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Some classes of four-weight spin models ☆

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Abstract

We classify four-weight spin models $(X; W_1, W_2, W_3, W_4)$ with sizes at most four. We also study the class of four-weight spin models with exactly two distinct values in W_4, reduce their existence to that of certain symmetric designs.

MSC: 05E

Keywords: Four-weight spin model; Bose–Mesner algebras; Symmetric design

1. Introduction

The notion of *spin models* is one of the statistical mechanical models introduced by Jones (1989) for constructing invariants of knots and links. It was generalized to *two-weight spin models* (also called generalized spin models) by Kawagoe et al. (1994) by removing the condition of symmetry. Recently, Bannai and Bannai (1995) made a further generalization by defining *four-weight spin models* (or generalized generalized spin models).

Jaeger (1992) first discovered that *association schemes* and their *Bose–Mesner algebras* are a natural place to look for spin models. There are many recent works on the connections between spin models and association schemes, see (Bannai, 1993) for a survey on this topic. A general theory of four-weight spin models in connection with Bose–Mesner algebras was developed by the present authors (Guo and Huang, 2000).

In this paper, two special classes of four-weight spin models are studied. In Section 2, we give necessary preliminaries for four-weight spin models, association schemes and Bose–Mesner algebras, and we then recall some general facts on four-weight spin

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models derived in Guo and Huang (2000). Section 3 deals with the classification of four-weight spin models of sizes at most four. In the final section, the class of four-weight spin models with exactly two distinct values on W_4 is studied and their existence is reduced to the existence of certain *symmetric designs*.

2. Preliminaries

In this section, we present some background on four-weight spin models, association schemes and Bose–Mesner algebras. For more details on these objects, see Bannai and Bannai (1995), Bannai and Ito (1984), Guo and Huang (2000).

Definition 2.1. Let X be a finite set with $|X|=n=D^2$, and let w_i $(i=1,2,3,4)$ be functions on $X\times X$ to the complex numbers. Then $(X;w_1,w_2,w_3,w_4)$ is a four-weight spin model of loop variable D of size n if the following conditions are satisfied for any α, β and $\gamma\in X$:

(1) $w_1(\alpha,\beta)w_3(\beta,\alpha)=w_2(\alpha,\beta)w_4(\beta,\alpha)=1$,
(2) $\sum_{x\in X}w_1(\alpha,x)w_3(x,\beta)=\sum_{x\in X}w_2(\alpha,x)w_4(x,\beta)=n\delta_{\alpha\beta}$,
(3a) $\sum_{x\in X}w_1(\alpha,x)w_1(x,\beta)w_4(\gamma,x)=Dw_1(\alpha,\beta)w_4(\gamma,\alpha)w_4(\gamma,\beta)$,
(3b) $\sum_{x\in X}w_1(x,\alpha)w_1(\beta,x)w_4(x,\gamma)=Dw_1(\beta,\alpha)w_4(\alpha,\gamma)w_4(\beta,\gamma)$.

Conditions (3a) and (3b) are called the *star-triangle conditions*. They play an important role in the study of four-weight spin models. The above definition can also be expressed in terms of matrices. For $i\in\{1,2,3,4\}$, let $W_i=(w_i(\alpha,\beta))_{\alpha,\beta\in X}$ denote complex matrices, I the identity matrix, and J the all-one matrix. Denote $Y^{ij}_{\alpha,\beta}$ the n-dimensional column vector whose x-entry is given by $Y^{ij}_{\alpha\beta}(x)=w_i(\alpha,x)w_j(x,\beta)$ for $i,j\in\{1,2,3,4\}$ and $\alpha,\beta\in X$. Then the expressions of conditions in Definition 2.1 are

1. ${}^tW_1\circ W_3={}^tW_2\circ W_4=J$,
2. $W_1W_3=W_2W_4=nI$,
3a. $W_1Y^{41}_{\alpha\beta}=DW_4(\alpha,\beta)Y^{41}_{\alpha\beta}$,
3b. ${}^tW_1Y^{14}_{\beta\alpha}=DW_4(\beta,\alpha)Y^{14}_{\beta\alpha}$.

Remarks. (1) If there are two matrices W_+ and W_- such that $W_1=W_2=W_+$ and $W_3=W_4=W_-$, then it is easily seen that the above definition is just the definition given for two-weight spin model in Kawagoe et al. (1994). Moreover, if W_+ and W_- are symmetric, then it is exactly the one introduced in Jones (1989).

(2) Let $(X;W_1,W_2,W_3,W_4)$ be a four-weight spin model, by definition, (W_3,W_2) can be determined by (W_1,W_4), so we will focus on the pair (W_1,W_4).

Bannai and Bannai (1995) showed that, assuming conditions (1) and (2) in Definition 2.1, there are 16 equations which can be separated into two groups of eight each including (3a) $(=\mathrm{III}_1)$ and (3b) $(=\mathrm{III}_{14})$, respectively, in such a way that all equations in one group are pairwise equivalent.

Theorem 2.2 (Bannai and Bannai, 1995, Theorem 1). *Under the condition* (1) *and* (2) *in Definition* 2.1, *the following conditions* III_1–III_8 *are equivalent to each other*, *as well as* III_9–III_{16}, *for any* $\alpha,\beta\in X$.

$$\mathrm{III}_1{:}W_1Y^{41}_{\alpha\beta}=DW_4(\alpha,\beta)Y^{41}_{\alpha\beta},\quad \mathrm{III}_2{:}W_4Y^{13}_{\alpha\beta}=DW_1(\alpha,\beta)Y^{24}_{\alpha\beta},$$
$$\mathrm{III}_3{:}{}^tW_3Y^{32}_{\alpha\beta}=DW_2(\alpha,\beta)Y^{32}_{\alpha\beta},\quad \mathrm{III}_4{:}{}^tW_2Y^{13}_{\alpha\beta}=DW_3(\alpha,\beta)Y^{24}_{\alpha\beta},$$
$$\mathrm{III}_5{:}W_3Y^{41}_{\alpha\beta}=DW_2(\beta,\alpha)Y^{41}_{\alpha\beta},\quad \mathrm{III}_6{:}W_2Y^{24}_{\alpha\beta}=DW_3(\beta,\alpha)Y^{13}_{\alpha\beta},$$
$$\mathrm{III}_7{:}{}^tW_1Y^{32}_{\alpha\beta}=DW_4(\beta,\alpha)Y^{32}_{\alpha\beta},\quad \mathrm{III}_8{:}{}^tW_4Y^{24}_{\alpha\beta}=DW_1(\beta,\alpha)Y^{13}_{\alpha\beta},$$

