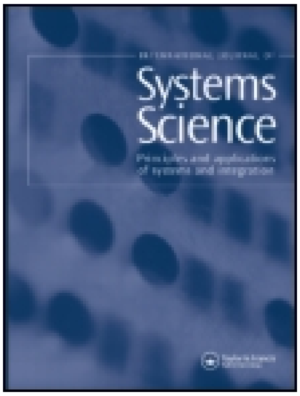

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Design of sampled-data systems with large plant uncertainty using quantitative feedback theory

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Design of sampled-data systems with large plant uncertainty using quantitative feedback theory

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This paper proposes a new quantitative feedback theory (QFT) design framework for dealing with sampled-data systems with large plant uncertainty. After the QFT-based design in the continuous-time domain is completed, the analogue controller can be transformed directly into a rational discrete-time transfer function via approximate Z transform, with the sampling time as a free parameter. The sampling time can therefore be adjusted to make the uncertain sampled-data system robustly stable. In comparison with other approaches, our approach is much more systematic without the solvability problem and yet significant enough to guide the designer to realize the physical controller in which the plant transfer function has prescribed bounds on its parameters. Several examples are used to illustrate the proposed approach and excellent results are obtained.

1. Introduction

In the 1960s, Issac Horowitz continued the pioneering work of Bode and introduced a frequency-domain design methodology (Horowitz 1963) that was refined in the 1970s to its present form, commonly referred to as the quantitative feedback theory (QFT) (Horowitz and Sidi 1972, Horowitz and Wang 1979a,b, Sidi 1973). The QFT is considered as a practical engineering method for the robust controller design of continuous-time feedback systems, based on frequency-domain design methodologies. In QFT, one of the main objectives is to design a simple low-order controller as a natural requirement in practice to avoid problems with noise amplification, resonance and unmodelled high-frequency dynamics. In any real life design, iterations in QFT design are inevitable and QFT can offer direct insight into the available trade-off between controller complexity and specifications during such iterations.

For the QFT design of robust sampled-data systems, Sidi (1977) applied the QFT design procedure for single-loop sampled feedback systems in which the plant transfer function has prescribed bounds on its parameters. Tsai and Wang (1987) extended Wiener's least-squares optimization with a quadratic constraint to the design of a digital controller with large plant uncertainty. Horowitz and Liao (1986) extended QFT to sampled-data structures by finding the minimum sampling frequency $(\omega_s)_{\min}$ by transformation from the z domain to the w domain. However, it is important to note that in the w domain any practical $L(w)$ (loop transmission) is a non-minimum phase. Contrary to the minimum-phase feedback problem, no uniqueness theorem can be expected for an optimal $L(w)$, since a solution for the problem is not guaranteed. It was demonstrated that a realistic relaxation of the design specifications could generally lead to the solvability of the problem (Sidi 1976). However, we do not know what minimum degradation is needed in the specifications so that the problem becomes solvable.

In this paper we propose another QFT design framework for dealing with robust sampled-data systems. This new design framework is based upon the digital redesign methodology. The approximate Z transform using higher-order integrators (Wang and Hsu 1998a,b) is adopted to convert the analogue controller

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$G_c(s)$ (obtained by QFT methodology) into a digital controller $G_c(z)$ with free sampling time T_s. The stable range of T_s can then be found through the Kharitonov-type methodology for closed-loop characteristic polynomials with perturbed coefficients. Finally, a pre-filter is used to make sure that the closed-loop transfer functions for the given plants lie within the performance specifications. In comparison with previous approaches, our proposed method provides a more systematic QFT design of robust sampled-data systems without the solvability problem. Excellent results are obtained for several fully illustrated numerical examples.

2. Quantitative design theory: two degrees of freedom

A general introduction to the QFT technique is presented in this section. This design is based upon specifying the tolerance in the frequency domain by means of the sets of plant transfer functions $\wp = \{P(j\omega)\}$ and closed-loop control ratios $\Im(j\omega) = \{T(j\omega)\}$ and finding the resulting bounds on the loop transfer functions $L(s) = G(s)P(s)$ and input filter transfer functions $F(s)$.

