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Forced symmetry-breaking via boundary conditions

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1. Introduction

Boundary conditions have sophisticated influence on behavior of reaction–diffusion equations. Since diffusion is the underlying mechanism for the spatial pattern formation in chemical reactions, spatial structure of solutions of reaction–diffusion equations can be sensitive to boundary conditions. Posing and realizing appropriate boundary conditions, for instance, for chemical reactions in open and large systems, is delicate (cf. [7]). Furthermore, components of a system of reaction–diffusion equations can be imposed with different boundary conditions. The behavior of system and that of scalar equation are very different in, for example, Hopf bifurcations, spiral waves and other pattern formations. Typically multiple bifurcations occur more likely in systems. Moreover, stability of the bifurcating solution branches varies considerably from a scalar equation to a system. To distinguish the influence of boundary conditions from that of

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interactions among the different species (components) of systems, we consider a scalar stationary reaction–diffusion equation

$$\Delta u + \lambda u = f(u, \lambda) \quad \text{in } \Omega := (0, \pi) \times (0, \pi). \tag{1}$$

We assume that the mapping $f : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ is sufficiently smooth and

$$f(0, \lambda) = 0, \qquad D_u f(0, \lambda) = 0, \tag{2}$$

that is, it describes the nonlinearity of problem (1) and implies that

$$u \equiv 0, \quad \lambda \in \mathbf{R}$$

is a trivial solution of (1).

We are interested in impact of symmetry-breaking in boundary conditions on the bifurcation scenarios. To this end we consider a square domain and impose the following conditions along its four sides

$$\begin{aligned} &h_0(\mu)u(x, 0) - h_1(\mu)\frac{\partial u}{\partial y}(x, 0) = 0, \\ &h_0(\mu)u(x, \pi) + h_1(\mu)\frac{\partial u}{\partial y}(x, \pi) = 0, \\ &\frac{\partial u}{\partial x}(0, y) = 0, \\ &\frac{\partial u}{\partial x}(\pi, y) = 0. \end{aligned} \tag{3}$$

Here $h_0,\ h_1 : [0, 1] \to R$ are smooth functions satisfying

$$\begin{aligned} &h_0(0) = h_1(1) = 0, \qquad h_0(\mu) \neq 0 \quad \text{for } \mu \in (0, 1], \\ &h_1(\mu) \neq 0 \quad \text{for } \mu \in [0, 1). \end{aligned} \tag{4}$$

Boundary conditions (3) break the D_4-symmetry into D_2-symmetry as μ varies. More precisely, properties of $h_0(\mu)$, $h_1(\mu)$ make (3) a homotopy from the homogeneous Neumann boundary conditions along the four sides of Ω at $\mu = 0$ to the mixed boundary conditions at $\mu = 1$, which are of the Neumann type along the sides $x = 0, \pi$; and of the Dirichlet type at $y = 0,\ \pi$ (see Fig. 1).

Early application of symmetry to bifurcation analysis dates back to Othmer [14,15]. The studies in influence of boundary conditions upon the solution structure of partial differential equation have been done by many scientists. For example, Mielke [13] has shown that some typical patterns in Ginzburg–Landau equation are not dominated by boundary effects. Homotopy of boundary conditions has been used by Fiedler [3] and Gardner [4] to study global attractors and nonsingular solutions of a class of reaction–diffusion equations. In these studies, detailed affection from boundary conditions on the structure of global attractor has been analyzed in different aspects. For example, Fiedler showed that the class of global attractors is independent of boundary conditions. Nevertheless, if the equilibrium is nonhyperbolic and a bifurcation occurs, the bifurcation scenario, for example, the structure of attractors, may vary with respect to boundary conditions. This has been observed by Dillon et al. [2] in the study of

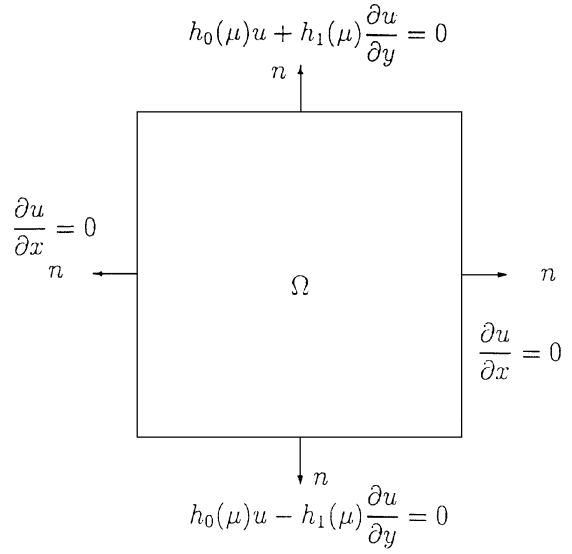

Fig. 1. A homotopy between homogeneous Neumann and mixed boundary conditions.

pattern formation in generalized one-dimensional Turing systems and by Mei and Theil [12] in the analysis of steady state bifurcations as well as by Holder and Schaeffer [9] and Schaeffer and Golubitsky [16] on mode-jumping of von Kármán equations. Using (3) as an example we study in this paper how reaction–diffusion equations react to a symmetry-breaking in boundary conditions.

An outline of this paper is as follows. In Section 2 we consider variational form and symmetries of Eq. (1). Section 3 describes bifurcation points of (1) along the trivial solution curve. In Section 4 problem (1) at bifurcation points is reduced to algebraic equations via the well-known Liapunov–Schmidt method. We derive the bifurcation scenario at simple and double bifurcation points in Section 5 and illustrate these with a simple example in Section 6.

2. Variational equations and symmetries

The classical regularity theory of elliptic problems ensures the $C^{2,s}$-Hölder continuity of solutions of the linear problem

$$\Delta u + \lambda u = g$$

with boundary conditions (3) on the square Ω (cf. [17]). However, differentiability of solutions at the four corners depends strongly on properties of g. This linear problem is involved in the analysis of bifurcations of (1) with various right-hand sides. Here we write problem (1) into variational form to avoid technicalities for the classical solutions. That is, we study bifurcations of its weak solutions.

2.1. Weak form

We consider the Sobolev space

$$X := H^1(\Omega) = \left\{ u \in L^2(\Omega); \quad \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} \in L^2(\Omega) \right\}$$

with the norm $\|\cdot\|_{1,\Omega}$, and for $u, v \in X$, $\mu \in [0,1)$ the parameter-dependent bilinear form

$$b_\mu(u,v) := -\int_\Omega (\nabla u \nabla v + uv)\,\mathrm{d}x\,\mathrm{d}y - \frac{h_0(\mu)}{h_1(\mu)} \int_0^\pi [u(x,0)v(x,0) + u(x,\pi)v(x,\pi)]\,\mathrm{d}x. \tag{5}$$

For $\mu = 1$ we choose the bilinear form

$$b_1(u,v) := -\int_\Omega (\nabla u \nabla v + uv)\,\mathrm{d}x\,\mathrm{d}y \tag{6}$$

defined in the space $\tilde{X} \times \tilde{X}$ and

$$\tilde{X} := \{u \in H^1(\Omega);\ u \text{ satisfies boundary conditions (3) for } \mu = 1\}.$$

The weak form of the linear problem

$$\Delta u - u = g \quad \text{in } \Omega \tag{7}$$

with boundary conditions (3) is

$$\text{Find } u \in H^1(\Omega) \quad \text{such that } b_\mu(u,v) = (g,v) \quad \text{for all } v \in H^1(\Omega). \tag{8}$$

In particular, weak form of the Neumann problem corresponds to $\mu = 0$.