$$\mathrm{III}_9{:}W_1Y^{23}_{\alpha\beta}=DW_4(\beta,\alpha)Y^{23}_{\alpha\beta},\quad \mathrm{III}_{10}{:}{}^tW_3Y^{14}_{\alpha\beta}=DW_2(\beta,\alpha)Y^{14}_{\alpha\beta},$$
$$\mathrm{III}_{11}{:}W_4Y^{42}_{\alpha\beta}=DW_1(\beta,\alpha)Y^{31}_{\alpha\beta},\quad \mathrm{III}_{12}{:}{}^tW_2Y^{42}_{\alpha\beta}=DW_3(\beta,\alpha)Y^{31}_{\alpha\beta},$$
$$\mathrm{III}_{13}{:}W_3Y^{23}_{\alpha\beta}=DW_2(\alpha,\beta)Y^{23}_{\alpha\beta},\quad \mathrm{III}_{14}{:}{}^tW_1Y^{14}_{\alpha\beta}=DW_4(\alpha,\beta)Y^{14}_{\alpha\beta},$$
$$\mathrm{III}_{15}{:}W_2Y^{31}_{\alpha\beta}=DW_3(\alpha,\beta)Y^{42}_{\alpha\beta},\quad \mathrm{III}_{16}{:}{}^tW_4Y^{31}_{\alpha\beta}=DW_1(\alpha,\beta)Y^{42}_{\alpha\beta}.$$

It is clear from III_1 and III_5 that $DW_4(\alpha,\beta)$ and $DW_2(\alpha,\beta)$ are eigenvalues of W_1 and W_3, respectively, in the four-weight spin model $(X;W_1,W_2,W_3,W_4)$. Furthermore, the following is known, for example see Guo and Huang (2000). This theorem is used in the classification of them with four vertices in Theorem 3.7.

Theorem 2.3 (Guo and Huang, 2000). *Each column and row of* W_4 *is a permutation of the multiset* D^{-1} *Spectrum*(W_1).

Let $\alpha=\beta$ in III_6, III_8, III_{11} and III_{15}, respectively, and apply the conditions (1) and (2), then there exists a non-zero complex number a, called the *modulus* of the four-weight spin model, such that

4. $W_1\circ I=aI,\ W_3\circ I=a^{-1}I,$
5. $W_4J=JW_4=DaJ,\ W_2J=JW_2=Da^{-1}J.$

Remark. Note that the diagonal elements of W_1 (resp., W_3) are constants, but not necessary for the diagonal elements of W_4 (resp., W_2).

From the star-triangle conditions and their equivalent equations, some more important properties of four-weight spin models can be derived.

Lemma 2.4 (Guo and Huang, 2000, Lemma 3.2). *For a four-weight spin model* $(X;W_1,W_2,W_3,W_4)$,

1. ${}^tW_1\circ W_1=aD^{-1}(W_4\circ W_4)W_2=aD^{-1}W_2(W_4\circ W_4),$
2. ${}^tW_4W_4=W_4\,{}^tW_4=a\,{}^tW_3\circ(W_1W_1).$

Proof. (1) Setting $\beta=\gamma$ in III_{11},

$$\sum_x W_2(x,\alpha)W_4(\beta,x)W_4(\beta,x)=DW_1(\alpha,\beta)W_1(\beta,\alpha)W_3(\beta,\beta),$$

it follows that

$$
\begin{aligned}
(W_4 \circ W_4)W_2(\beta,\alpha) &= \sum_x W_2(x,\alpha)W_4(\beta,x)W_4(\beta,x) \\
&= Da^{-1}W_1(\beta,\alpha)W_1(\alpha,\beta) \\
&= Da^{-1}({}^tW_1 \circ W_1)(\beta,\alpha).
\end{aligned}
$$

(2) Summing both sides of III_1 over all γ, we have

$$\sum_x W_1(\alpha,x)W_1(x,\beta)\sum_\gamma W_4(\gamma,x) = DW_1(\alpha,\beta)\sum_\gamma W_4(\gamma,\alpha)W_4(\gamma,\beta).$$

The left-hand side equals $Da\sum_x W_1(\alpha,x)W_1(x,\beta) = Da({}^tW_1W_1)(\alpha,\beta)$ and the right-hand side equals $DW_1(\alpha,\beta)({}^tW_4W_4)(\alpha,\beta)$, it follows that

$${}^tW_4W_4(\alpha,\beta) = aW_1^{-1}(\alpha,\beta)(W_1W_1)(\alpha,\beta) = a\,{}^tW_3 \circ (W_1W_1)(\alpha,\beta)$$

and hence ${}^tW_4W_4 = a^tW_3 \circ (W_1W_1)$. Similarly, summing both sides of III_{14} over all γ, we have $W_4^tW_4 = aW_3 \circ {}^t(W_1W_1)$. Since ${}^tW_4W_4$ is symmetric, ${}^tW_4W_4 = aW_3 \circ {}^t(W_1W_1) = W_4^tW_4$ as required. □

Lemma 2.5. *For a four-weight spin model* $(X; W_1, W_2, W_3, W_4)$, *and any* $x \in X$,

1. $Y^{11}_{\alpha\beta}(x) = D^{-1}W_1(\alpha,\beta)\sum_r \dfrac{W_4(\gamma,\alpha)W_4(\gamma,\beta)}{W_4(\gamma,x)}$,
2. $W_1(\alpha,\beta) = \dfrac{W_1(\alpha,x)}{W_1(\beta,x)} \dfrac{a\sum_r W_4(\gamma,\beta)^2/W_4(\gamma,x)}{\sum_r W_4(\gamma,\alpha)W_4(\gamma,\beta)/W_4(\gamma,x)}$.

