a nonlinear affine system $\hat{\Theta}$ to be conjugate $(-J,-J')$-lossless which is well known in linear case (see e.g. Hong *et al.* [7]). The same as the $(J,J')$-lossless system, this conjugate $(-J,-J')$-lossless system also involves the validity of an energy storage balance equality in integral form. The nonlinear $(I,I')$ case was proposed by Scherpen and Van der Schaft [19], who called it 'co-inner'.

*Definition 7*

A nonlinear $C^{\infty}$ chain-scattering system $\hat{\Theta}$ given by




$$\hat{\Theta}:=\begin{cases}\dot{x}=a(x)+b(x)u\\ y=c(x)+d(x)u\end{cases}$$

with $y_1\in\mathbb{R}^m$, $y_2\in\mathbb{R}^n$, $u_1\in\mathbb{R}^p$, and $u_2\in\mathbb{R}^q$, is called a conjugate $(-J,-J')$-lossless system, if $\hat{\Theta}$ has an equilibrium point at $x=0$, with $c(0)=0$, and the input–output map of system Equation (10) from $u_a$ to $y$, with $x(0)=0$ and $p(0)=0$, is equal to $-J$ (i.e., $-\hat{\Theta}J'\hat{\Theta}^{*}=-J$), and there exists a smooth storage function $W(x)\geqslant 0$, $W(0)=0$, and $T\geqslant 0$ such that:

$$W(x(T))-W(x(0))=\frac{1}{2}\int_0^{\mathrm{T}}(y^{\mathrm{T}}(t)Jy(t)-u^{\mathrm{T}}(t)J'u(t))\,\mathrm{d}t\geqslant 0 \quad (12)$$

Equation (12) shows that the output energy is less than or equal to the input energy.

From the Hamiltonian system for $-\hat{\Theta}J'\hat{\Theta}^{*}$ in Equation (10), since $u=-J'y_a$ and the conjugate $(-J,-J')$-lossless system $\hat{\Theta}$ has $-\hat{\Theta}J'\hat{\Theta}^{*}=-J$ (i.e. $y=-Ju_a$), the above equation is equal to

$$W(x(T))-W(x(0))=\frac{1}{2}\int_0^{\mathrm{T}}(u_a^{\mathrm{T}}(t)Ju_a(t)-y_a^{\mathrm{T}}(t)J'y_a(t))\,\mathrm{d}t\geqslant 0 \quad (13)$$

Also, the following theorem gives a state-space criterion for a nonlinear affine system $\hat{\Theta}$ to be conjugate $(-J,-J')$-lossless.

*Theorem 1*

System $\hat{\Theta}$ is conjugate $(-J,-J')$-lossless with respect to a smooth function $W(x)$, if there is an invariant manifold $\{(x,p):p=W_x^{\mathrm{T}}\}$ for $(\hat{\Theta}J'\hat{\Theta}^{*})^{-1}$ (with $y=0$) such that:

(i) $-a(x)+\frac{1}{2}b(x)J'b^{\mathrm{T}}(x)W_x^{\mathrm{T}}(x)=0$;

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(ii) $-c(x) + d(x)J'b^{\mathrm{T}}(x)W_x^{\mathrm{T}}(x) = 0$;
(iii) $d(x)J'd^{\mathrm{T}}(x) = J$;
(iv) $W(x) \geqslant 0$, $W(0) = 0$.

*Proof.* Replacing the right-hand side of $y = c(x) - d(x)J'b^{\mathrm{T}}(x)p - d(x)J'd^{\mathrm{T}}(x)u_a$ in Equation (10) with (ii), (iii), and $p = W_x^{\mathrm{T}}$, one obtains $y = -Ju_a$ (i.e. $-\hat{\Theta}J'\hat{\Theta}^* = -J$). Furthermore, from Equation (7), since $y_a = b^{\mathrm{T}}(x)p + d^{\mathrm{T}}(x)u_a = b^{\mathrm{T}}(x)W_x^{\mathrm{T}}(x) + d^{\mathrm{T}}(x)u_a$, one has

$$\begin{aligned} y_a^{\mathrm{T}}J'y_a &= (u_a^{\mathrm{T}}d(x) + W_x(x)b(x))J'(b^{\mathrm{T}}(x)W_x^{\mathrm{T}}(x) + d^{\mathrm{T}}(x)u_a) \\ &= 2W_x(x)b(x)J'd^{\mathrm{T}}(x)u_a + W_x(x)b(x)J'b^{\mathrm{T}}(x)W_x^{\mathrm{T}}(x) + u_a^{\mathrm{T}}d(x)J'd^{\mathrm{T}}(x)u_a \end{aligned}$$

