A new index of component importance

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Abstract

Several indices of component importance have been proposed in the literature. The index most relevant to system reliability is the Birnbaum importance, but a comparison between two components is hard to prove and even hard to verify by computation. In this paper we propose a new index which is stronger than the Birnbaum comparison, but always verifiable by computing. In many cases, it could even be easier to prove. We also show the relation of this index with other indices. © 2001 Published by Elsevier Science B.V.

Keywords: Component importance; Birnbaum importance; System reliability; Cutset

1. Introduction

An importance index measures the relative importance of a component with respect to other components in a system. Here, a component has two aspects: the structural aspect and the reliability aspect. The former refers to the location of the component in the system, and the latter refers to the reliability of the physical unit installed at that location. The structural aspect is relevant in building a system when several components with distinct reliabilities can be arbitrarily assigned to several locations in the system. Presumably, we want to assign the more reliable components to the more important locations. The reliability aspect comes into the picture when the components are already installed in the system but there is budget to improve the system reliability through the improvement of the reli-

ability of a component. Here, the global improvement as a consequence of the local improvement, depends not only on the location of the local improvement, but also on the component reliabilities.

Many importance indices have been proposed. Some, like the Birnbaum importance and the Fussell–Vesely importance, are more oriented towards the reliability aspect. Others, like the criticality importance, the structural importance, the cut importance and the structural Fussell–Vesely importance, have removed the component reliabilities from consideration and compare only the structural aspect. In this paper we focus on the structural aspect. To bring the importance indices oriented towards the reliability aspect into line, we consider the i.i.d. model, i.e., each component has the same reliability p. Note that even though the comparison of two components now reflects only the strengths of their relative locations, this comparison, unlike those importance indices oriented towards the structural aspect, is still a function of p.

To eliminate this difference, we call a comparison *universal* if it holds for all p.

The importance index most relevant to system reliability is the Birnbaum importance since its very definition implies that the system improvement is monotone in the index order (all indices have this monotonicity as a goal). However, the Birnbaum importance is computed through computing the reliabilities of some subsystems, which themselves are computed by recursive equations. Since an explicit expression of the Birnbaum importance is usually not available, it is difficult to compare the Birnbaum importance. Even when a comparison holds for all p, we cannot verify this through computation since there are infinitely many values of p (in the unlikely case that an explicit expression is available, then some numerical procedures to test the roots of the difference function of the two compared importances can be used). On the other hand, other importance indices cannot replace the Birnbaum importance since either they are too strong, like the criticality importance, such that not many comparisons can be made, or they are too weak, like the structural importance, such that the above-mentioned monotonicity relation is much in doubt.

In this paper we propose a new importance index which seems to preserve the "relevance" of the Birnbaum index, and yet removes its weakness. More specifically, dominance in this new importance implies dominance in the universal Birnbaum comparison (so that the monotonicity property is preserved), but it requires only finite computation to confirm a comparison. Also, it does not seem harder to prove this comparison than the Birnbaum comparison. In fact, in a companion paper [5], we show how some Birnbaum comparison hanging open in the literature can be proved through this new index.

Finally, we mention that the notion of component importance also appears in evaluating the strength of a variable in Boolean functions [7], in particular, in evaluating the power of a player in a winner-takes-all game [8].

2. A review of the importance indices

Many importance indices are closely related to the notion of cutsets. A cutset is a subset of components whose collective failures will cause the system failure. A cutset is minimal if it has no proper subset as a cutset. Let $C(\bar{C})$ denote the set of (minimal) cutsets, $C(d)$ $(\bar{C}(d))$ denotes the set of (minimal) cutsets of cardinality d, where $C_i(d)$ and $C_{(i)}(d)$ $(\bar{C}_i(d)$ and $\bar{C}_{(i)}(d))$ are, respectively, the set of (minimal) cutsets of size d that contains and that does not contain component i. Finally, A pathset is a subset of components whose collective successes will cause the system success. We define $P, \bar{P}, P_i(d), P_{(i)}(d), \bar{P}_i(d), \bar{P}_{(i)}(d)$ similarly.

Let I_i^x denote the importance index of component i under the type x. Sometimes, an importance index is defined implicitly through a comparison. We will give complete definitions of those indices commonly discussed in the literature. In particular, we will express them in terms of cutsets or pathsets for easier comparisons.

Criticality importance [3]: Let S denote a subset containing neither i nor j. Then $I_i^t \geqslant I_j^t$ if $\{j\} \cup S \in C_j \Rightarrow \{i\} \cup S \in C_i$ (t denotes criticality).

Let $R(n, p)$ denote the reliability of a system with n components each independently has reliability p.