The QFT technique can be viewed by considering the unity-feedback cascade compensated control system in figure 1, where G is a compensator and P is the plant, in which the plant parameters vary over some known range or there is plant parameter uncertainty. Since the design goal is to decide $G(s)$ and $F(s)$, we define that there are two degrees of freedom for the QFT design in figure 1. The loop transmission L is defined as

$$L = GP, \tag{1}$$

and the control ratio of the unity-feedback system of figure 1 is

$$T = \frac{Y}{R} = F\frac{L}{1+L}. \tag{2}$$

The plant with nominal plant parameters is denoted as P_0; thus, $L_0 = GP_0$. For a given $G(j\omega)$ and $P_0(j\omega)$, a plot of $\ln[L_0(j\omega)]$ versus $\angle L_0(j\omega)$ on the Nichols chart (NC) can be obtained. From this plot on the NC, the closed-loop frequency data can be obtained by plotting $M_0(j\omega)\angle\alpha(j\omega)$ versus ω, where

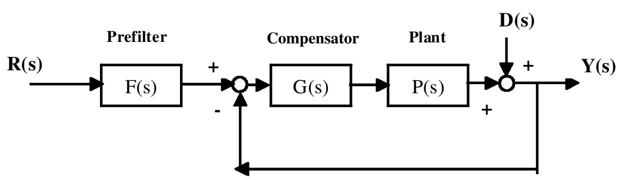

Figure 1. Structure of a two-degrees-of-freedom system.

$$M_0(j\omega)\angle\alpha(j\omega) = \frac{Y(j\omega)}{R(j\omega)} = \frac{L_0(j\omega)}{1+L_0(j\omega)}. \tag{3}$$

Note that, for the nominal plant $P_0(j\omega)$, the nominal loop transmission is

$$\ln L_0 = \ln(GP_0) = \ln G + \ln P_0 \tag{4}$$

whereas, for all other plants $P(j\omega)$,

$$\ln L = \ln(GP) = \ln G + \ln P. \tag{5}$$

Thus, for $\omega = \omega_i$, the variation $\delta_p(j\omega_i)$ in $\ln[L(j\omega_i)]$ is given by

$$\begin{aligned}\delta_p(j\omega_i) &= \ln[L(j\omega_i)] - \ln[L_0(j\omega_i)]\\ &= \ln[P(j\omega_i)] - \ln[P_0(j\omega_i)]\end{aligned} \tag{6}$$

and

$$\begin{aligned}\angle\Delta P(j\omega_i) &= \angle L - \angle L_0\\ &= (\angle G + \angle P) - (\angle G + \angle P_0)\\ &= \angle P - \angle P_0\end{aligned} \tag{7}$$

A variation in P results in a horizontal translation in the phase angle of P (see (7)), and a vertical translation in the logarithmic magnitude value of P (see (6)). We can therefore obtain the corner points on the NC from the bounds of uncertainty parameters in P. The essence of QFT is therefore to determine the variations in the system due to the plant uncertainty from the corner plot on the NC. Therefore, at each ω_i, the optimal bounds on $L(j\omega)$ can be determined. Design of a proper $L_0(s)$ guarantees only that the variation in $|T_R(j\omega)|$ is less than or equal to that allowed. The purpose of the pre-filter in figure 1 is to position $\ln[T(j\omega)]$ with the frequency-domain specifications. This graphical description of the effect of plant uncertainty is the basis of the QFT technique.

3. Quantitative feedback theory for the sampled-data system

QFT was extended to the synthesis of a sampled-data feedback system for prescribed tolerance (Sidi 1976, 1977, Horowitz and Liao 1984, 1986, Tsai and Wang 1987). For the design of sampled-data feedback systems, a pulse transfer function $P^*(s)$ can be described in three domains, which are s, z and w domains. The following transformations are commonly used:

$$P^*(s) \xrightarrow{z=e^{T_s}\Leftrightarrow s=(1/T_s)\ln z} P(z) \xrightarrow{z=(w+1)/(w-1)} P(w),$$