Proof. (1) From III_1,

$$\sum_x W_1(\alpha,x)W_1(x,\beta)W_4(\gamma,x) = (W_4Y^{11}_{\alpha\beta})(\gamma) = DW_1(\alpha,\beta)W_4(\gamma,\alpha)W_4(\gamma,\beta),$$

and $W_2W_4 = nI$, thus

$$
\begin{aligned}
Y^{11}_{\alpha\beta}(x) &= Dn^{-1}W_1(\alpha,\beta)\sum_r W_2(x,\gamma)W_4(\gamma,\alpha)W_4(\gamma,\beta) \\
&= D^{-1}W_1(\alpha,\beta)\sum_r \frac{W_4(\gamma,\alpha)W_4(\gamma,\beta)}{W_4(\gamma,x)}.
\end{aligned}
$$

(2) Note that $Y^{11}_{\beta\beta}(x) = D^{-1}a\sum_r W_4^2(\gamma,\beta)/W_4(\gamma,x)$ from (1), then

$$
\begin{aligned}
W_1(\alpha,\beta) &= \frac{DY^{11}_{\alpha\beta}(x)}{\sum_r W_4(\gamma,\alpha)W_4(\gamma,\beta)/W_4(\gamma,x)} \\
&= \frac{DW_1(\alpha,x)W_1(x,\beta)}{\sum_r W_4(\gamma,\alpha)W_4(\gamma,\beta)/W_4(\gamma,x)}
\end{aligned}
$$

$$= \frac{W_1(\alpha,x)}{W_1(\beta,x)} \frac{DY^{11}_{\beta\beta}(x)}{\sum_r W_4(\gamma,\alpha)W_4(\gamma,\beta)/W_4(\gamma,x)}$$

$$= \frac{W_1(\alpha,x)}{W_1(\beta,x)} \frac{a\sum_r W_4^2(\gamma,\beta)/W_4(\gamma,x)}{\sum_r W_4(\gamma,\alpha)W_4(\gamma,\beta)/W_4(\gamma,x)}. \qquad \square$$

Based on this lemma, W_1 can be decomposed into the product of two diagonal matrices and a symmetric matrix, which is significant for later developments of four-weight spin models. With respect to, a fixed point $x\in X$, we define a complex number c_α as

$$c_\alpha = \sqrt{Y^{11}_{\alpha\alpha}(x)} = \sqrt{aD^{-1}\sum_{\gamma\in X}\frac{W_4^2(\gamma,\alpha)}{W_4(\gamma,x)}}.$$

Let Δ be a diagonal matrix with the diagonal $\Delta_\alpha = c_\alpha^{-1}W_1(\alpha,x)$ for each $\alpha\in X$. In this notation, we have

$$\begin{aligned} W_1(\alpha,\beta) &= a\frac{W_1(\alpha,x)}{W_1(\beta,x)} \frac{\sum_r W_4^2(\gamma,\beta)/W_4(\gamma,x)}{\sum_r W_4(\gamma,\alpha)W_4(\gamma,\beta)/W_4(\gamma,x)} \\ &= a\frac{c_\alpha\Delta_\alpha}{c_\beta\Delta_\beta} \frac{Da^{-1}c_\beta^2}{\sum_r W_4(\gamma,\alpha)W_4(\gamma,\beta)/W_4(\gamma,x)} \\ &= a\frac{\Delta_\alpha}{\Delta_\beta} \frac{Da^{-1}c_\alpha c_\beta}{\sum_r W_4(\gamma,\alpha)W_4(\gamma,\beta)/W_4(\gamma,x)}. \end{aligned}$$

With the same notation, the above observation proves the following lemma which will be used in Theorem 3.1.

Theorem 2.6. *For a four-weight spin model $(X;W_1,W_2,W_3,W_4)$ and a fixed point $x\in X$, let S be the symmetric matrix with (α,β)-entry*

$$S(\alpha,\beta) = \frac{Da^{-1}c_\alpha c_\beta}{\sum_r W_4(\gamma,\alpha)W_4(\gamma,\beta)/W_4(\gamma,x)}.$$

Then $W_1 = a\Delta S\Delta^{-1}$.

Recall that a $(d+1)$-tuple $\mathscr{A}=(A_0,A_1,\ldots,A_d)$ of $(0,1)$-matrices indexed by X is called a d-class *association scheme* on X if

(1) $A_0 = I$,
(2) $\sum_{i=0}^d A_i = J$,
(3) for every i, there exists i' with ${}^tA_i = A_{i'}$, and
(4) there exist integers p_{ij}^k for all $i,j,k\in\{0,l,\ldots,d\}$ such that $A_iA_j = \sum_{k=0}^d p_{ij}^k A_k$.

It is called *commutative* if $p_{ij}^k = p_{ji}^k$ for all $i,j,k\in\{0,1,\ldots,d\}$, and *symmetric* if ${}^tA_i = A_i$ for every i. For an association scheme $\mathscr{A}=(A_0,A_1,\ldots,A_d)$, the matrix $\sum_{i=0}^d iA_i$, is called the relation matrix of $\mathscr{A}$. The *Bose–Mesner algebra* of an association scheme $\mathscr{A}=(A_0,A_1,\ldots,A_d)$ is the linear span of the adjacency matrices $\{A_i \mid i=0,1,\ldots,d\}$.

The following theorem describes some combinatorial structures of four-weight spin models in terms of Bose–Mesner algebras.

Theorem 2.7 (Guo and Huang, 2000, Theorem 4.4). *${}^tW_4W_4 = W_4{}^tW_4, {}^tW_2W_2 = W_2{}^tW_2$, ${}^tW_1 \circ W_1$ and ${}^tW_3 \circ W_3$ are in the Bose–Mesner algebra of a symmetric association scheme.*

3. The case of size at most four

In this section, we study four-weight spin models of small sizes. The conditions of Definition 2.1 are rather easy when it consists of only two or three vertices. In the following theorem, we present all such possible four-weight spin models in the form given in Lemma 2.6. The proof is straightforward and hence is omitted.

Theorem 3.1. (1) *If $(X; W_1, W_2, W_3, W_4)$ is a four-weight spin model with two vertices, then*

$$W_1 = a\begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix}\begin{pmatrix} 1 & \gamma \\ \gamma & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix}^{-1},$$

$$W_4 = Da\frac{1-i}{2}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$$

where $i^2 = -1$, $\gamma^2 = -i$, and a, t are non-zero complex numbers.