For domains $\Omega \in C^{0,1}$, typically rectangle and L-domains (cf. [8, pp. 118]), we have

$$\|u\|_{H^{1/2}(\partial\Omega)} \le C\|u\|_{1,\Omega} \quad \text{for all } u \in H^1(\Omega),$$

$$\left\| \frac{\partial u}{\partial n} \right\|_{H^{1/2}(\partial\Omega)} \le C\|u\|_{2,\Omega} \quad \text{for all } u \in H^2(\Omega),$$

where $C > 0$ is a constant. Note that

$$\|\phi\|_{L^2(\partial\Omega)} \le \|\phi\|_{H^{1/2}(\partial\Omega)} = \inf_{u|_{\partial\Omega}=\phi} \|u\|_{1,\Omega} \quad \text{for all } \phi \in H^{1/2}(\partial\Omega).$$

Thus the bilinear form $b_\mu(\cdot,\cdot)$ is continuous and coercive on $X \times X$. Moreover, if the inequality $h_0(\mu)h_1(\mu) \ge 0$ holds, it is elliptic. Therefore, problem (8) has a unique solution $u(\mu) \in X$ for every $\mu \in [0,1)$ and all $g \in H^{-1}(\Omega)$. The solution $u(\mu)$ satisfies boundary conditions (3).

Denote the solution operator of (8) as

$$T(\mu)\colon g \in H^{-1}(\Omega) \mapsto T(\mu)g = u(\mu) \in H^1(\Omega). \tag{9}$$

The operator $T(\mu)$ is linear and bounded for any fixed $\mu \in [0,1)$. Furthermore, owing to the symmetry of Laplacian it is self-adjoint. This can be seen from the following

equalities for all $f,\ g\in H^{-1}(\Omega)$:

$$\begin{aligned}
&\langle T(\mu)f,\ g\rangle_{H^1(\Omega)\times H^{-1}(\Omega)}\\
&\quad=\int_\Omega (T(\mu)f)g\,\mathrm{d}x\,\mathrm{d}y\\
&\quad=\int_\Omega (\nabla(T(\mu)f)\nabla(T(\mu)g)+(T(\mu)f)(T(\mu)g)\,\mathrm{d}x\,\mathrm{d}y\\
&\qquad+\frac{h_0(\mu)}{h_1(\mu)}\int_0^\pi [(T(\mu)f)(T(\mu)g)|_{y=0}+(T(\mu)f)(T(\mu)g)|_{y=\pi}]\,\mathrm{d}x\\
&\quad=\int_\Omega (-\Delta+I)(T(\mu)f)(T(\mu)g)\,\mathrm{d}x\,\mathrm{d}y+\int_{\partial\Omega}(T(\mu)g)\frac{\partial}{\partial n}T(\mu)f\,\mathrm{d}s\\
&\qquad+\frac{h_0(\mu)}{h_1(\mu)}\int_0^\pi [(T(\mu)f)(T(\mu)g)|_{y=0}+(T(\mu)f)(T(\mu)g)|_{y=\pi}]\,\mathrm{d}x\\
&\quad=\int_\Omega f\cdot(T(\mu)g)\,\mathrm{d}x\,\mathrm{d}y\\
&\quad=\langle f,\ T(\mu)g\rangle_{H^{-1}(\Omega)\times H^1(\Omega)}.
\end{aligned}$$

More importantly is the fact that together with the Gelfand relation

$$H^1(\Omega)\overset{c}{\hookrightarrow}L^2(\Omega)\hookrightarrow H^{-1}(\Omega)$$

the Riesz–Schauder theory is applicable to the operator $T(\mu)$ for all $\mu\in[0,1)$, so that $T(\mu)$ has the following properties.

- The spectrum of operator $T(\mu)$ consists of eigenvalues. There are maximally countable eigenvalues and can be ordered as

  $$\lambda_1\geq\lambda_2\geq\cdots\to 0.$$

  The eigenspace associated to each eigenvalue $\lambda_i,\ i=1,2,\ldots$ is finite dimensional.
- For $i=1,2,\ldots$ the equation $T(\mu)u-\lambda u=f$ is solvable if and only if $f\perp\mathrm{Ker}(T(\mu)-\lambda I)$.

For $\mu=1$ we obtain the same conclusions with bilinear form (6). In the sequel we consider the weak form

$$G(u,\lambda,\mu):=u+(\lambda+1)T(\mu)u-T(\mu)f(u,\lambda)=0. \tag{10}$$

The mapping $G:X\times R\to X$ is obviously as smooth as f in (u,λ). It is also continuously differentiable in μ due to the following property of $T(\mu)$.

Lemma 1 (Mei [11]). *The operator $T(\mu)$ is continuous and differentiable with respect to μ in $[0,1)$. Furthermore, the derivative $u'(\mu)=T'(\mu)g=:v(\mu)$ for all $g\in Y$ is*

given as the weak solution of

$$\begin{aligned}
&\Delta v - v = 0 \quad \text{in } \Omega,\\
&\frac{h_0(\mu)}{h_1(\mu)} v(x,0) - \frac{\partial v}{\partial y}(x,0) = -\left(\frac{h_0(\mu)}{h_1(\mu)}\right)' u(x,0),\\
&\frac{h_0(\mu)}{h_1(\mu)} v(x,\pi) + \frac{\partial v}{\partial y}(x,\pi) = -\left(\frac{h_0(\mu)}{h_1(\mu)}\right)' u(x,\pi),\\
&\frac{\partial v}{\partial x}(0,y) = 0, \quad \frac{\partial v}{\partial x}(\pi,y) = 0.
\end{aligned} \tag{11}$$

To calculate $T'(\mu)g$, we denote by $u = T(\mu)g$ the weak solution of the equation $\Delta u - u = g$ with boundary condition (3). Define

$$\hat{v} := \left(\frac{h_0(\mu)}{h_1(\mu)}\right)' \left(-\frac{y^2}{\pi} + y\right) u(x,y). \tag{12}$$