where T_s is the sampling period. The Z transformation $P(z)$ is very difficult to obtain analytically, if not impossible. Thus we adopt the approximate Z transform using higher-order integrators in this paper to obtain the approximate $P(z)$. Further, owing to the similarity between the w domain and the s domain, design in the w domain is a general practice in a sampled-data system. (Horowitz and Liao 1986) showed that, if the continuous transfer function $P(s)$ is of an order higher than that at high frequencies, and $P(s)$ does not contain a pure time delay, then, in the w domain, $P(w)$ will have one non-minimum-phase zero located at $w = 1$. Then the loop transmission around $P(w)$ can be rewritten as

$$L(w) = \frac{w-1}{w+1} L_{\mathrm{mp}}(w) = A(w) L_{\mathrm{mp}}(w)$$

where $L_{\mathrm{mp}}(w)$ is a minimum-phase transfer function and $A(w)$ is a dipole $(w-1)/(w+1)$. $A(w)$ is an all-pass transfer function because $|A(\mathrm{j}\omega)| = 1$ for all ω. However, $\arg[A(\mathrm{j}\omega)] = -2\arctan(\omega)$; so its phase lag increases from zero at $\omega = 0$ to 90° at $\omega = 1$ and approaches 180° as ω approaches ∞. It is well known (Sidi 1976, 1977, Horowitz and Liao 1984, 1986, Tsai and Wang 1987) that this phase lag (delay) limits heavily the achievable bandwidth which can be obtained in a stable feedback system having such a non-minimum-phase zero. It was also shown in (Sidi 1976, 1977, Horowitz and Liao 1984, 1986, Tsai and Wang 1987) that no uniqueness theorem can be expected for an optimal $L(w)$ in the non-minimum-phase system, since a solution to the problem is not guaranteed. For the solvability of the problem, it was demonstrated (Sidi 1976) that a realistic relaxation of the design specifications is needed. To bypass the above difficulties, we propose the digital redesign framework in figure 2.

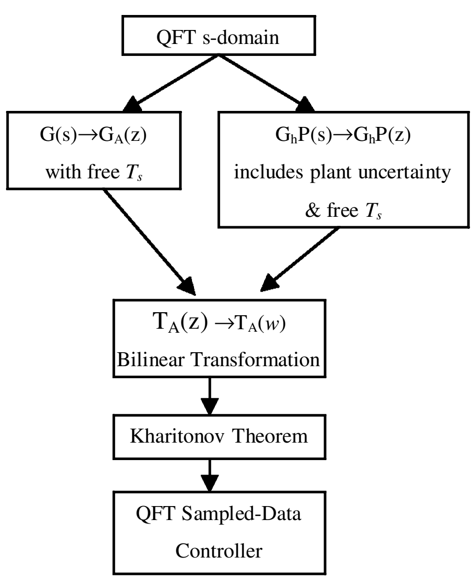

Figure 2. Digital redesign framework for QFT sampled-data design.

The redesign procedure can be briefly described as follows. First the discrete equivalent controller $G_{\mathrm{A}}(z)$ with free sampling time T_{s} is obtained by converting the continuous-time controller (obtained from QFT design in the s domain). The zero-order hold (ZOH) and plant $P(s)$, $G_{\mathrm{h}}(s)P(s)$, is also converted into $G_{\mathrm{h}}P(z)$ by the approximate Z transform. The $G_{\mathrm{h}}P(z)$ contains free T_{s} and plant uncertainties. Finally by transforming the discretized system from the z plane to the w plane, the maximum range of the sampling time T_{s} of the closed-loop sampled-data system, which meets all design requirements, can then be determined by the Routh–Hurwitz criterion and the Kharitonov theorem.