Birnbaum importance [2] $I_i^{b}(p) = (\partial R / \partial p_i)(n, p)$.

We now show that the Birnbaum importance can be expressed in terms of pathsets. Let $R(n, p{:}1_i)$ and $R(n, p{:}0_i)$ denote the system reliability conditional on component i being working or failed. Let $\bar{R} = 1 - R$ denote the unreliability.

Lemma 1.

$$I_i^{b}(p) = \sum_{d=0}^{n} (|P_i(d)| - |P(d)|) p^{n-d} q^d. \tag{2.1}$$

Proof.

$$I_i^{b}(p) = \frac{\partial}{\partial p_i}[p_i R(n, p{:}1_i) + (1 - p_i) R(n, p{:}0_i)]$$

$$= p_i R(n, p{:}1_i) - R(n, p{:}0_i)$$

$$= \sum_{d=0}^{n} |P_i(d)| p^{n-d} q^d - \sum_{d=0}^{n} |P_{(i)}(d)| p^{n-d} q^d$$

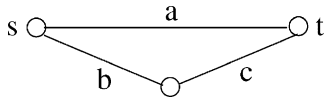$$= \sum_{d=0}^{n} |P_i(d)| - |P(d)|) p^{n-d} q^d. \qquad \square$$

Fig. 1.

Structural importance [1]: The structural importance I_i^s is the special case of the Birnbaum importance with $p = 1/2$.

Lemma 2. $I_i^s \geqslant I_j^s$ if and only if $|P_i| \geqslant |P_j|$.

Proof. By Lemma 1,

$$I_i^s(p) - I_p^s(p) = \left(\frac{1}{2}\right)^n \sum_{d=0}^n (|p_i(d)| - |p_j(d)|)$$

$$= (1/2)^n (|p_i| - |p_j|). \quad \square$$

Let $N_i(d,s)$ denote the number of unions of s sets in $\bar{C}$ such that the union has cardinality d and contains i. Define

$$N_i(d) = \sum_{s \geqslant 1} (-1)^{s+1} N_i(d,s)$$

Cut importance [4]. $I_i^c = (N_i(1), N_i(2), \ldots)$. $I_i^c \geqslant I_j^c$ means that I_i^c is lexicographically larger than I_j^c.

Butler [4] proved that the cut comparison is identical to the Birnbaum comparison with $p \to 1$.

Fussell–Vesely importance [6,9]: $I_i^{fv} = P(\exists\, c \in \bar{C}_i$ s.t. $c \subseteq \{i\} \cup X \,|\,$ a random (under uniform distribution) X is a cutset).

Structural Fussell–Vesely importance [6,9]:

$$I_i^{sfv} = \frac{|X \in C : \exists\, c \in \bar{C}_i \text{ s.t. } c \subseteq \{i\} \cup X|}{|X \in C|}.$$

I_i^{sfv} is a special case when $p = 1/2$.

The two Fussell–Vesely importance indices are not very useful since they cannot even compare a and b in Fig. 1. Suppose the network in Fig. 1 works if there exists a path connecting s and t. Then $C = \{ab, ac, abc\}$, $\bar{C}_a = \{ab, ac\}$, $\bar{C}_b = \{ab\}$, $\bar{C}_c = \{ac\}$. It is easily verified that for any $X \in C$, $\exists\, c \in \bar{C}_i$ s.t. $c \subseteq \{i\} \cup X$. Hence $I_a^{fv} = I_b^{fv} = I_c^{fv}$.

For u and v two types of importance, let $u \Rightarrow v$ mean that $I_j^u \geqslant I_j^u \Rightarrow I_i^v \geqslant I_j^v$. For $v \in \{b, fv\}$, if $I_i^v \geqslant I_j^v$ for all p, we call v strong v. Clearly, strong $v \Rightarrow v$.

3. A new index

We define a new importance index , $I_i^h = \{|C_i(d)|\}$, $d = 1, 2, \ldots,$ where $I_i^h \geqslant I_j^h$ means $|C_i(d)| \geqslant |(C_j(d))|$ for all d. A comparison of this index does not require a containment relation, only a numerical dominance which has to hold for every d.

We first show that I_i^h passes some elementary tests of reasonableness. For example, if i is an irrelevant component, then for every cutset $i \cup s$ where $j \notin s$, $j \cup s$ is also a cutset since s is. Therefore,

$$|C_i(d)| \leqslant |C_j(d)| \text{ for all } d.$$

On the other hand, if $\{i\}$ is a cutset, then for any cutset $j \cup s$ where $i \notin s$, $i \cup s$ is also a cutset. Therefore,

$$|C_i(d)| \leqslant |C_j(d)| \quad \text{for all } d.$$

For a component i not necessarily so extreme, we prove that a comparison on the new index is between the criticality comparison and the strong Birnbaum comparison.