It is easy to verify that $\hat{v}$ satisfies the boundary conditions in (11). Let $v = w + \hat{v}$ and substitute it into (11). We obtain the equation

$$\Delta w - w = -(\Delta \hat{v} - \hat{v})$$

with boundary conditions (3). Furthermore,

$$-(\Delta \hat{v} - \hat{v}) = -\left(\frac{h_0(\mu)}{h_1(\mu)}\right)' \left[\left(-\frac{y^2}{\pi} + y\right) g - \frac{2}{\pi} u + 2\left(-\frac{2y}{\pi} + 1\right) \frac{\partial u}{\partial y}\right].$$

Hence, the weak solution v of Eq. (11) is

$$\begin{aligned}
v &= -T(\mu)(\Delta \hat{v} - \hat{v}) + \hat{v}\\
&= \left(\frac{h_0(\mu)}{h_1(\mu)}\right)' \left\{ T(\mu)\left[\frac{2}{\pi} T(\mu)g + 2\left(\frac{2y}{\pi} - 1\right) \frac{\partial}{\partial y}(T(\mu)g) + \left(\frac{y^2}{\pi} - y\right) g\right]\right.\\
&\quad \left. + \left(-\frac{y^2}{\pi} + y\right) T(\mu)g \right\}.
\end{aligned} \tag{13}$$

Remark. Note that $T(\mu)$ is self-adjoint, so is its derivative $T'(\mu)$. Based on formulation (11), one can calculate the higher-order derivatives of $u(\mu)$ in a similar manner.

2.2. Symmetries

Let D_4 be the dihedral group of the square Ω and

$$S_1(x,y) = (\pi - x, y), \qquad R(x,y) = (\pi - y, x)$$

be its generators. With $Z_2 := \{1, -1\}$, we define $Z_2 \times D_4 = \{\pm\delta;\ \delta \in D_4\}$ and its actions on $Y := L^2(\Omega)\ (\supset X)$ as

$$\gamma u(x,y) = \pm u(\delta^{-1}(x,y)) \quad \text{for all } \gamma = \pm\delta,\ \delta \in D_4 \text{ and } u \in Y. \tag{14}$$

The function spaces X, Y are obviously $Z_2 \times D_4$-invariant. Similarly, the L^2-product is also $Z_2 \times D_4$-invariant. Corresponding to boundary condition (3) we are particularly interested in the subgroup

$$D_2 := \{S_1, R^2;\ S_1R^2, I\}. \tag{15}$$

Let

$$\Gamma := \begin{cases} Z_2 \times D_4 & \text{if } f(u,\lambda) \text{ is an odd function in } u \text{ and } \mu = 1, \\ D_4 & \text{if } f(u,\lambda) \text{ is not odd in } u \text{ and } \mu = 1, \\ Z_2 \times D_2 & \text{if } f(u,\lambda) \text{ is an odd function in } u \text{ and } \mu \neq 1, \\ D_2 & \text{if } f(u,\lambda) \text{ is not odd in } u \text{ and } \mu \neq 1. \end{cases}$$

The Γ-*equivariance* of the mapping G, i.e.,

$$G(\gamma u, \lambda, \mu) = \gamma G(u, \lambda, \mu) \quad \text{for all } \gamma \in \Gamma,\ u \in X,\ \lambda \in \mathbf{R}$$

can be verified directly via the generators $\pm S_1$ and $\pm R$, respectively.

3. Bifurcation points

Since $D_\lambda G(0,\lambda,\mu) = D_\mu G(0,\lambda,\mu) \equiv 0$ and $D_u G(0,\lambda,\mu) = I + (\lambda+1)T(\mu)$ for all $\lambda,\ \mu \in \mathbf{R}$, a bifurcation occurs at a point $(0,\lambda,\mu)$ on the trivial solution manifold $\{(0,\lambda,\mu);\ \lambda \in \mathbf{R},\ \mu \in [0,1]\}$ of (10) if the linearized problem

$$D_u G(0,\lambda,\mu)u = u + (\lambda+1)T(\mu)u = 0 \tag{16}$$

has nontrivial solutions. By definition (9) of $T(\mu)$ this equation is the weak form of the eigenvalue problem

$$\Delta u + \lambda u = 0 \quad \text{in } \Omega = (0,\pi) \times (0,\pi)$$

with boundary condition (3). To solve this problem with the rule of separating variables, we take the ansatz $u(x,y) = u_1(x)u_2(y) \not\equiv 0$ and derive

$$\frac{u_1''}{u_1} + \frac{u_2''}{u_2} + \lambda = 0 \quad \text{in } \Omega = (0,\pi) \times (0,\pi).$$

Hence, u_1 satisfies the equation

$$u_1'' + k_1 u_1 = 0 \quad \text{for some } k_1 \in \mathbf{R} \tag{17}$$

with boundary conditions $u_1'(0) = u_1'(\pi) = 0$. Similarly, u_2 is a solution of

$$u_2'' + k_2 u_2 = 0 \quad \text{for some } k_2 \in \mathbf{R} \tag{18}$$

with the boundary conditions

$$h_0(\mu)u_2(0) - h_1(\mu)u_2'(0) = 0,$$
$$h_0(\mu)u_2(\pi) + h_1(\mu)u_2'(\pi) = 0.$$

These are eigenvalue problems of the one-dimensional differential operator d^2/dx^2 with two different boundary conditions. Solutions of these problems are of the form

$$(u_1, k_1) = (\cos nx,\ n^2), \quad n \in \mathbf{N},$$

$$(u_2, k_2) = (h_0(\mu)\sin(k(\mu)y) + h_1(\mu)k(\mu)\cos(k(\mu)y),\ k(\mu)^2),$$

where $k(\mu) \in \mathbf{R}$ satisfies

$$2h_0(\mu)h_1(\mu)k\cos(k\pi) + (h_0^2(\mu) - h_1^2(\mu)k^2)\sin(k\pi) = 0. \tag{19}$$

Thus eigenvalues of the Laplacian $-\Delta$ are given as

$$\lambda = n^2 + k(\mu)^2 \tag{20}$$

with the corresponding eigenfunction

$$\phi(\mu) := \tilde{\phi}/\|\tilde{\phi}\|, \qquad \tilde{\phi} := (\cos nx)[h_0(\mu)\sin(k(\mu)y) + h_1(\mu)k(\mu)\cos(k(\mu)y)]. \tag{21}$$

As a function of the homotopy parameter μ, the function $\kappa(\mu)$ has the following properties (cf. [12]).

Lemma 2. *Under assumption* (4), *Eq.* (19) *does not have integer solution for* $\mu \in (0,1)$. *Moreover, if* $[h_1(\mu)/h_0(\mu)]' < 0$ *for all* $\mu \in (0,1)$, *then the solution* $k(\mu)$ *of* (19) *increases monotonously from* $m \in \mathbf{N}$ *at* $\mu = 0$ *to* $(m+1) \in \mathbf{N}$ *at* $\mu = 1$.