4. Approximate Z transform using higher-order integrators (Wang *et al*. 1990, 1994, Wang and Hsu 1998 a, b)

In recent years, computers have become indispensable in the analysis and design of control systems. A digital computer can accept only sequences of numbers, and its outputs again consist only of sequences of numbers. Many numerical methods have been proposed to approximate a differential equation by a difference equation. The approximate z transform of a continuous-time system $G(s)$ $(G(s) = L\{g(t)\})$, can be written as (Wang and Hsu 1998 a, b)

$$G(z) = Z[G(s)] \approx Z_{\mathrm{A}}[G(s)] = G_A(z)$$
$$= G(s)|_{s^{-k}=(T_{\mathrm{s}}/2)^k[R_k(z^{-1})/(1-z^{-1})^k]} \frac{1}{T_{\mathrm{s}}} \tag{8}$$

where Z and Z_{A} are the exact and approximate Z transform operations respectively and T_{s} is the sampling period; s^{-k} is the higher-order integrator of power k defined as (Wang *et al.* 1990):

$$s^{-k} \approx \left(\frac{T_{\mathrm{s}}}{2}\right)^k (\nu_0 + \nu_1 u^{-1} + \nu_2 u^{-2} + \cdots + \nu_k u^{-k})$$
$$= \left(\frac{T_{\mathrm{s}}}{2}\right)^k \frac{R_k(z^{-1})}{(1-z^{-1})^k}, \tag{9}$$

where $R_k(z^{-1})$ has been given by Wang *et al.* (1990). The approximate Z transform via higher-order integrators provides a strong correspondence between the s domain and the z domain with the sampling time T_{s} as a free parameter for adjustment of performance matching.

5. Maximum stable sampling time T_{s} of the redesigned systems

If the ZOH is used as a digital-to-analogue converter, the plant is embedded in a linear two-degrees-of-freedom sampled feedback configuration as shown in

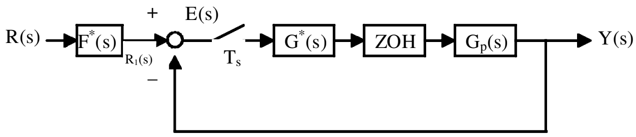

Figure 3. Two-degrees-of-freedom sampled-data control system.

figure 3. In the discrete time domain, we can facilitate the analysis of the sampled-data control system, since z-domain tools are readily available. Therefore, the approximate Z transform using higher-order integrators of $G_h G_p(z)$ is defined as

$$\begin{aligned} G_h G_p(z) &= Z\left(\frac{1-e^{-T_s s}}{s} G_p(s)\right) \\ &= (1-z^{-1}) Z\left(\frac{G_p(s)}{s}\right) \\ &\approx (1-z^{-1}) \frac{G_p(s)}{s}\bigg|_{s^{-k}=(T_s/2)^k[R_k(z^{-1})/(1-z^{-1})^k]} \frac{1}{T_s}. \end{aligned}$$

It is well known that theoretical difficulties exist in the sampled-data feedback systems. Contrary to the minimum-phase feedback problem, no uniqueness theorem can be expected for an optimal loop transmission $L(w)$, since a solution to the problem is not guaranteed. In order to lead generally to the solvability of the problem, it was elucidated that realistic relaxation of the design specifications is needed. However, we do not know what minimum degradation is needed in the specifications to solve the problem. To overcome this difficulty, we propose a digital redesign framework in this paper for dealing with QFT sampled-data systems. We shall convert directly an analogue controller into a digital controller by the approximate Z transform using higher-order integrators. Then the range of stable sampling times T_s can be determined by the Kharitonov theorem.

We first consider a continuous-time single-input single-output (SISO) negative unity feedback system as shown in figure 1. The plant uncertainty is defined by a set $\wp = \{P(s)\}$ of possible plants, where $P(s)$ is a strictly proper transfer function. A controller $G(s)$ and pre-filter $F(s)$ is designed by the s-domain QFT technique to satisfy the following system requirements:

(i) robust stability;

(ii) robust margins

$$|T(j\omega)| = \left|\frac{G(j\omega)P(j\omega)}{1+G(j\omega)P(j\omega)}\right| < \gamma;$$

(iii) robust tracking (related to tracking step response)

$$a(\omega) < \left|F(j\omega)\frac{G(j\omega)P(j\omega)}{1+G(j\omega)P(j\omega)}\right| < b(j\omega),$$

where $T(s)$ is a closed-loop transfer function.