Theorem 3. $t \Rightarrow h \Rightarrow$ strong b.

Proof. By definition of t, every cutset in C_j has a distinct counterpart in C_i. Clearly,

$$|C_i(d)| \leqslant |C_j(d)| \quad \text{for all } d.$$

Hence $t \Rightarrow h$.

Note that S is a cutset in C_i if and only if the complement of S is not a pathset. Furthermore, the complementary set does not contain i and has size $n - d$. Thus

$$|C_i(d)| + |P_{(i)}(n-d)| = 2^{n-1}.$$

By Lemma 1

$$|C_i(d)| \geqslant |C_j(d)| \text{ for all } d$$

$$\Rightarrow |P_{(j)}(n-d)| \geqslant |P_{(i)}(n-d)| \text{ for all } d$$

$$\Rightarrow |P(n-d) - |P_j(n-d)| \geqslant |P(n-d)|$$

$$- |P_i(n-d)| \text{ for all } d$$

$$\Rightarrow |P_i(n-d)| \geqslant |P_j(n-d)| \text{ for all } d$$

$$\Rightarrow I_i^b(p) \geqslant I_j^b(p).$$

Since the above holds for all p, $h \Rightarrow$ strong b. $\quad \square$

We can also compute $|C_i(d)|$ from $N_i(d)$.

Lemma 4.

$$|C_i(d)| = \sum_{k \leqslant d} \left[N_i(k) \binom{n-k-1}{d-k} + N(k) \binom{n-k-1}{d-k-1} \right].$$

Proof. Each cutset counted in $N_i(k,1)$, i.e., each minimal cutset, can become a cutset in $C_i(d)$ by adding some $d-k$ components from the remaining $n-k$ components, and there are

$$\binom{n-k}{d-k}$$

such choices. Similarly, each cutset counted in $N_{(i)}(k,1)$ can become a cutset in $C_i(d)$ by adding component i and some other $d-k-1$ components from the remaining $n-k-1$ components, and there are

$$\binom{n-k-1}{d-k-1}$$

such choices. However, these additions could induce the same cutset in $C_i(d)$. $N_i(d,2)$ counts the number of pairs of minimal cutsets whose additions overlap. But if three cutsets overlap, then $N_i(d,2)$ will over-correct; thus we add back $N_i(d,3)$. By the inclusion-exclusion principle, we have

$$|C_i(d)|$$

$$= \sum_{k \leqslant d} \sum_{s \geqslant 1} (-1)^{s+l}$$

$$\times \left[N_i(k,s) \binom{n-k}{d-k} + N_{(i)}(k,s) \binom{n-k-1}{d-k-1} \right]$$

$$= \sum_{k \leqslant d} \sum_{s \geqslant 1} (-1)^{s+l}$$

$$\times \left[N_i(k,s) \binom{n-k}{d-k} + N(k,s) - N_i(k,s) \binom{n-k-1}{d-k-1} \right]$$

$$= \sum_{k \leqslant d} \sum_{s \geqslant 1} (-1)^{s+l}$$

$$\times \left[N_i(k,s) \binom{n-k-1}{d-k} + N(k,s) \binom{n-k-1}{d-k-1} \right]$$

$$= \sum_{k \leqslant d}$$

$$\times \left[N_i(k) \binom{n-k-1}{d-k} + N(k) \binom{n-k-1}{d-k-1} \right]. \quad \square$$

Corollary 5. $h \Rightarrow c$.

Proof. Assume $I_i^h \geqslant I_j^h$, i.e., $|C_i(d)| \geqslant |C_j(d)|$ for all d. Suppose to the contrary that $I_i^c < I_j^c$, i.e., there exists a $d \geqslant 1$ such that

$$N_i(k) = N_j(k) \quad \text{for } k = 1, \ldots, d-1$$

and

$$N_i(k) < N_j(k).$$

Then by Lemma 4

$$|C_i(d)| = \sum_{k \leqslant d} \left[N_i(k) \binom{n-k-1}{d-k} + N(k) \binom{n-k-1}{d-k-1} \right]$$

$$< \sum_{k \leqslant d} \left[N_j(k) \binom{n-k-1}{d-k} + N(k) \binom{n-k-1}{d-k-1} \right]$$

$$= |C_j(d)|.$$

This contradicts our assumption $|C_i(d)| \geqslant |C_j(d)|$ for all d. $\square$

With h added, the relations among the importance comparisons are:

$$t \Rightarrow h \Rightarrow \text{strong } b \Rightarrow s \qquad \text{strong } fv \Rightarrow sfv$$
$$\searrow\!\!\!\searrow \ c$$

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