Note that after multiplying the factor $\sin(k\pi)$ to the both sides of Eq. (19) we can rewrite it as

$$[h_0(\mu)\sin(k\pi) - h_1(\mu)k(1 - \cos(k\pi))][h_0(\mu)\sin(k\pi) + h_1(\mu)k(1 + \cos(k\pi))] = 0.$$

We use the following definition of parities of $k(\mu)$ in [1], which are consistent with the parities of wavenumbers of the Neumann problem at $\mu = 0$.

Definition 3. The parities of the wavenumber $k(\mu)$ for Robin boundary conditions are defined as

$$\kappa(\mu) = \begin{cases} \text{EVEN} & \text{if } h_0(\mu)\sin(k(\mu)\pi) = h_1(\mu)k(\mu)[1 - \cos(k(\mu)\pi)] \\ & \quad \text{for all } \mu \in [0,1], \\ \text{ODD} & \text{if } h_0(\mu)\sin(k(\mu)\pi) = -h_1(\mu)k(\mu)[1 + \cos(k(\mu)\pi)]. \end{cases} \tag{22}$$

In the rest of this paper we restrict the discussion to the case

$$\frac{h_0(\mu)}{h_1(\mu)} \geqslant 0, \qquad \left(\frac{h_1(\mu)}{h_0(\mu)}\right)' < 0.$$

We conclude that bifurcation points of (10) on the trivial solution manifold are

$$\{(0,\ \lambda(\mu),\ \mu);\ \lambda(\mu) = n^2 + k(\mu)^2,\ n \in \mathbf{N},\ \text{and } k(\mu) \text{ satisfying (19)}\} \tag{23}$$

(see Fig. 2). The kernel $\mathrm{Ker}(D_u G(0, \lambda(\mu), \mu))$ is generically one-dimensional and

$$\mathrm{Ker}(D_u G(0, \lambda(\mu), \mu)) = \mathrm{span}[\phi(\mu)].$$

From statement (20) and Fig. 2 it is evident that two curves of bifurcation points may intersect. In fact, these are generic as μ approaches zero and one, respectively. An

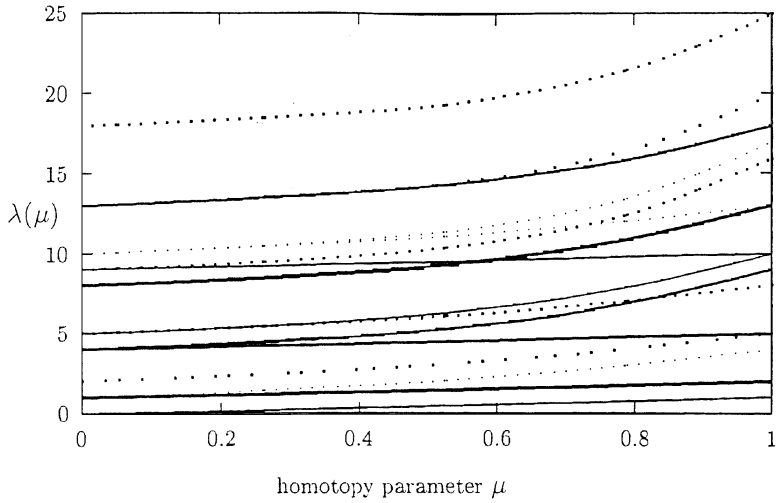

Fig. 2. Bifurcation points of (10) in the parameter space (λ, μ). Here we have chosen $h_0(\mu)=\mu,\ h_1(\mu)=1-\mu$.

intersection point corresponds to a multiple bifurcation of (10). At $\mu = 0, 1$ solutions of Eq. (19) are integers. In particular, at $\mu = 0$ boundary conditions (3) reduce to homogeneous Neumann type. Thus the eigenvalues of $-\Delta$ are of the form

$$\lambda_0 = n^2 + k^2, \quad n,\ k \in \mathbf{N} \cup \{0\}.$$

Such an eigenvalue is generically double and the associated eigenspace is

$$\mathrm{Ker}(D_u G(0, \lambda_0, 0)) = \mathrm{span}[\phi_1, \phi_2],$$

where

$$\phi_1 = \frac{2}{\pi}\cos(nx)\cos(ky), \qquad \phi_2 = \frac{2}{\pi}\cos(kx)\cos(ny).$$

Consequently, as $\mu \to 0$, two different curves of simple bifurcation points approach the same double bifurcation point. However, with λ as the bifurcation parameter, there is exactly one solution branch bifurcating from the trivial solution curve at every simple bifurcation point, while at a double bifurcation point for $\mu = 0$ the Neumann problem has up to four different nontrivial solution branches (cf. [10]). On the other hand, as we have seen before, the D_4-symmetry of the Neumann problem breaks into $D_2 \times D_2$-symmetry as μ moves away from 0. This leads to the question how the bifurcation scenario reacts to such a symmetry-breaking in the boundary conditions. We want to investigate which solution branches of the Neumann problem and what symmetries of the solutions persist as the parameter μ varies.

4. Liapunov–Schmidt reduction

To investigate the solutions of (10) at a bifurcation point $(0, \lambda_0, \mu_0)$ on the curve $(0, \lambda(\mu), \mu)$ in (23), we use the well-known Liapunov–Schmidt method to reduce problem (10) to an algebraic system (cf. [5,6]).

According to the Fredholm properties of $D_uG(0,\lambda_0,\mu_0)=I+(\lambda_0+1)T(\mu_0)$, we have the decomposition

$$X = \mathrm{Ker}(D_uG(0,\lambda_0,\mu_0)) \oplus \mathrm{Im}(D_uG(0,\lambda_0,\mu_0)).$$

Suppose that the kernel $\mathrm{Ker}(D_uG(0,\lambda_0,\mu_0))=\mathrm{span}[\phi_1,\ldots,\phi_l]$ is l-dimensional ($l=1,2$ generically). We write elements $(u,\lambda,\mu)\in X\times\mathbf{R}\times\mathbf{R}$ as

$$u=\sum_{i=1}^{l} z_i\phi_i + w = z\cdot\phi + w,$$

$$\lambda = \lambda_0 + \sigma,$$

$$\mu = \mu_0 + v,$$

where $z=(z_1,\ldots,z_l)$, $\phi=(\phi_1,\ldots,\phi_l)$, $z_i,\ \sigma, v\in\mathbf{R}$ and $w\in\mathrm{Im}(D_uG(0,\lambda_0,\mu_0))$. Consider the projection $Q:=I-\sum_{i=1}^{l}\langle\phi_i,\cdot\rangle\phi_i$ from X onto $\mathrm{Im}(D_uG(0,\lambda_0,\mu_0))$. We rewrite the equation $G(u,\lambda,\mu)=0$ into a system

$$QG(z\cdot\phi+w,\ \lambda_0+\sigma,\ \mu_0+v)=0, \tag{24}$$

$$(I-Q)G(z\cdot\phi+w,\ \lambda_0+\sigma,\ \mu_0+v)=0. \tag{25}$$

Solving w uniquely from (24) as a function of z, σ, v and substituting it into (25), we obtain the *reduced bifurcation equation* for z, σ, v:

$$(I-Q)G(z\cdot\phi+w(z,\sigma,v),\ \lambda_0+\sigma,\ \mu_0+v)=0. \tag{26}$$

We note that $w(0,0,0)=0$, $D_zw(0,0,0)=0$, from (24). In the coordinate system

$$z\cdot\phi\in\mathrm{Ker}(D_uG(0,\lambda_0,\mu_0))\leftrightarrow z=(z_1,\ldots,z_l)\in\mathbf{R}^l,$$

operator Eq. (26) becomes a system of l algebraic equations

$$B(z,\sigma,v):=(\langle\phi_i,\ G(z\cdot\phi+w(z,\sigma,v),\ \lambda_0+\sigma,\ \mu_0+v)\rangle)_{i=1}^{l}=0. \tag{27}$$