(2) *If $(X; W_1, W_2, W_3, W_4)$ is a four-weight spin model with three vertices, then*

$$W_1 = a\begin{pmatrix} 1 & 0 & 0 \\ 0 & t_1 & 0 \\ 0 & 0 & t_2 \end{pmatrix}\begin{pmatrix} 1 & \alpha & \alpha \\ \alpha & 1 & \alpha \\ \alpha & \alpha & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & t_1 & 0 \\ 0 & 0 & t_2 \end{pmatrix}^{-1},$$

$$W_4 = \frac{Da}{2+\alpha}W_4',$$

where, up to simultaneous permutation of rows and columns, W_4' is one of the following three matrices:

$$\begin{pmatrix} \alpha & 1 & 1 \\ 1 & \alpha & 1 \\ 1 & 1 & \alpha \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & \alpha \\ \alpha & 1 & 1 \\ 1 & \alpha & 1 \end{pmatrix}, \quad \begin{pmatrix} \alpha & 1 & 1 \\ 1 & 1 & \alpha \\ 1 & \alpha & 1 \end{pmatrix},$$

α is a primitive third root of unity, and a, t_1, t_2 are non-zero complex numbers.

From now on, we suppose $(X; W_1, W_2, W_3, W_4)$ is a four-weight spin model with four vertices in the rest of this section. By Theorem 2.3, each row and column of W_4 consists of a multiset, say $\{a, b, c, d\}$. Without loss of generality, we may assume that the first row of W_4 is (a,b,c,d). Based on the fact that the 2nd, 3rd and 4th rows of W_4 are permutations of the 1st row (a,b,c,d), an exhaustive search shows that all other rows of W_4 must be one of the following types:

$$\begin{aligned}
\text{(i)}&: \{(b,a,d,c),\ (c,d,a,b),\ (d,c,b,a)\},\\
\text{(ii)}&: \{(b,a,d,c),\ (c,d,b,a),\ (d,c,a,b)\},\\
\text{(iii)}&: \{(b,c,d,a),\ (c,d,a,b),\ (d,a,b,c)\},\\
\text{(iv)}&: \{(b,d,a,c),\ (c,a,d,b),\ (d,c,b,a)\}.
\end{aligned}$$

Since $W_4W_2 = nI$, we have the following equations corresponding to these four cases, respectively,

$$\text{(i)}: \begin{cases} ba^{-1}+ab^{-1}+dc^{-1}+cd^{-1}=0 & (1)\\ ca^{-1}+db^{-1}+ac^{-1}+bd^{-1}=0 & (2)\\ da^{-1}+cb^{-1}+bc^{-1}+ad^{-1}=0 & (3)\end{cases}$$

$$\text{(ii)}: \begin{cases} ba^{-1}+ab^{-1}+dc^{-1}+cd^{-1}=0 & (4)\\ ac^{-1}+bd^{-1}+cb^{-1}+da^{-1}=0 & (5)\\ ad^{-1}+bc^{-1}+ca^{-1}+db^{-1}=0 & (6)\end{cases}$$

$$\text{(iii)}: \begin{cases} ba^{-1}+cb^{-1}+dc^{-1}+ad^{-1}=0 & (7)\\ ca^{-1}+db^{-1}+ac^{-1}+db^{-1}=0 & (8)\\ da^{-1}+ab^{-1}+bc^{-1}+cd^{-1}=0 & (9)\end{cases}$$

$$\text{(iv)}: \begin{cases} ba^{-1}+db^{-1}+ac^{-1}+cd^{-1}=0 & (10)\\ ca^{-1}+ab^{-1}+dc^{-1}+bd^{-1}=0 & (11)\\ da^{-1}+cb^{-1}+bc^{-1}+ad^{-1}=0 & (12)\end{cases}$$

It follows that

$$\begin{aligned}
\text{(i)}&: (a-d)(b-c)(ad+bc)=0 && \text{by (1)–(2)}\\
&\ (a-c)(b-d)(ac+bd)=0 && \text{by (1)–(3)}\\
&\ (a-b)(c-d)(ab+cd)=0 && \text{by (2)–(3)}\\
\text{(ii)}&: (a-b)(c-d)(ab-cd)=0 && \text{by (5)–(6)}\\
\text{(iii)}&: (a-c)(b-d)(ac-bd)=0 && \text{by (7)–(9)}\\
\text{(iv)}&: (a-d)(b-c)(ad-bc)=0 && \text{by (10)–(11).}
\end{aligned}$$

The case of pairwise distinct a,b,c,d will be ruled out in the following way:

1. *Case (i)*: since $ad+bc=ac+bd=ab+cd=0$, and then $a^2=b^2=c^2=d^2$, a contradiction.

2. *Case* (*ii*): since $ab = cd$, then we have $bc^{-1} + cb^{-1} + db^{-1} + bd^{-1} = 0$ from (6) and then $(a + b)(c + d) = 0$, it follows that either $a + b = 0$ or $c + d = 0$, and hence either $c = d$ or $a = b$, respectively, by (4), a contradiction.
3. *Cases* (*iii*) *and* (*iv*) can be ruled out as done in case (ii).

Whereas the case that $a = b = c = d$ can be ruled out simply by $W_4W_2 = nI$. The above observations are summarized in the following lemma.

Lemma 3.2. *Without loss of generality, the multiset* $\{a, b, c, d\}$ *must be one of the following,*

(1) $a = b = c = -d$,
(2) $a = c$, $b = -d$, *but* $a^2 \neq b^2$.

As for case (1) of Lemma 3.2, we have the following proposition.

Proposition 3.3. *Let* $(X; W_1, W_2, W_3, W_4)$ *be a four-weight spin model with* $|X| = 4$. *Then the multiset* $\{a, b, c, d\}$ *satisfies condition* (1) *of Lemma* 3.2 *if and only if*

$$W_1 = \frac{2a}{D} T(2I - J)T^{-1},$$

$$W_4 = a(2E_1 - J),$$

where T *is an invertible diagonal matrix and* E_1 *a permutation matrix of size four.*

Now we consider case (2) and assume that each column and each row consists of the multiset $\{a, a, b, -b\}$ $(a^2 \neq b^2)$, hence W_4 can be written as $aE_1 + bE_2 + (-b)E_3$, where each E_i is a $(0, 1)$-matrix with $E_1J = JE_1 = 2J$, $E_1 + E_2 + E_3 = J$, and moreover E_2 and E_3 are permutation matrices. Clearly $E_2\,{}^tE_2 = E_3\,{}^tE_3 = I$. To determine $E_1\,{}^tE_1$, let $E_1 = A_1 + A_2$ for some permutation matrices A_1 and A_2, then

$$\begin{aligned} E_1\,{}^tE_1 - 2I &= (A_1 + A_2)({}^tA_1 + {}^tA_2) - 2I \\ &= A_1\,{}^tA_2 + A_2\,{}^tA_1. \end{aligned}$$

Let $B = A_1\,{}^tA_2$, then B is a permutation matrix since both $A_1\,{}^tA_2$ and $A_2\,{}^tA_1$ are. Therefore,

$$E_1\,{}^tE_1 - 2I = \begin{cases} B + {}^tB & \text{if } B \text{ is not symmetric}, \\ 2B & \text{if } B \text{ is symmetric}. \end{cases}$$

The case of non-symmetric B will be ruled out by Lemma 3.4, and the case of symmetric B will be treated later in Lemmas 3.5 and 3.6.