From (i) and (iii), this implies that

$$W_x(x)[a(x) - b(x)J'b^{\mathrm{T}}(x)W_x^{\mathrm{T}}(x) - b(x)J'd^{\mathrm{T}}(x)u_a] = \tfrac{1}{2}(u_a^{\mathrm{T}}Ju_a - y_a^{\mathrm{T}}J'y_a)$$

Integrating both side with respect to $t$ (from 0 to $T$), together with $\dot{x} = a(x) - b(x)J'b^{\mathrm{T}}(x)p - b(x)J'd^{\mathrm{T}}(x)u_a$ in Equation (10), Equation (13) follows immediately.

Note that, the differentiability for $V(x)$ or $W(x)$ is an artificial hypothesis imposed for all solutions of the Hamilton–Jacobi equations in this paper; however, there might exist some *viscosity solutions* to admit nonsmooth $V(x)$ or $W(x)$ (see e.g. Başar *et al.* [9] or Van der Schaft [25]).

## 4. THE CSMD APPROACH FOR DERIVING $H^{\infty}$ CONTROLLERS

This paper proposes an alternative method for designing nonlinear $H^{\infty}$ controllers. This method is based on a combination of a chain-scattering matrix description (CSMD) together with the $(J, J')$-lossless and conjugate $(-J, -J')$-lossless properties.

From Equation (1) and the properties in Definition 5, let $P = NM^{-1}$ be a right-coprime factorization, in which one chooses $F(x)$ to be a stabilizing feedback control for the pair $(A(x), B(x))$, and hence $N$ and $M$ are stable. This is analogous to linear system theory, thus giving

$$N := \begin{cases} \dot{x} = A(x) + B(x)F(x) + B(x)U_a(x)\begin{bmatrix} z' \\ w' \end{bmatrix} \\ \begin{bmatrix} z \\ y \end{bmatrix} = C(x) + D(x)F(x) + D(x)U_a(x)\begin{bmatrix} z' \\ w' \end{bmatrix} \end{cases}$$

$$M := \begin{cases} \dot{x} = A(x) + B(x)F(x) + B(x)U_a(x)\begin{bmatrix} z' \\ w' \end{bmatrix} \\ \begin{bmatrix} w \\ u \end{bmatrix} = F(x) + U_a(x)\begin{bmatrix} z' \\ w' \end{bmatrix} \end{cases}$$

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where

$$B(x) = [B_1(x)\ B_2(x)], \quad D(x) = \begin{bmatrix} 0 & D_{12}(x) \\ D_{21}(x) & 0 \end{bmatrix}, \quad C(x) = \begin{bmatrix} C_1(x) \\ C_2(x) \end{bmatrix}, \quad F(x) = \begin{bmatrix} F_1(x) \\ F_2(x) \end{bmatrix}$$

and

$$U_a(x) = \begin{bmatrix} U_{a11}(x) & U_{a12}(x) \\ U_{a21}(x) & U_{a22}(x) \end{bmatrix}$$

One further defines $G_1$ and $G_2$ as

$$G_1 := \begin{cases} \dot{x} = A(x) + B(x)F(x) + B(x)U_a(x)\begin{bmatrix} z' \\ w' \end{bmatrix} \\ \begin{bmatrix} z \\ w \end{bmatrix} = \begin{bmatrix} C_1(x) \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & D_{12}(x) \\ I & 0 \end{bmatrix}\begin{bmatrix} F_1(x) \\ F_2(x) \end{bmatrix} + \begin{bmatrix} 0 & D_{12}(x) \\ I & 0 \end{bmatrix} U_a(x)\begin{bmatrix} z' \\ w' \end{bmatrix} \end{cases} \tag{14}$$

$$G_2 := \begin{cases} \dot{x} = A(x) + B(x)F(x) + B(x)U_a(x)\begin{bmatrix} z' \\ w' \end{bmatrix} \\ \begin{bmatrix} u \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ C_2(x) \end{bmatrix} + \begin{bmatrix} 0 & I \\ D_{21}(x) & 0 \end{bmatrix}\begin{bmatrix} F_1(x) \\ F_2(x) \end{bmatrix} + \begin{bmatrix} 0 & I \\ D_{21}(x) & 0 \end{bmatrix} U_a(x)\begin{bmatrix} z' \\ w' \end{bmatrix} \end{cases} \tag{15}$$