By definition the projection Q is Γ-equivariant. Thereafter the mapping $B(z,\sigma,v)$ is also Γ-equivariant with respect to the induced action of Γ in $\mathbf{R}^l$, i.e.,

$$B(\gamma z,\sigma,v)=\gamma B(z,\sigma,v)\quad\text{for all } (z,\sigma,v)\in\mathbf{R}^l\times\mathbf{R}\times\mathbf{R},\ \gamma\in\Gamma.$$

We take the Taylor expansion of the components of $(\lambda_0+1)B(z,\sigma,v)=0$ at the point $(z,\sigma,v)=(0,0,0)$ and consider the truncated form

$$\begin{aligned}&-\sigma z_i+(\lambda_0+1)^2\langle\phi_i,T'(\mu_0)(z\cdot\phi)\rangle v\\&\quad+\left\langle\phi_i,\frac{1}{2}D_{uu}f_0(z\cdot\phi)^2+D_{uu}f_0(z\cdot\phi)\left(\frac{1}{2}\sum_{|\alpha|=2}D^\alpha w_0 z^\alpha\right)+\frac{1}{6}D_{uuu}f_0(z\cdot\phi)^3\right\rangle\\&\quad=0,\quad i=1,\ldots,l.\end{aligned} \tag{28}$$

Here and in the sequel $D_{uu}f_0$ and $D_{uuu}f_0$ denote the derivatives of f at $(u,\lambda)=(0,\lambda_0)$; $\alpha\in\mathbf{N}^l$ is a multi-index and

$$\frac{1}{2}\sum_{|\alpha|=2} D^\alpha w_0 z^\alpha$$

represents the second-order terms in the Taylor expansion of w at $(z,\sigma,v)=(0,0,0)$.

We recall that in the singularity theory a problem $B=0$ is finitely determined if there exists $k\in\mathbf{N}$, $k<\infty$, such that the bifurcation scenarios of $B=0$ and its k-jets $j_k(B)=0$, the Taylor expansion of B truncated at kth order, are equivalent. The determinacy of a general problem at a bifurcation point is characterized by its reduced bifurcation equations. We refer to Golubitsky and Schaeffer [5] for more detailed discussions. For 3-determined problems solutions of system (28) correspond one-to-one to those of original problem (10) and contain all information of bifurcations of (10) at $(0,\lambda_0,\mu_0)$.

With the knowledge of the bifurcation point $(0,\lambda_0,\mu_0)$ and the kernel $\mathrm{Ker}(D_uG_0)$, the terms $\langle\phi_i, D_{uu}f_0(z\cdot\phi)^2\rangle$ and $\langle\phi_i, D_{uuu}f_0(z\cdot\phi)^3\rangle$ in (28) can be calculated directly. The other terms involve the derivatives $T'(\mu_0)$ and $D_{z_iz_j}w_0$. Since the function $w(z,\sigma,v)$ is defined implicitly by Eq. (24), the term $D_{z_iz_j}w_0$ is the unique solution v of the linear problem

$$D_uG_0v = QT(\mu_0)D_{uu}f_0\phi_i\phi_j, \quad v\in\mathrm{Im}(D_uG_0). \tag{29}$$

The term $T'(\mu_0)(z\cdot\phi)$ is calculated as a solution of Eq. (11). In fact, via (13) we derive

$$\begin{aligned}
&\langle\phi_i,\ T'(\mu_0)(z\cdot\phi)\rangle\\
&\quad=\tilde h(\mu_0)\left\langle\phi_i,\left(-\frac{y^2}{\pi}+y\right)T(\mu_0)(z\cdot\phi)\right.\\
&\qquad+T(\mu_0)\left[\frac{2}{\pi}T(\mu_0)(z\cdot\phi)+2\left(\frac{2y}{\pi}-1\right)\frac{\partial}{\partial y}(T(\mu_0)(z\cdot\phi))\right.\\
&\qquad\left.\left.+\left(\frac{y^2}{\pi}-y\right)(z\cdot\phi)\right]\right\rangle\\
&\quad=\tilde h(\mu_0)\frac{1}{(\lambda_0+1)^2}\left\langle\phi_i,\frac{2}{\pi}(z\cdot\phi)-2(\lambda_0+1)\left(\frac{2y}{\pi}-1\right)\frac{\partial}{\partial y}(T(\mu_0)(z\cdot\phi))\right\rangle\\
&\quad=\tilde h(\mu_0)\frac{2}{(\lambda_0+1)^2}\left[\frac{z_i}{\pi}+\left\langle\phi_i,\left(\frac{2y}{\pi}-1\right)\frac{\partial}{\partial y}(z\cdot\phi)\right\rangle\right],
\end{aligned}$$

where $\tilde h(\mu_0)=(\frac{h_0}{h_1})'(\mu_0)$.

5. Bifurcation scenarios

5.1. Simple bifurcations for $\mu \in (0,1)$

Let $\lambda(\mu) = n^2 + k(\mu)^2$ be a homotopy of simple eigenvalues of the Laplacian and $\lambda(0) = n^2 + m^2$, $\lambda(1) = n^2 + (m+1)^2$. Then $(0, \lambda(\mu), \mu)$ is a curve of simple bifurcation points of (10) and

$$\mathrm{Ker}(D_u G)(0, \lambda(\mu), \mu) = \mathrm{span}[\phi]$$

is one-dimensional and ϕ is given in (21). To obtain the generic bifurcation diagram of (10) at $(0, \lambda(\mu_0), \mu_0)$ for an arbitrary $\mu_0 \in (0,1)$, we consider Eq. (28), that is, the 3-jet of the reduced bifurcation equation,

$$\begin{aligned} 0 = &-\sigma z + (\lambda_0 + 1)^2 \langle \phi, T'(\mu_0)\phi \rangle vz \\ &+ \langle \phi, \tfrac{1}{2}(D_{uu} f_0)\phi^2 \rangle z^2 + \langle \phi, \tfrac{1}{2} D_{uu} f_0 (D_{zz} w_0)\phi + \tfrac{1}{6}(D_{uuu} f_0)\phi^3 \rangle z^3. \end{aligned} \tag{30}$$