Lemma 3.4. *If* B *is not symmetric, then*

(1) $E_1\,{}^tE_2 \neq E_1\,{}^tE_3$,
(2) *neither* $E_1\,{}^tE_2$ *nor* $E_1\,{}^tE_3$ *is symmetric*,
(3) $a^2 = b^2$.

Proof. (1) Suppose, to the contradictory, that $E_1\,{}^tE_2 = E_1\,{}^tE_3$. Note that if a permutation matrix B of order four is not symmetric, then B and ${}^tB$ have no common non-zero entries and

$$E_1J = E_1\,{}^tE_1 + 2E_1\,{}^tE_2 = 2I + B + {}^tB + 2E_1\,{}^tE_2,$$

contradicting $E_1J = 2J$.

(2) Assume that either $E_1\,{}^tE_2$ or $E_1\,{}^tE_3$ is symmetric. Observe that

$$E_3\,{}^tE_2 = (J - E_1 - E_2)\,{}^tE_2 = J - E_1\,{}^tE_2 - I,$$

$$E_2\,{}^tE_3 = (J - E_1 - E_3)\,{}^tE_3 = J - E_1\,{}^tE_3 - I,$$

so one of $E_2\,{}^tE_3$ and $E_3\,{}^tE_2$ is also symmetric, i.e., $E_2\,{}^tE_3 = E_3\,{}^tE_2$. Hence, $E_1\,{}^tE_2 = E_1\,{}^tE_3$, a contradiction of (1).

(3) Let $P_1 = J - I - E_1\,{}^tE_3$, $P_2 = J - I - E_1\,{}^tE_2$ and $P_3 = J + I - E_1\,{}^tE_1 = J - I - (B + {}^tB)$. Then P_i, $i = 1, 2, 3$, are permutation matrices and P_3 is symmetric. Furthermore, $P_1 + P_2 = B + {}^tB$, $P_1 + P_2 + P_3 = J - I$. By (2), neither $E_1\,{}^tE_2 = P_1 + P_3$ nor $E_1\,{}^tE_3 = P_2 + P_3$ is symmetric, forcing that ${}^tP_1 = P_2$. Moreover

$$E_1\,{}^tE_2 - E_1\,{}^tE_3 = P_3 + P_1 - P_3 - P_2 = P_1 - P_2,$$

$$E_2\,{}^tE_1 + E_3\,{}^tE_1 = P_3 + P_1 + P_3 + P_2 = 2P_3 + P_1 + P_2,$$

$$E_2\,{}^tE_3 = P_1 \quad \text{and} \quad E_3\,{}^tE_2 = P_2$$

and hence

$$\begin{aligned}(W_4 \circ W_4)W_2 &= (a^2E_1 + b^2E_2 + b^2E_3)(a^{-1}\,{}^tE_1 + b^{-1}\,{}^tE_2 - b^{-1}\,{}^tE_3)\\ &= aE_1\,{}^tE_1 + a^2b^{-1}E_1\,{}^tE_2 - a^2b^{-1}E_1\,{}^tE_3 + b^2a^{-1}E_2\,{}^tE_1\\ &\quad + bE_2\,{}^tE_2 - bE_2\,{}^tE_3 + b^2a^{-1}E_3\,{}^tE_1 + bE_3\,{}^tE_2 - bE_3\,{}^tE_3\\ &= aE_1\,{}^tE_1 + a^2b^{-1}(E_1\,{}^tE_2 - E_1\,{}^tE_3)\\ &\quad + b^2a^{-1}(E_2\,{}^tE_1 + E_3\,{}^tE_1) - b(E_2\,{}^tE_3 - E_3\,{}^tE_2)\\ &= 2aI + a(P_1 + P_2) + a^2b^{-1}(P_1 - P_2)\\ &\quad + b^2a^{-1}(2P_3 + P_1 + P_2) - b(P_1 - P_2)\\ &= 2aI + 2b^2a^{-1}P_3 + (a + a^2b^{-1} + b^2a^{-1} - b)P_1\\ &\quad + (a - a^2b^{-1} + b^2a^{-1} + b)P_2.\end{aligned}$$

Since ${}^tW_1 \circ W_1 = aD^{-1}(W_4 \circ W_4)W_2$ by Lemma 2.4, it is clear that $(W_4 \circ W_4)W_2$ is symmetric. Taking the transpose of $(W_4 \circ W_4)W_2$, since ${}^tP_1 = P_2$, we have

$$\begin{aligned}&2aI + 2b^2a^{-1}P_3 + (a + a^2b^{-1} + b^2a^{-1} - b)P_1 + (a - a^2b^{-1} + b^2a^{-1} + b)P_2\\ &\quad = 2aI + 2b^2a^{-1}P_3 + (a + a^2b^{-1} + b^2a^{-1} - b)P_2 + (a - a^2b^{-1} + b^2a^{-1} + b)P_1.\end{aligned}$$

But P_1 and P_2 have no common non-zero entries, so $a + a^2b^{-1} + b^2a^{-1} - b = a - a^2b^{-1} + b^2a^{-1} + b$, and hence $a^2 = b^2$. □

Lemma 3.5. *If B is symmetric, then*

(1) $E_1\,{}^tE_2 = E_1\,{}^tE_3 = E_2\,{}^tE_1 = E_3\,{}^tE_1$ $(= J - I - B)$,
(2) $E_2\,{}^tE_3 = E_3\,{}^tE_2$ $(= B)$.