As for the redesigned digital system of the continuous system by approximate Z transform using higher-order integrators, the following theorem gives the result.

Theorem 1: The two-degrees-of-freedom continuous system as shown in figure 3 has the following approximate Z transform of the digital redesigned closed-loop transfer function $T_A(z) = Y(z)/R(z)$ using higher-order integrators:

$$\begin{aligned} T_A(z) &= F(s)|_{s^{-k}=(T_s/2)^k[R_k(z^{-1})/(1-z^{-1})^k]} \\ &\times \frac{[G(s)|_{s^{-k}=(T_s/2)^k[R_k(z^{-1})/(1-z^{-1})^k]}]\,[(1-z^{-1})(G_p(s)/s)|_{s^{-k}=(T_s/2)^k[R_k(z^{-1})/(1-z^{-1})^k]}1/T_s)]}{1+[G(s)|_{s^{-k}=(T_s/2)^k[R_k(z^{-1})/(1-z^{-1})^k]}]\,[(1-z^{-1})(G_p(s)/s)|_{s^{-k}=(T_s/2)^k[R_k(z^{-1})/(1-z^{-1})^k]}(1/T_s)]} \\ &= F_A(z)\frac{G_A(z)G_h G_p(z)}{1+G_A(z)G_h G_p(z)}, \end{aligned} \quad (10)$$

where $G_h(s) = (1-e^{-T_s s})/s$ is a ZOH and T_s is the sampling period. $G(s)$ and $F(s)$ are obtained to achieve all requirements by the conventional QFT methodology.

Proof: From figure 3, the Z transforms of the error signal and the output signal are

$$\begin{aligned} E(s) &= R_1(s) - Y(s), \\ E^*(s) &= R_1^*(s) - Y^*(s) \end{aligned} \quad (11)$$

and

$$Y(s) = E^*(s)G_h(s)G_p(s). \quad (12)$$

Substituting (11) into (12) yields

$$Y(s) = [R_1^*(s) - Y^*(s)]G^*(s)G_h(s)G_p(s). \quad (13)$$

Hence

$$Y^*(s) = [R_1^*(s) - Y^*(s)]G^*(s)[G_h(s)G_p(s)]^*.$$

Since

$$R_1^*(s) = F^*(s)R^*(s),$$

simple manipulation yields

$$\begin{aligned} T_A(z) &= \frac{Y(z)}{R(z)} \\ &= F_A(z)\frac{G_A(z)G_h G_p(z)}{1+G_A(z)G_h G_p(z)}, \end{aligned} \quad (14)$$

where $G_A(z)$ and $F_A(z)$ are discrete equivalents using higher-order integrators of the analogue controller and pre-filter and $G_h G_p(z)$ is the approximate Z transform of $G_h G_p(s)$, that is

$$\begin{aligned} G_A(z) &= Z_A\{G(s)\} \\ &= G(s)|_{s^{-k}=(T_s/2)^k[R_k(z^{-1})/(1-z^{-1})^k]} \end{aligned} \tag{15}$$

$$\begin{aligned} F_A(z) &= Z_A\{F(s)\} \\ &= F(s)|_{s^{-k}=(T_s/2)^k[R_k(z^{-1})/(1-z^{-1})^k]} \end{aligned} \tag{16}$$

and

$$\begin{aligned} G_h G_p(z) &= Z\left(\frac{1-e^{-T_s s}}{s} G_p(s)\right) \\ &= (1-z^{-1})Z\ \frac{G_p(s)}{s} \\ &= (1-z^{-1})\ \frac{G_p(s)}{s}\bigg|_{s^{-k}=(T_s/2)^k[R_k(z^{-1})/(1-z^{-1})^k]} \frac{1}{T_s}. \end{aligned} \tag{17}$$