Here $z \in \mathbf{R}$ is a scalar and

$$\langle \phi, T'(\mu_0)\phi \rangle = \tilde{h}(\mu_0) \frac{2}{(\lambda_0+1)^2} \left[\frac{1}{\pi} + \left\langle \phi, \left(\frac{2y}{\pi} - 1 \right) \frac{\partial}{\partial y} \phi \right\rangle \right].$$

Theorem 4. *Problem* (10) *undergoes a pitchfork bifurcation at all points on the curve* $(0, \lambda(\mu), \mu)$, $\mu \in (0,1)$, *that is, the simple bifurcation points. Moreover, the truncated bifurcation Eq.* (30) *reduces to*

$$j_3[(\lambda_0 + 1)B(z, \sigma, v)] = (-\sigma + av)z + cz^3 = 0 \tag{31}$$

with

$$\begin{aligned} a &= 2\tilde{h}(\mu_0) \left[\frac{1}{\pi} + \left\langle \phi, \left(\frac{2y}{\pi} - 1 \right) \frac{\partial}{\partial y} \phi \right\rangle \right], \\ c &= \langle \phi, \tfrac{1}{2} D_{uu} f_0 (D_{zz} w_0)\phi + \tfrac{1}{6} D_{uuu} f_0 \phi^3 \rangle. \end{aligned} \tag{32}$$

Proof. It is easy to verify that the eigenfunction ϕ of the Laplacian has the property $\langle \phi, \phi^2 \rangle = 0$. Thus the z^2 term in (30) vanishes. The conclusion follows directly from Eqs. (30) and (31) consecutively. □

The nontrivial solution of (31) is given as

$$z = \pm \left(\frac{\sigma - av}{c} \right)^{1/2}.$$

5.2. Double bifurcations of the Neumann problem

For the Neumann problem ($\mu = 0$) a generic double bifurcation point $(0, \lambda_0, 0)$ has the property that $\lambda_0 = n^2 + k^2(0)$ with the wavenumbers n, $k(:=k(0))$, $(n \neq k)$ as integers. Furthermore, we can choose

$$\mathrm{Ker}(D_u G_0) = \mathrm{span}[\phi_1, \phi_2]$$

with

$$\phi_1(x,y):=\begin{cases}\dfrac{2}{\pi}\cos(nx)\cos(ky) & \text{for } n\cdot k\neq 0,\\ \dfrac{\sqrt{2}}{\pi}\cos(nx) & \text{for } n\neq 0,\ k=0,\\ \dfrac{\sqrt{2}}{\pi}\cos(ky) & \text{for } n=0,\ k\neq 0,\end{cases}$$

and

$$\phi_2(x,y):=\phi_1(y,x).$$

On the other hand, taking into account the homotopy parameter μ in the boundary conditions, we see this double bifurcation point is split into two simple bifurcation points $(0,\lambda_i(\mu),\mu)$, $i=1,2$ with $\lambda_1(\mu)=n^2+k^2(\mu)$ and $\lambda_2(\mu)=k^2+n^2(\mu)$ for $\mu\neq 0$. We want to investigate bifurcation scenario of (10) at a double bifurcation point $(0,\lambda_0,0)$ and its variation with respect to the homotopy parameter μ.

Note that $\langle\phi_i,\phi_j\phi_l\rangle=0$ for all $i,j,l=1,2$. Eq. (29) can be solved analytically (cf. [10]). Together with the statements

$$\langle\phi_i^4,1\rangle=\begin{cases}\dfrac{9}{4\pi^2} & \text{for } n\cdot k\neq 0,\\ \dfrac{3}{2\pi^2} & \text{for } n\cdot k=0,\ n^2+k^2\neq 0\end{cases}$$

and $\langle\phi_i^2,\phi_j^2\rangle=1/\pi^2$ for $i\neq j$, we simplify Eq. (28) into

$$\begin{aligned}-\sigma z_1+(\lambda_0+1)^2\langle\phi_1,T'(\mu_0)(z_1\phi_1+z_2\phi_2)\rangle v+c_1z_1^3+c_2z_1z_2^2&=0,\\ -\sigma z_2+(\lambda_0+1)^2\langle\phi_2,T'(\mu_0)(z_1\phi_1+z_2\phi_2)\rangle v+c_2z_1^2z_2+c_1z_2^3&=0.\end{aligned}\tag{33}$$

Here c_1, c_2 are constants. More precisely, if $n\cdot k\neq 0$, we have

$$c_1=\frac{1}{6\pi^2}\left[\frac{9}{4}D_{uuuu}f_0-\frac{1}{4}(D_{uu}f_0)^2\frac{45(k^2-n^2)^2+4k^2n^2}{(k^2-3n^2)(n^2-3k^2)(n^2+k^2)}\right],$$

$$c_2=\frac{1}{6\pi^2}\left[3D_{uuuu}f_0-6(D_{uu}f_0)^2\frac{1}{n^2+k^2}\left(\frac{(k^2-n^2)^2-4k^2n^2}{[(k^2+n^2)^2-16k^2n^2]}-\frac{1}{2}\right)\right].$$

If $n=0$, $k\neq 0$, then

$$c_1=\frac{1}{6\pi^2}\left[\frac{3}{2}D_{uuuu}f_0+\frac{5}{2k^2}(D_{uu}f_0)^2\right],$$

$$c_2=\frac{1}{2\pi^2}D_{uuuu}f_0.$$

If $v=0$, Eq. (33) coincide with those in [10], and yield four nontrivial solutions of (10) with symmetries the isotropy subgroups of ϕ_1,ϕ_2 and $\phi_1\pm\phi_2$, respectively.

For $v \neq 0$, the forced symmetry-breaking in boundary conditions introduces in (33) the terms

$$\begin{aligned}&(\lambda_0+1)^2\langle\phi_i, T'(\mu_0)(z_1\phi_1+z_2\phi_2)\rangle v\\&\quad=2\tilde{h}(0)\left[\frac{z_i}{\pi}+\left\langle\phi_i,\left(\frac{2y}{\pi}-1\right)\frac{\partial}{\partial y}(z_1\phi_1+z_2\phi_2)\right\rangle\right]\\&\quad=: d_i z_i, \quad i=1,2.\end{aligned}$$

Here,

$$d_1=d_2=\frac{4}{\pi}\tilde{h}(0) \quad \text{if } n\cdot k\neq 0,$$

$$d_1=\frac{4}{\pi}\tilde{h}(0) \qquad d_2=0, \quad \text{if } n=0,\ k\neq 0.$$

System (33) reduces to

$$\begin{aligned}&[-\sigma+d_1 v+c_1 z_1^2+c_2 z_2^2]z_1=0,\\&[-\sigma+d_2 v+c_2 z_1^2+c_1 z_2^2]z_2=0.\end{aligned} \tag{34}$$