Proof. Since $E_1J = JE_1 = 2J$, $E_2J = JE_2 = E_3J = JE_3 = J$, we have $(E_1\,{}^tE_2)J = J(E_1\,{}^tE_2) = (E_1\,{}^tE_3)J = J(E_1\,{}^tE_3) = 2J$. Note that $E_1\,{}^tE_2 + E_1\,{}^tE_3 = 2J - 2I - 2B$ under the assumption and B is symmetric, it forces that $E_1\,{}^tE_2 = E_1\,{}^tE_3 = J - I - B$ must be symmetric. This proves (1). On the other hand, $E_2\,{}^tE_3 = E_2(J - {}^tE_1 - {}^tE_2) = J - E_2\,{}^tE_1 - I = J - (J - I - B) - I = B$, this proves (2). □

Corollary 3.6. *If B is symmetric, then*

(1) $W_4\,{}^tW_4 = 2(a^2 + b^2)I + 2(a^2 - b^2)B$,
(2) *B is one of the following*:

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

(3) *Up to simultaneous permutation of rows and columns, E_1 is one of the following three matrices*:

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix}.$$

Proof. (1) By Lemma 3.5,

$$\begin{aligned} W_4\,{}^tW_4 &= (aE_1 + bE_2 - bE_3)(a\,{}^tE_1 + b\,{}^tE_2 - b\,{}^tE_3) \\ &= a^2E_1\,{}^tE_1 + abE_1\,{}^tE_2 - abE_1\,{}^tE_3 + abE_2\,{}^tE_1 \\ &\quad + b^2E_2\,{}^tE_2 - b^2E_2\,{}^tE_3 - abE_3\,{}^tE_1 - b^2E_3\,{}^tE_2 + b^2E_3\,{}^tE_3 \\ &= 2(a^2 + b^2)I + 2(a^2 - b^2)B. \end{aligned}$$

(2) Since B is a symmetric permutation matrix whose diagonal entries are all zero, we have (2).

(3) Observe that $E_1\,{}^tE_1 = 2I + 2B$, up to simultaneous permutation of rows and columns,

$$E_1\,{}^tE_1 = \begin{pmatrix} 2 & 2 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 2 & 2 \end{pmatrix}.$$

By exhaustive searching, E_1 is one of the three matrices given in (3). □

From Lemma 3.6, W_4 must be of the form

$$W_4 = aE_1 + bE_2 - bE_3,$$

where E_1 is one of three matrices given in (3), E_2 and E_3 are permutation matrices of order four with $E_1 + E_2 + E_3 = J$. Since $W_4J = 2aJ$, the modulus is equal to $2a/D$. In Section 2, we decompose W_1 into the product of diagonal matrices and a symmetric matrix, i.e., $W_1 = (2a/D)\Delta S \Delta^{-1}$ with

$$S(\alpha,\beta) = \frac{D((2a/D))^{-1}c_\alpha c_\beta}{\sum_r W_4(\gamma,\alpha)W_4(\gamma,\beta)/W_4(\gamma,x)}$$

for some fixed point $x \in X$. Now let $x = 1$, then for all cases of W_4, $c_1 = c_2 = a$, $c_3 = c_4 = b$ and

$$\sum_r \frac{W_4(\gamma,\alpha)W_4(\gamma,\beta)}{W_4(\gamma,1)} = \begin{cases} 2a & \text{if } (\alpha,\beta) = (1,2) \text{ or } (1,3) \text{ or } (1,4), \\ -2a & \text{if } (\alpha,\beta) = (2,3) \text{ or } (2,4), \\ -2b^2a^{-1} & \text{if } (\alpha,\beta) = (3,4). \end{cases}$$

Hence

$$S = \begin{pmatrix} 1 & 1 & ba^{-1} & ba^{-1} \\ 1 & 1 & -ba^{-1} & -ba^{-1} \\ ba^{-1} & -ba^{-1} & 1 & -1 \\ ba^{-1} & -ba^{-1} & -1 & 1 \end{pmatrix}.$$

It is straightforward to check that the pair (W_1, W_4) defined above satisfies the star-triangle conditions, and hence $(X; W_1, W_2, W_3, W_4)$ is a four-weight spin model. The following theorem follows immediately.

Theorem 3.7. *If* $(X; W_1, W_2, W_3, W_4)$ *is a four-weight spin model with four vertices, then* W_1 and W_4 *are one of the following cases*:

(1) *Those given in Proposition* 3.3,

(2) $W_1 = \Delta \begin{pmatrix} a & a & b & b \\ a & a & -b & -b \\ b & -b & a & -a \\ b & -b & -a & a \end{pmatrix} \Delta^{-1},$

$W_4 = aE_1 + bE_2 - bE_3,$

where a and b are non-zero complex numbers with $a^2 \neq b^2$, Δ *is an invertible diagonal matrix, and* E_1 *is one of the following three matrices*:

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix}$$

and E_2, E_3 *are permutation matrices of order four with* $E_1 + E_2 + E_3 = J$.

Remark. Jaeger proved that for two four-weight spin models $(X; W_1, W_2, W_3, W_4)$ and $(X; W_1', W_2', W_3', W_4')$ with $W_1 = W_1'$, $W_3 = W_3'$, then these two spin models have the same associated link invariant. Hence, the four-weight spin models with two or three vertices associate a unique link invariant, and those with four vertices associate two link invariants with respect to those two classes of four-weight spin models given in Theorem 3.7.

4. The case with two values on W_4

In this section, we suppose $(X; W_1, W_2, W_3, W_4)$ is a four-weight spin model of size n, in which each row and column of W_4 consists of the multiset for distinct non-zero complex numbers α, β with multiplicities k and $n-k$, respectively, and where $1 \leqslant k \leqslant n-1$. Hence $W_4 = \alpha E_1 + \beta E_2$, where E_1, E_2 are $(0,1)$-matrix such that $E_1 + E_2 = J$, $E_1 J = J E_1 = kJ$ and $E_2 J = J E_2 = (n-k)J$. In the rest of this section, let $t_0 = \alpha$ and $t = \alpha^{-1}\beta$. Hence W_4 can be expressed as $W_4 = t_0(E_1 + tE_2)$ with $t_0 \neq 0$, and $t \neq 0, 1$.

First of all, we shall collect some information on $E_i\,{}^tE_j$ for $i,j \in \{1,2\}$. As a consequence, we show that the non-diagonal entries of $E_1\,{}^tE_1$ is a constant, and then derive a quadratic equation in t which pose some constraints over their existence.

Lemma 4.1. *There is a positive integer* λ *such that*

(1) $E_1\,{}^tE_1 = kI + \lambda(J - I)$,
(2) $E_2\,{}^tE_2 = (n-k)I + (n + \lambda - 2k)(J - I)$,
(3) $E_1\,{}^tE_2 = (k - \lambda)(J - I)$, *and*
(4) $t^2 + (n(k-\lambda)^{-1} - 2)t + 1 = 0$.