Substituting (15)–(17) into (14), the proof is completed. □

Since the sampling time is T_s and the plant uncertainty is defined by a set $\wp$, (11) can be rewritten as

$$T_A(z) = F_A(z)\frac{G_A(z, T_s, q)G_h G_p(z, T_s, q)}{1 + G_A(z, T_s, q)G_h G_p(z, T_s, q)}, \tag{18}$$

where $q \in \wp$. For robust stability checking, the robust stability analysis can be performed by the Jury stability criterion and the Kharitonov theorem (Yeung and Wang 1987, Chapellat and Bhattacharyya 1989, Barmish 1994). In order to apply the Kharitonov theorem, we should use the Mobius transformation $z = (w+1)/(w-1)$ to transform $T_A(z)$ to $T_A(w)$ and then apply the Routh–Hurwitz criterion to four Kharitonov polynomials to find the desired sampling time range to achieve robust stability. Associated with the interval polynomial $\sum_{i=0}^{n} [q_i^-, q_i^+]w^i$, the four fixed Kharitonov polynomials are defined as

$$\begin{aligned} K_1(w) &= q_0^- + q_1^- w + q_2^+ w^2 + q_3^+ w^3 \\ &\quad + q_4^- w^4 + q_5^- w^5 + q_6^+ w^6 + \cdots, \\ K_2(w) &= q_0^+ + q_1^+ w + q_2^- w^2 + q_3^- w^3 \\ &\quad + q_4^+ w^4 + q_5^+ w^5 + q_6^- w^6 + \cdots, \\ K_3(w) &= q_0^+ + q_1^- w + q_2^- w^2 + q_3^+ w^3 \\ &\quad + q_4^+ w^4 + q_5^- w^5 + q_6^- w^6 + \cdots, \\ K_4(w) &= q_0^- + q_1^+ w + q_2^+ w^2 + q_3^- w^3 \\ &\quad + q_4^- w^4 + q_5^+ w^5 + q_6^+ w^6 + \cdots. \end{aligned}$$

An interval polynomial family $\wp$ with invariant degree is robustly stable if and only if its four Kharitonov polynomials are stable.

The maximum stable sampling time T_s of the redesigned system can be obtained by applying the Kharitonov theorem to four fixed Kharitonov polynomials defined in above equations. The intersection range of four stable ranges corresponding to each Kharitonov polynomial is our final result.

6. Examples

In order to demonstrate the effectiveness of our digital redesign framework for QFT sampled-data systems, two examples will be considered in this section. Example 1 has two free parameters, whereas example 2 is a more complicated systems with three free parameters.

6.1. *Example 1* (*Borghesani et al. 1994*)

Consider a continuous-time SISO negative unit feedback system. The plant $G_p(s)$ has a parametric uncertainty model:

$$G_p(s) = \left\{\frac{ka}{s(s+a)} : k \in [1, 10],\ a \in [1, 10]\right\}.$$

The performance specifications are to design a controller $G(s)$ and a pre-filter $F(s)$ such that they achieve the following:

(i) robust stability:

(ii) robust margins (via closed-loop magnitude peaks)

$$\left|\frac{G(j\omega)G_p(j\omega)}{1+G(j\omega)G_p(j\omega)}\right| < 1.2, \quad \omega > 0;$$

(iii) robust tracking (related to the tracking of step responses)

$$a(\omega) < \left|F(j\omega)\frac{G(j\omega)G_p(j\omega)}{1+G(j\omega)G_p(j\omega)}\right| < b(\omega), \quad \omega < 10,$$

$$B_L(\omega) = a(\omega) = \left|\frac{120}{(j\omega)^3 + 17(j\omega)^2 + 828(j\omega) + 120}\right|$$

$$B_U(\omega) = b(\omega) = \left|\frac{0.6584(j\omega+30)}{(j\omega)^2 + 4(j\omega) + 19.752}\right|.$$

The above $B_L(\omega)$ and $B_U(\omega)$ are used in figures 6 and 9 later.

The objective is first to design a controller $G(s)$ and a pre-filter $F(s)$ to meet all requirements by using QFT methodology. Then we convert $G(s)$ into digital equivalent $G_A(z)$ and $G_p(s)$ into $G_h G_p(z)$ by the approximate Z transform and redesign the system as shown in figure 3.

Finally we can find the range of stable sampling times so that the design performance can be achieved.