It is obvious that the standard nonlinear $H^\infty$ set-up as shown in Figure 1 is thus transformed into the chain-scattering matrix description as in Figure 2.

*Remark 4*

If one rewrites $G_1$ in Equation (14) as

$$G_1 := \begin{cases} \dot{x} = \hat{A}(x) + B(x)U_a(x)\begin{bmatrix} z' \\ w' \end{bmatrix} \\ \begin{bmatrix} z \\ w \end{bmatrix} = \hat{C}(x) + \hat{D}(x)U_a(x)\begin{bmatrix} z' \\ w' \end{bmatrix} \end{cases}$$

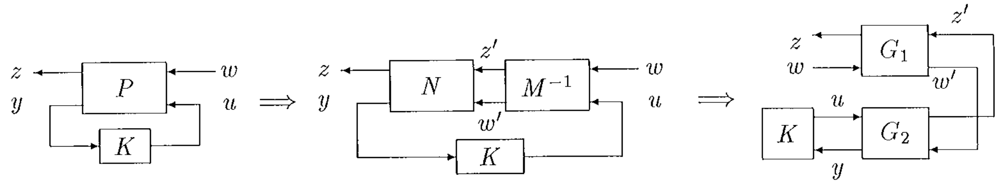


Figure 2.

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then, from Assumptions A1–A4 and Lemma 3, $G_1$ will be a $(J, J')$-lossless system if the following properties hold (an equivalent version of these properties can be found in Isidori [4]).

(i) One chooses $U_a(x) = \left[\begin{smallmatrix} 0 & -I \\ I & 0 \end{smallmatrix}\right]$ such that $U_a(x)^{\mathrm{T}}\hat{D}(x)^{\mathrm{T}}J\hat{D}(x)U_a(x) = J'$.

(ii) there exists a $C^2$ nonnegative differentiable function $V(x)$ (with $V(0) = 0$) that is locally defined in a neighbourhood of the origin and $V(x)$ and satisfies the Hamilton–Jacobi equation

$$V_x(x)\hat{A}(x) + \tfrac{1}{2}\hat{C}(x)J\hat{C}(x) = 0 \tag{16}$$

such that

$$V_x(x)B(x)U_a(x) + \hat{C}^{\mathrm{T}}(x)J\hat{D}(x)U_a(x) = 0 \tag{17}$$

This also implies that the stabilizing state feedback gain $F(x)$ can also be obtained from Equation (17). That is

$$\begin{aligned}
&V_x(x)B(x) + \hat{C}^{\mathrm{T}}(x)J\hat{D}(x) = 0\\
&\Rightarrow V_x(x)B(x) + \left(\begin{bmatrix} C_1(x) \\ 0 \end{bmatrix} + \hat{D}(x)F(x)\right)^{\mathrm{T}} J\hat{D}(x) = 0\\
&\Rightarrow B^{\mathrm{T}}(x)V_x^{\mathrm{T}}(x) + \hat{D}^{\mathrm{T}}(x)J\begin{bmatrix} C_1(x) \\ 0 \end{bmatrix} + \hat{D}^{\mathrm{T}}(x)J\hat{D}(x)F(x) = 0\\
&\Rightarrow F(x) = -R^{-1}(x)\left[B^{\mathrm{T}}(x)V_x^{\mathrm{T}}(x) + \begin{bmatrix} 0 \\ D_{12}^{\mathrm{T}}(x)C_1(x) \end{bmatrix}\right]\\
&\Rightarrow \begin{bmatrix} F_1(x) \\ F_2(x) \end{bmatrix} = \begin{bmatrix} B_1^{\mathrm{T}}(x)V_x^{\mathrm{T}}(x) \\ -B_2^{\mathrm{T}}(x)V_x^{\mathrm{T}}(x) + D_{12}^{\mathrm{T}}(x)C_1(x) \end{bmatrix}
\end{aligned}$$

where

$$R(x) = \hat{D}^{\mathrm{T}}(x)J\hat{D}(x) = \begin{bmatrix} -I_{m_1} & 0 \\ 0 & D_{12}^{\mathrm{T}}(x)D_{12}(x) \end{bmatrix} \quad \text{and } \hat{D}(x) = \begin{bmatrix} 0 & D_{12} \\ I & 0 \end{bmatrix}$$