Remark. The coefficients in Eqs. (32) and (34) are related as follows:

$$\lim_{\mu_0\to 0} a=d_i \quad \text{for } \phi=\phi_i,\ i=1,2,$$

$$\lim_{\mu_0\to 0} c=c_1.$$

Solutions of system (34) are

$$\text{(a)}\ \left(\pm\left(\frac{\sigma-d_1 v}{c_1}\right)^{1/2},0\right), \qquad \left(0, \quad \pm\left(\frac{\sigma-d_2 v}{c_1}\right)^{1/2}\right) \tag{35}$$

$$\text{(b)}\ \left(\pm\left(\frac{(c_1-c_2)\sigma-(c_1d_1-c_2d_2)v}{c_1^2-c_2^2}\right)^{1/2}, \quad \pm\left(\frac{(c_1-c_2)\sigma+(c_2d_1-c_1d_2)v}{c_1^2-c_2^2}\right)^{1/2}\right).$$

These lead to four bifurcating solution branches of original problem (10), that is, (1). The solutions in (35a) are pure-mode solution branches with the isotropy groups of ϕ_1, ϕ_2, respectively. They correspond to those bifurcating solutions at the simple bifurcation points on the curves $(0, \lambda_i(\mu), \mu)$, $i=1,2$. The solutions in (35b) involve both ϕ_1 and ϕ_2 modes. They are called the mixed-mode branches.

If $n\cdot k=0$, the terms $d_1 v$, $\mathrm{d}_2 v$ break the D_4-symmetry of the Neumann problem and the mixed-mode solution branches have merely the trivial symmetry. Moreover, the pure mode and mixed-mode solution branches may intersect at

$$\sigma=\frac{(c_1d_1-c_2d_2)v}{(c_1-c_2)} \quad \text{or} \quad \sigma=\frac{(c_2d_1-c_1d_2)v}{(c_1-c_2)}$$

and induce a secondary bifurcation, respectively.

If $n \cdot k \neq 0$, then $d_1 = d_2$ and the D_4-symmetry is preserved in (34) and mixed-mode solutions (35b) becomes

$$\pm\left(\left(\frac{\sigma - d_1 v}{c_1 + c_2}\right)^{1/2}, \quad \pm\left(\frac{\sigma - d_1 v}{c_1 + c_2}\right)^{1/2}\right).$$

Symmetries of these solutions are the isotropy groups of the eigenfunctions $\phi_1 + \phi_2$ and $\phi_1 - \phi_2$, respectively.

As a conclusion, we have seen that all four bifurcating solution branches of the Neumann problem at a double-bifurcation point persist if we vary both λ and μ as bifurcation parameters. Moreover, symmetry of these bifurcating solution branches is preserved for those with the wavenumbers $n \cdot k \neq 0$ and is broken for those with the wavenumbers $n = 0$ or $k = 0$.

6. A simple example

Choose

$$f(u, \lambda) = \lambda(u^2 + u^3). \tag{36}$$

We consider the bifurcation scenarios at the corank-2 bifurcation points $u_0 = 0, \mu_0 = 0$ and $\lambda_0 = 5, 1$, respectively. In particular, we aim to examine variations of the bifurcation scenarios as the homotopy parameter μ moves away from zero, that is, as the homogeneous Neumann boundary conditions with D_4-symmetry are perturbed. To simplify the discussion, we take $h_0(\mu) = \mu, h_1(\mu) = 1 - \mu$. Note that at $\mu = 0$ we have

$$\mathrm{Ker}(D_u G_0) = \mathrm{span}[\phi_1, \phi_2]$$

and the inequalities $c_1 \neq 0, c_2 \neq 0$ and $c_1^2 - c_2^2 \neq 0$ hold for all $n, k \in \mathbf{N} \cup \{0\}$.

(1) *Wavenumber* $n = 1,\ k = 2$: For $\lambda(0) = 5$ and $n = 1,\ k = 2$ we have

$$\phi_1 = 2/\pi \cos x \cos 2y, \qquad \phi_2 = 2/\pi \cos 2x \cos y.$$

Furthermore, $d_1 = d_2 = 4/\pi,\ c_1 = 5695/132\pi^2,\ c_2 = 110\,220/132\pi^2$. Solutions (z_1, z_2) in (35) become

$$\left(\pm\left(\frac{132\pi(\pi\sigma - 4v)}{5695}\right)^{1/2}, 0\right), \left(0, \pm\left(\frac{132\pi(\pi\sigma - 4v)}{5695}\right)^{1/2}\right),$$

$$\pm\left(\left(\frac{132\pi(\pi\sigma - 4v)}{115\,915}\right)^{1/2}, \pm\left(\frac{132\pi(\pi\sigma - 4v)}{115\,915}\right)^{1/2}\right).$$

Figs. 3 and 4 show the pure and mixed mode solution branches.

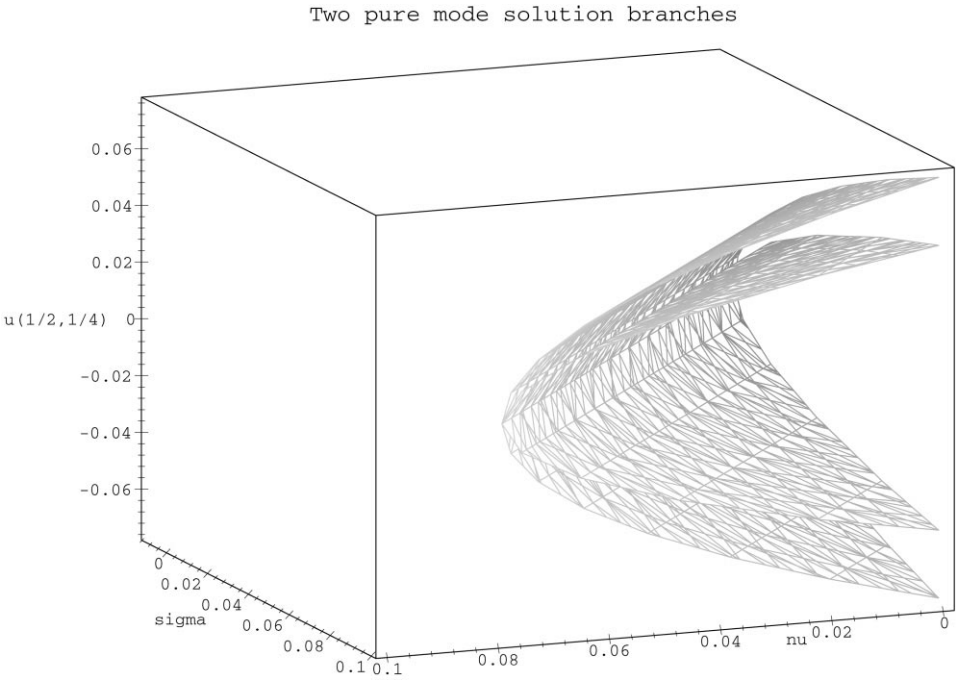

Fig. 3. Two solution branches of pure ϕ_1 and ϕ_2 modes.