Proof. It is clear that the diagonals of $E_1\,{}^tE_1$, $E_2\,{}^tE_2$ and $E_1\,{}^tE_2$ are k, $n-k$ and 0, respectively. Denote λ_{ij}, α_{ij}, and β_{ij} (i,j)-entry of $E_1\,{}^tE_1$, $E_2\,{}^tE_2$ and $E_1\,{}^tE_2$, respectively, for $i \neq j$. Observe that $E_1 J = E_1({}^tE_1 + {}^tE_2) = kJ$ and $E_2 J = E_2({}^tE_1 + {}^tE_2) = (n-k)J$, so

$$\lambda_{ij} + \beta_{ij} = k,$$

$$\beta_{ij} + \alpha_{ij} = n - k.$$

Hence $\lambda_{ij} + \alpha_{ij} = n - 2\beta_{ij}$. Since

$$\begin{aligned} W_4 W_2 &= t_0(E_1 + tE_2)t_0^{-1}({}^tE_1 + t^{-1}\,{}^tE_2), \\ &= E_1\,{}^tE_1 + tE_2\,{}^tE_1 + t^{-1}E_1\,{}^tE_2 + E_2\,{}^tE_2, \\ &= nI + (E_1\,{}^tE_1 + E_2\,{}^tE_2 - nI) + (t + t^{-1})E_1\,{}^tE_2, \\ &= nI, \end{aligned}$$

we have

$$(E_1\,{}^tE_1 + E_2\,{}^tE_2 - nI) + (t + t^{-1})E_1\,{}^tE_2 = 0,$$

i.e., for any $i \neq j$,

$$\begin{aligned} \lambda_{ij} + \alpha_{ij} + (t + t^{-1})\beta_{ij} &= n - 2\beta_{ij} + (t + t^{-1})\beta_{ij}, \\ &= n + (t + t^{-1} - 2)\beta_{ij}, \\ &= 0. \end{aligned}$$

This means that β_{ij} is uniform for any $i \neq j$. Therefore λ_{ij} and α_{ij} are also uniform. Then (1), (2) and (3) follow by denoting $\lambda_{ij} = \lambda$. Furthermore, (4) follows by substituting $\beta_{ij} = k - \lambda$ into $n + (t + t^{-1} - 2)\beta_{ij} = 0$. □

Note that E_1 and E_2 are complementary each other, and that E_1 is simply a permutation matrix when $k = 1$. The following is such an example of four-weight spin models.

Example 4.2. Let X be a set of n elements, E_1 a permutation matrix of order n and $E_2 = J - E_1$. Let

$$\begin{aligned} W_1 &= aT^{-1}(I + t^{-1}(J - I))T, \\ W_4 &= t_0(E_1 + tE_2), \end{aligned}$$

where t_0 is a non-zero complex number, t is a root of $x^2 + (n-2)x + 1 = 0$ and T is an invertible diagonal matrix. Then $(X; W_1, W_2, W_3, W_4)$ is a four-weight spin model with modulus $a = D^{-1}t_0(1 + (n-1)t)$.

Lemma 4.1 shows that E_i $(i = 1, 2)$ can be interpreted in terms of the notion of combinatorial designs. A *block design* with parameters (n, k, λ) is a pair $(X, \mathscr{B})$ where X is an n-set (whose elements are called points) and $\mathscr{B}$ is a collection of some k-subsets (called blocks) of X such that any pair elements of X is contained in exactly λ blocks. It is well known that $|\mathscr{B}| \geqslant |X|$. A design is said to be *a symmetric design* if $|\mathscr{B}| = |X|$. For a symmetric design, the number of the common points of any two blocks is exactly λ. A block design is called a *quasi-symmetric design* with intersection x and y $(y > x)$ if the number of common points of any two blocks is either x or y. Let $(X, \mathscr{B})$ be a symmetric design (n, k, λ) and B be a block of $\mathscr{B}$, then the points of B and the intersections of B with the remaining blocks form a design with parameters $(k,\lambda,\lambda-1)$, which is called the *derived design* with respective to B.

Without loss of generality, we assume $k>2$ in the rest of this section and focus on the incidence structure $(X,\mathscr{B})$ with an incidence matrix E_1 which rows and columns indexed by $\mathscr{B}$ and X respectively. Note that E_1 is a square matrix and $E_1\,{}^tE_1=kI+\lambda(J-I)$, hence E_1 gives a 2-(n,k,λ) design with equal number of blocks and points, and hence a symmetric 2-(n,k,λ) design. This gives $k^2-n\lambda=k-\lambda$. Note also that $W_4J=DaJ$ in Theorem 2.3, it follows that $Da=t_0(k+(n-k)t)$. The expressions of $W_1\circ{}^tW_1$, $W_1(\alpha,\beta)$ given in Lemmas 2.4(1) and 2.5(2) can be made more precisely in terms of these conditions:

$$k^2-n\lambda=k-\lambda \qquad \text{(a)}$$

$$Da=t_0(k+(n-k)t) \qquad \text{(b)}$$

$$t^2+(n(k-\lambda)^{-1}-2)t+1=0 \qquad \text{(c)}$$

which in turn provides constraints on the existence of four-weight spin models with exactly two values in W_4.

Lemma 4.3. (1) $W_1\circ{}^tW_1=a^2I-t_0^2t(J-I)$.

(2) *For any three pairwise distinct blocks x, α, and β,*

(i)

$$\sum_r \frac{W_4(\gamma,\alpha)W_4(\gamma,\beta)}{W_4(\gamma,x)}=\left(k-\frac{n(\lambda-s)}{k-\lambda}\right)t_0(1-t),$$

where s is the number of common points in x, α and β,

(ii)

$$\left(\sum_r \frac{W_4(\gamma,\alpha)W_4(\gamma,\beta)}{W_4(\gamma,x)}\right)^2=-nt_0^2(1-t).$$

Proof. (1) Substituting

$$W_4\circ W_4=t_0^2(E_1+t^2E_2),$$

$$W_2=t_0^{-1}({}^tE_1+t^{-1}\,{}^tE_2)$$

into $W_1\circ{}^tW_1=aD^{-1}(W_4\circ W_4)W_2$ given in Lemma 2.4(1), we have

$$W_1\circ{}^tW_1=aD^{-1}t_0\{(k+(n-k)t)I-(kt+(n-k))(J-I)\}$$

by Lemma 4.1. In terms of conditions (a)–(c), it is straightforward to check that the coefficients of I, $J-I$ above are a^2 and $-t_0^2t$, respectively, as required.