However, as stated on p.11 in Section 3.1, the existence of $V_x^{\mathrm{T}}(x)$ such that $A(x) + B(x)F(x)$ is locally asymptotically stable corresponds to the Jacobian matrix of the Hamiltonian flow associated with $G_1^* JG_1$ (with $z' = 0,\ w' = 0$) at equilibrium belonging to dom(Ric). Direct computation yields that such a Jacobian matrix is as

$$H_\infty = \begin{bmatrix} A & B_1B_1^{\mathrm{T}} - B_2B_2^{\mathrm{T}} \\ -C_1^{\mathrm{T}}C_1 & -A^{\mathrm{T}} \end{bmatrix}$$

which is equal to the Hamiltonian matrix '$H_\infty$' proposed by Doyle *et al.* [20].

### *4.1. Local disturbance attenuation by measurement feedback*

Before discussing the nonlinear output-feedback control problem, first consider the linear case. As proposed by Hong and Teng [7], the linear 4-block $H^\infty$ controllers are obtained directly by inverting one of the $(J, J')$-coprime factors of $G_2$. That is, if the linear version of $G_2$ in Equation (15) has an outer-(conjugate $(J, J')$-inner) factorization of $G_2 = \tilde{\Pi}^{-1}\tilde{\Theta}$ so that $\tilde{\Theta}$ is

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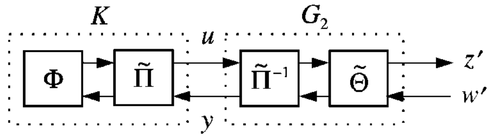
conjugate $(-J, -J')$-lossless and both $\tilde{\Pi}$ and $\tilde{\Pi}^{-1}$ are stable, then the linear 4-block $H^\infty$ controllers can be described as $K = F_L(\tilde{\Pi}, \Phi)$, i.e.




where $\|\Phi\|_\infty \leqslant 1$ and $F_L(\cdot,\cdot)$ indicates left CSMD. Definitions of left and right CSMD are reported in Reference [7].

For a nonlinear system, as shown in Crouch *et al.* [24], Scherpen *et al.* [19], and Van der Schaft [10, 11, 25], there locally exists an outer-(conjugate $(J, J')$-inner) factorization for $G_2$ (Equation (15)), assuming that there exists solutions of the relevant Hamilton–Jacobi equations. However, the outer-(conjugate $(J, J')$-inner) factorization does not exist in nonlinear systems in general. Hence, it is natural to replace $x$ by some estimate $\xi$ provided by a proper auxiliary dynamics. One then finds an appropriate nonlinear system $\tilde{\Pi}$ constructed by this estimate state $\xi$ such that $\tilde{\Pi}G_2$ satisfies the conjugate $(-J, -J')$-lossless properties. This also implies that locally one has the outer-(conjugate $(J, J')$-inner) factorization for $G_2$.

Now, one rewrites $G_2$ as

$$G_2 := \begin{cases} \dot{x} = \hat{A}(x) + B(x)U_a(x)\begin{bmatrix} z' \\ w' \end{bmatrix} \\ \begin{bmatrix} u \\ y \end{bmatrix} = \tilde{C}(x) + \tilde{D}(x)U_a(x)\begin{bmatrix} z' \\ w' \end{bmatrix} \end{cases}$$

and define system $\tilde{\Pi}$ given as follows:

$$\tilde{\Pi} := \begin{cases} \dot{\xi} = \hat{A}(\xi) + H(\xi)\tilde{C}(\xi) + H(\xi)\begin{bmatrix} u \\ y \end{bmatrix} \\ \begin{bmatrix} v \\ \sigma \end{bmatrix} = U_z(\xi)\tilde{C}(\xi) + U_z(\xi)\begin{bmatrix} u \\ y \end{bmatrix} \end{cases} \tag{18}$$