(2) *Wavenumber* $n = 0,\ k = 1$: For $\lambda(0) = 1$ and $n = 0,\ k = 1$, moreover,

$$\phi_1 = \frac{\sqrt{2}}{\pi}\cos y, \qquad \phi_2 = \frac{\sqrt{2}}{\pi}\cos x.$$

Simple calculations show $d_1 = 4/\pi$, $\mathrm{d}_2 = 0$, $c_1 = 19/6\pi^2$, and $c_2 = 3/\pi^2$. The solution branches described by (z_1, z_2) in (35) are

$$\left(\pm\left(\frac{6\pi(\pi\sigma - 4v)}{19}\right)^{1/2}, 0\right), \qquad \left(0, \pm\left(\frac{6\pi^2\sigma}{19}\right)^{1/2}\right),$$

$$\pm\left(\left(\frac{6\pi(\pi\sigma - 76v)}{37}\right)^{1/2}, \pm\left(\frac{6\pi(\pi\sigma + 72v)}{37}\right)^{1/2}\right).$$

The pure ϕ_2-mode solution branch meets a mixed-mode solution branch at $\sigma = 76v/\pi$ and a secondary bifurcation is induced.

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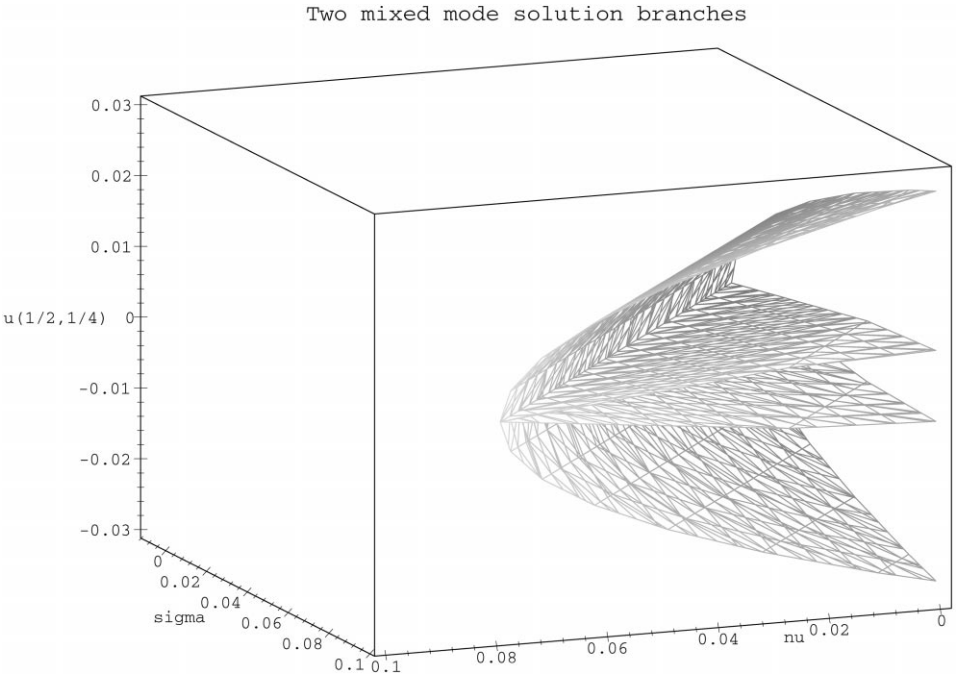

Fig. 4. Two solution branches of mixed ϕ_1 and ϕ_2 modes.

References

[1] P. Ashwin, Z. Mei, A Hopf bifurcation with Robin boundary conditions, J. Dynamics Differential Equations 6 (1994) 487–503.
[2] R. Dillon, P.K. Maini, H.G. Othmer, Pattern formation in generalized Turing systems, I. steady-state patterns in systems with mixed boundary conditions, J. Math. Biol. 32 (1994) 345–393.
[3] B. Fiedler, Do global attractors depend on boundary conditions? Doc. Math. J. DMV. 1 (1996) 215–228.
[4] R.A. Gardner, Global continuation of branches of nondegenerate solutions, J. Differential Equations 61 (1986) 321–334.
[5] M. Golubitsky, D. Schaeffer, Groups and singularities in bifurcation theory I, Appl. Math. Sci., vol. 51, Springer, New York, 1986.
[6] M. Golubitsky, I.N. Stewart, D. Schaeffer, Singularities and groups in bifurcation theory II, Appl. Math. Sci., vol. 69, Springer, New York, 1988.
[7] P. Gray, S.K. Scott, Chemical reaction in isothermal system, in: S. Sarkar (Ed.), Nonlinear Phenomena and Chaos: Oscillations and Instabilities, Malvern Phys. Ser., Hilger, Bristol, 1986, pp. 70–95.
[8] W. Hackbusch, Theorie und Numerik elliptischer Differentialgleichungen, Teubner Verlag, Stuttgart, 1986.
[9] E.J. Holder, D.G. Schaeffer, Boundary condition and mode jumping in the von Kármán equations, SIAM J. Math. Anal. 15 (1984) 446–458.
[10] Z. Mei, Solution branches of a semilinear elliptic problem at corank-2 bifurcation points with Neumann boundary conditions, Roy. Soc. Edinburgh Proc. A 123A (1993) 479–495.
[11] Z. Mei, Numerical Analysis of Bifurcation Problems of Reaction–Diffusion Equations, Habilitationsschrift, University of Marburg, 1997.

[12] Z. Mei, F. Theil, Variation of bifurcations along a homotopy from Neumann to Dirichlet problems, Nonlinear Anal. Theory Methods Appl. 27 (1996) 1381–1395.
[13] A. Mielke, The complex Ginzburg–Landau equation on large and unbounded domains: sharper bounds and attractors, Nonlinearity 10 (1997) 199–222.
[14] H.G. Othmer, Applications of bifurcation theory in the analysis of spatial and temporal pattern formation, in: O. Gurel, O.K. Rossler (Eds.), Bifurcation Theory and Application in Scientific Disciplines, New York Academy of Sciences, New York, 1979, pp. 64–77.
[15] H.G. Othmer, The analysis of nonlinear equations with symmetries, in: P. Holmes (Ed.), New Approaches to Nonlinear Problems in Dynamics, SIAM, Philadelphia, 1980, pp. 437–447.
[16] D. Schaeffer, M. Golubitsky, Boundary conditions and mode jumping in the buckling of a rectangular plate, Comm. Math. Phys. 69 (1979) 209–236.
[17] N.M. Wigley, Mixed boundary value problems in plane domain with corners, Math. Z. 115 (1970) 33–52.