(2) (i). Note that for those k points of block x, there are s points contained in both blocks α and β, $2(\lambda-s)$ points contained in either block α or block β but not both, and $k-s-2(\lambda-s)(=k+s-2\lambda)$ points contained in neither block α nor block β. Similarly for those $n-k$ points not contained in x, there are $\lambda-s$ points contained in both blocks α and β, $k-s-2(\lambda-s)$ $(=k+s-2\lambda)$ points contained in either block

α or block β but not both, and $n+3\lambda-3k-s$ points contained in neither block α nor block β. Combining these information, we have

$$\sum_r \frac{W_4(\gamma,\alpha)W_4(\gamma,\beta)}{W_4(\gamma,x)} = (s+2(\lambda-s)t+(k+s-2\lambda)t^2+(\lambda-s)t^{-1}$$

$$+2(k+s-2\lambda)+(n+3\lambda-3k-s)t)t_0,$$

$$=(k-((\lambda-s)n(k-\lambda)^{-1}))t_0(1-t),$$

in terms of condition (c).

(2) (ii). By Lemma 2.5 (2),

$$W_1(\alpha,\beta) = a\frac{W_1(\alpha,x)}{W_1(\beta,x)}\frac{\sum_r W_4(\gamma,\beta)^2/W_4(\gamma,x)}{\sum_r W_4(\gamma,\alpha)W_4(\gamma,\beta)/W_4(\gamma,x)},$$

$$W_1(\beta,\alpha) = a\frac{W_1(\beta,x)}{W_1(\alpha,x)}\frac{\sum_r W_4(\gamma,\alpha)^2/W_4(\gamma,x)}{\sum_r W_4(\gamma,\beta)W_4(\gamma,\alpha)/W_4(\gamma,x)},$$

it follows that

$$W_1(\alpha,\beta)W_1(\beta,\alpha)=a^2\frac{(\sum_r W_4(\gamma,\beta)^2/W_4(\gamma,x))(\sum_r W_4(\gamma,\alpha)^2/W_4(\gamma,x))}{(\sum_r W_4(\gamma,\beta)W_4(\gamma,\alpha)/W_4(\gamma,x))^2},$$

Moreover, by (1) and Lemma 2.4 (1),

$$W_1(\alpha,\beta)W_1(\beta,\alpha) = aD^{-1}\sum_r \frac{W_4(\gamma,\beta)^2}{W_4(\gamma,\alpha)}$$

$$= -t_0^2 t,$$

hence $\sum_r W_4(\gamma,\beta)^2/W_4(\gamma,x)=(-Dt_0^2t)/a$ is a constant whenever $\alpha\neq\beta$. This gives

$$\left(\sum_r \frac{W_4(\gamma,\alpha)W_4(\gamma,\beta)}{W_4(\gamma,x)}\right)^2 = -nt_0^2 t$$

as required. □

Lemma 4.4. *Following the same notations, we have*

1. $s=n^{-1}(k\lambda+\lambda-k\pm(k-\lambda)\sqrt{k-\lambda})$,
2. $k-\lambda$ *is a square.*

Proof. Combining (2) (i) and (2) (ii) in Lemma 4.3, we have

$$\left(\sum_r \frac{W_4(\gamma,\alpha)W_4(\gamma,\beta)}{W_4(\gamma,x)}\right)^2 = ((k-(\lambda-s)n(k-\lambda)^{-1})^2t_0^2(1-t)^2,$$

$$= -nt_0^2 t.$$

Since $(1-t)^2=-nt(k-\lambda)^{-1}$ by condition (c), and $k^2-n\lambda=k-\lambda$ by condition (a), it follows that

$$\left(\frac{k-\lambda-k\lambda+sn}{k-\lambda}\right)^2\frac{1}{k-\lambda}=1$$

and hence

$$s = n^{-1}(k\lambda + \lambda - k \pm (k-\lambda)\sqrt{k-\lambda}),$$

must be a non-negative integer, it follows immediately that $k-\lambda$ is a square. □

We now summarize the main results of this section in the following theorem.

Theorem 4.5. *For a four-weight spin model* $(X; W_1, W_2, W_3, W_4)$ *with* $W_4 = t_0(E_1 + tE_2)$, *then* E_1 *is the incidence matrix of a symmetric design* (n, k, λ) *with the following properties*:

(1) $k-\lambda$ *is a square*,
(2) *its derived design with respect to any block is a quasi-symmetric design with intersection numbers*

$$x = n^{-1}(k\lambda + \lambda - k - (k-\lambda)\sqrt{k-\lambda}),$$

$$y = n^{-1}(k\lambda + \lambda - k + (k-\lambda)\sqrt{k-\lambda}),$$

respectively.

Remark. (1) Since $k-\lambda$ is a square, the possibility of both *projective planes* and *Hadamard designs* as symmetric designs corresponding to E_1 can be ruled out. It means that $\lambda > 1$ generally.

(2) Certain symmetric designs coming from *Menon difference sets* satisfy the conditions given in Theorem 4.5. Their parameters are $(n, k, \lambda) = (4u^2, 2u^2 - u, u^2 - u)$ for $u = 2, 4$. The following is such an example of two-valued four-weight spin models defined over a set of 16 points associated with a symmetric 2-(16, 6, 2) design.

Example 4.6. Let $(X, \mathscr{B})$ be a symmetric 2-(16, 6, 2) design with an incidence matrix E_1 and $(B_0, \mathscr{B}')$ be its derived design with respective to the block $B_0 \in \mathscr{B}$. Let $W_4 = t_0(2E_1 - J)$ and W_1 be a matrix indexed by blocks of $\mathscr{B}$ such that,

$$W_1(\alpha, \alpha) = W_1(B_0, \alpha) = W_1(\alpha, B_0) = -t_0 \quad \text{for all } \alpha \in \mathscr{B},$$

$$W_1(\alpha, \beta) = \begin{cases} t_0 & \text{if } \alpha, \beta \text{ have no common points in } (B_0, \mathscr{B}'), \\ -t_0 & \text{if } \alpha, \beta \text{ have a unique common points in } (B_0, \mathscr{B}'), \end{cases}$$

where t_0 is any non-zero complex number, then $(X; W_1, W_2, W_3, W_4)$ is a four-weight spin model.

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