where $\xi$ is an an estimate of $x$,

$$H(\cdot) = [H_1(\cdot)\; H_2(\cdot)], \quad \tilde{C}(\cdot) = \begin{bmatrix} \tilde{C}_1(\cdot) \\ \tilde{C}_2(\cdot) \end{bmatrix} = \begin{bmatrix} F_2(\cdot) \\ C_2(\cdot) + D_{21}(\cdot)F_1(\cdot) \end{bmatrix}$$

and $c = \Phi(\sigma)$ ($\Phi(\sigma)$ is a free stable system with $\|\Phi(\sigma)\|_{L_2} \leqslant 1$).

Therefore, the state-space representation of $\tilde{\Pi}G_2$ is given by

$$\tilde{\Pi}G_2 := \begin{cases} \begin{bmatrix} \dot{x} \\ \dot{\xi} \end{bmatrix} = \begin{bmatrix} \hat{A}(x) \\ \hat{A}(\xi) + H(\xi)\tilde{C}(\xi) + H(\xi)\tilde{C}(x) \end{bmatrix} + \begin{bmatrix} B(x)U_a(x) \\ H(\xi)\tilde{D}(x)U_a(x) \end{bmatrix}\begin{bmatrix} z' \\ w' \end{bmatrix} \\ \begin{bmatrix} v \\ \sigma \end{bmatrix} = U_z(\xi)[\tilde{C}(x) + \tilde{C}(\xi)] + U_z(\xi)\tilde{D}(x)U_a(x)\begin{bmatrix} z' \\ w' \end{bmatrix} \end{cases}$$

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Rewrite it as

$$\tilde{\Pi}G_2 := \begin{cases} \dot{x}_e = A_e(x_e) + B_e(x_e)\begin{bmatrix} z' \\ w' \end{bmatrix} \\ \begin{bmatrix} v \\ \sigma \end{bmatrix} = C_e(x_e) + D_e(x_e)\begin{bmatrix} z' \\ w' \end{bmatrix} \end{cases}$$

*Remark 5*

From Assumptions A1–A4 and Theorem 1, if one chooses $U_z(x) = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$ such that

$$D_e(x_e)JD_e^{\mathrm{T}}(x_e) = U_z(x)\tilde{D}(x)U_a(x)JU_a^{\mathrm{T}}(x)\tilde{D}^{\mathrm{T}}(x)U_z^{\mathrm{T}}(x) = J$$

and if there exists a $C^2$ non-negative function $W(x_e) = Q(x-\xi)$ locally defined in a neighbourhood of $(x, \xi) = (0, 0)$ with $W(0) = 0$ and $W(x_e)$ satisfies the Hamilton–Jacobi equation

$$W_{x_e}A_e(x_e) - \tfrac{1}{2}W_{x_e}B_e(x_e)JB_e^{\mathrm{T}}(x_e)W_{x_e}^{\mathrm{T}} = 0 \tag{19}$$

such that

$$C(x_e) - D_e(x_e)JB_e^{\mathrm{T}}(x_e)W_{x_e}^{\mathrm{T}} = 0 \tag{20}$$

where

$$W_{x_e} = [W_x(x_e)\ \ W_\xi(x_e)] = [Q_x(x-\xi)\ \ Q_\xi(x-\xi)]$$

then $\tilde{\Pi}G_2$ will be a conjugate $(-J, -J')$-lossless system. Together with the properties in Equation (12), this also implies that

$$\int_0^{\mathrm{T}} (\|v(t)\|^2 - \|\sigma(t)\|^2)\,\mathrm{d}t \leqslant 0 \Rightarrow \int_0^{\mathrm{T}} (\|z'(t)\|^2 - \|w'(t)\|^2)\,\mathrm{d}t \leqslant 0$$

As we know, from Ball and Van der Schaft [10, 17] if the corresponding Jacobian matrix of the Hamiltonian flow associated with $(G_2JG_2^*)^{-1}$ (with $y_a = 0$) at equilibrium belongs to dom(Ric), then there exists such $W(x_e) = Q(x-\xi)$ so that $A_e(x_e)$ is locally asymptotically stable in a neighbourhood of $(x, \xi) = (0, 0)$. Direct computation yields that such a Jacobian matrix indicated by $A_{H_z}$ is similar to the Hamiltonian matrix $J_\infty$ given by Doyle *et al.* [20], where

$$J_\infty = \begin{bmatrix} A^{\mathrm{T}} & C_1^{\mathrm{T}}C_1 - C_2^{\mathrm{T}}C_2 \\ -B_1B_2^{\mathrm{T}} & -A \end{bmatrix} \qquad \square$$

Furthermore, suppose $\hat{Z}$ represents the solution of such $A_{H_z}$, then, comparing $Q(x-\xi)$ with $\hat{Z}$, it follows that

$$\hat{Z}^{-1} = \frac{1}{2}\left[\frac{\partial^2 Q}{\partial x^2}\right]_{x=0}$$

and $Q(x-\xi) = (x-\xi)^{\mathrm{T}}\hat{Z}^{-1}(x-\xi)$ is one of such solutions.

For this reason, although the nonlinear function $W(x_e)$ can be expanded as $W(x_e) = Q(x-\xi) = (x-\xi)^{\mathrm{T}}\hat{Z}^{-1}(x-\xi) + O(x-\xi)$ locally defined in a neighbourhood of $(x, \xi) = (0, 0)$ with $O(0) = 0$ and $W(x_e)$ satisfies Equations (19) and (20), we take $W(x_e) = Q(x-\xi) = (x-\xi)^{\mathrm{T}}\hat{Z}^{-1}(x-\xi)$ as its quadratic approximation at the origin.

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Since $Q(x-\xi)=(x-\xi)^{\mathrm{T}}\hat{Z}^{-1}(x-\xi)$, and then $[Q_x(x-\xi)\ \ Q_\xi(x-\xi)]=[Q_x\ \ -Q_x]$, one has the following derivation for the measurement feedback gain $H(\xi)$. From Equation (20), one has

$$-C(x_e)+D_e(x_e)JB_e^{\mathrm{T}}(x_e)\begin{bmatrix} Q_x^{\mathrm{T}} \\ -Q_x^{\mathrm{T}}\end{bmatrix}=0$$

$$\Rightarrow -(\tilde{C}(x)+\tilde{C}(\xi)+\tilde{D}(x)U_a(x)JU_a^{\mathrm{T}}(x)(B^{\mathrm{T}}(x)-\tilde{D}^{\mathrm{T}}(x)H^{\mathrm{T}}(\xi))Q_x^{\mathrm{T}}=0$$

and multiplying the right-hand side by $Q_x^{-\mathrm{T}}$, one obtains

$$-(\tilde{\mathrm{C}}(x)+\tilde{C}(\xi))Q_x^{-\mathrm{T}}+\tilde{D}(x)U_a(x)JU_a^{\mathrm{T}}(x)(B^{\mathrm{T}}(x)-\tilde{D}^{\mathrm{T}}(x)H^{\mathrm{T}}(\xi))=0$$

$$\Rightarrow -Q_x^{-1}(\tilde{C}^{\mathrm{T}}(x)+\tilde{C}^{\mathrm{T}}(\xi))+B(x)U_a(x)JU_a^{\mathrm{T}}(x)\tilde{D}^{\mathrm{T}}(x)-H(\xi)\tilde{D}(x)U_a(x)JU_a^{\mathrm{T}}(x)\tilde{D}^{\mathrm{T}}(x)=0$$

$$\Rightarrow H(\xi)\tilde{D}(x)U_a(x)JU_a^{\mathrm{T}}(x)\tilde{D}^{\mathrm{T}}=-Q_x^{-1}(\tilde{C}^{\mathrm{T}}(x)+\tilde{C}^{\mathrm{T}}(\xi))+B(x)U_a(x)JU_a^{\mathrm{T}}(x)\tilde{D}^{\mathrm{T}}(x)$$

$$\Rightarrow [H_1(\xi)\ \ H_2(\xi)]=[Q_x^{-1}(\bar{C}_1^{\mathrm{T}}(x)+\bar{C}_1^{\mathrm{T}}(\xi))-\mathrm{B}_2(x)\ Q_x^{-1}(\tilde{C}_2^{\mathrm{T}}(x)+\tilde{C}_2^{\mathrm{T}}(\xi))+B_1(x)D_{21}^{\mathrm{T}}(x)]\tilde{R}^{-1}$$

where

$$\tilde{R}=\begin{bmatrix} I & 0 \\ 0 & -D_{21}(x)D_{21}^{\mathrm{T}}(x)\end{bmatrix}=\begin{bmatrix} I & 0 \\ 0 & -I\end{bmatrix}=J$$

Specifically, $H_2(\xi)=[Q_x^{-1}(\tilde{C}_2^{\mathrm{T}}(x)+\bar{C}_2^{\mathrm{T}}(\xi))+B_1(x)D_{21}^{\mathrm{T}}(x)]\tilde{R}^{-1}$ is the measurement feedback gain of the central controller as shown by Isidori [4].

As a summary of the discussion so far, we state the following theorem as a conclusion.

*Theroem 2*

Under Assumptions $A1$–$A4$.

(i) Suppose there exists a $C^1$ nonnegative function $V(x)$ locally defined in a neighbourhood of the origin (with $V(0)=0$), and $V(x)$ satisfies Equations (16), (17) such that $G_1$ as shown in Remark 4 is $(J, J')$-lossless.

(ii) Suppose there exists a $C^2$ nonnegative function $W(x_e)=Q(x-\xi)$ that is locally defined in a neighbourhood of $(x,\xi)=(0,0)$ and vanishes at $(x,\xi)=(0,0)$, and $W(x_e)$ satisfies Equations (19), (20) such that $\tilde{\Pi}G_2$ as shown in Remark 5 is conjugate $(-J,-J')$-lossless.

Then, the problem of local disturbance attenuation with internal stability is solved by a family of output feedback controllers $\tilde{\Pi}$ (with a free stable system $\Phi(\sigma)$) as shown in Equation (18).

*A brief sketch of the proof* From Equation (18), since $v=\Phi(\sigma)$ with $\|\Phi(\sigma)\|_{L_2}\leqslant 1$ and $\tilde{\Pi}G_2$ is conjugate $(-J,-J')$-lossless, it immediately follows from Remark 5 that

$$\int_0^{\mathrm{T}}(\|v(t)\|^2-\|\sigma(t)\|^2\,\mathrm{d}t\leqslant 0\Rightarrow\int_0^{\mathrm{T}}(\|z'(t)\|^2-\|w'(t)\|^2)\,\mathrm{d}t\leqslant 0$$

Furthermore, as proposed in Remark 2, having $G_1$ is $(J,J')$-lossless implies

$$\int_0^{\mathrm{T}}(\|z'(t)\|^2-\|w'(t)\|^2)\,\mathrm{d}t\leqslant 0\Rightarrow\int_0^{\mathrm{T}}(\|z(t)\|^2-\|w(t)\|^2\,\mathrm{d}t\leqslant 0$$

This means that the $L_2$-gain of the closed-loop system (from $w$ to $z$) is less than or equal to a prescribed number $\gamma$ ($\gamma=1$).

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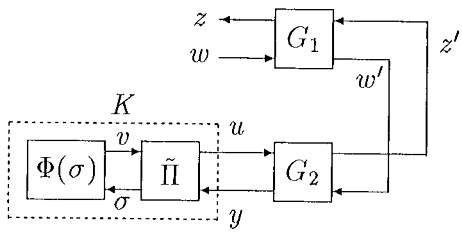


Figure 3.

To prove the internal stability, it suffices to prove the exponential stability of the closed-loop system, which is obtained from the linear approximation result (see Hong and Teng [7]). □

Theorem 2 can be illustrated in Figure 3.

## 5. CONCLUSION

We have extended a class of the chain-scattering approach from the linear $H^\infty$ control problem to the case of local disturbance attenuation with internal stability, via measurement feedback, in nonlinear affine systems. We have also stated the sufficient conditions for the existence of output feedback controllers. Because the fictitious signals introduced by traditional chain-scattering approach are thought superfluous in the $H^\infty$ control problem, a nonlinear outer-(conjugate $(J, J')$-inner) coprime factorization is proposed. As shown in the block diagrams, the nonlinear plant is thus described as serial energy-losslessness systems, which simplifies the solving process and provides deeper insight in the synthesis of the controllers.

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