Design procedure:

Step 1. *QFT design for continuous systems*. The nominal plant is chosen as $k = 1$, $a = 1$. From the above discussion of the QFT design framework, the achieved loop transmission $L(s) = G(s)G_p(s)$ is shown in figure 4, and the control network is found to be

$$G(s) = \frac{9\left(\dfrac{s}{1.1}+1\right)\left(\dfrac{s}{113.8}+1\right)}{\left(\dfrac{s}{42.81}+1\right)\left(\dfrac{s^2}{1000^2}+\dfrac{1.486s}{1000}+1\right)}.$$

Next, we shall use the second degree of freedom in order to attain the filter specifications, so that all $|F(j\omega)T(j\omega)|$ should lie within the permitted bounds. The desired filter is obtained as

$$F(s) = \frac{1}{\dfrac{s^2}{4^2}+\dfrac{1.4s}{4}+1},$$

and the frequency responses of the final results for the four extremes of the uncertain plant $|F(j\omega)T(j\omega)|$ are shown in figures 5 and 6:

(i) robust margin;

(ii) robust tracking;

Step 2. *redesign digital control system*. The discrete equivalent approximate Z transform of the analogue controller $G(s)$ using high-order integrators is obtained as

$$G_A(z) = G(s)|_{s^{-k}=(T_s/2)^k[R_k(z^{-1})/(1-z^{-1})^k]} = \frac{\alpha(b_0z^3+b_1z^2+b_2z+b_3)}{a_0z^3+a_1z^2+a_2z+a_2},$$

where

$$\alpha = \frac{(1155.87\times 10^6)T_s}{250.36},$$

$$b_0 = 2+38.3T_s,$$
$$b_1 = -2+244.7T_s,$$
$$b_2 = -2-344.7T_s+250.36T_s^2,$$
$$b_3 = 2-38.3T_s,$$

and

$$a_0 = 12+9172.86T_s+1\,063\,615.66T_s^2,$$
$$a_1 = 26-9172.86T_s+9\,572\,540.94T_s^2 + 25.686\times 10^7T_s^3,$$
$$a_2 = 36-9172.86T_s-9\,572\,540.94T_s^2 + 25.585\times 10^7T_s^3,$$
$$a_3 = 12+9172.86T_s-1\,063\,625.66T_s^2.$$

The approximate Z transform of the plant $G_p(s)$ is expressed as

$$G_hG_p(z) = (1-z^{-1}) \times \frac{G_p(s)}{s}\bigg|_{s^{-k}=(T_s/2)^k[R_k(z^{-1})/(1-z^{-1})^k]} \frac{1}{T_s} = \frac{\dfrac{kT_s^2}{2}(z+1)}{\left(1+\dfrac{aT_s}{2}\right)z^2-2+\left(1-\dfrac{aT_s}{2}\right)}.$$

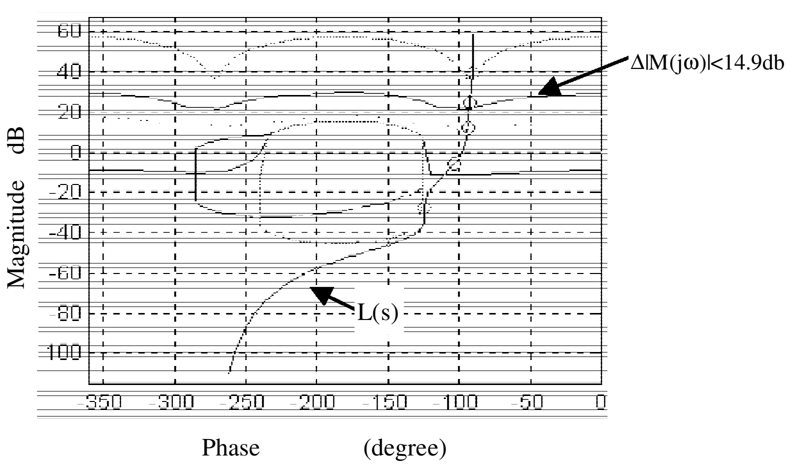

Figure 4. Bounds on the NC and $L(s)$.

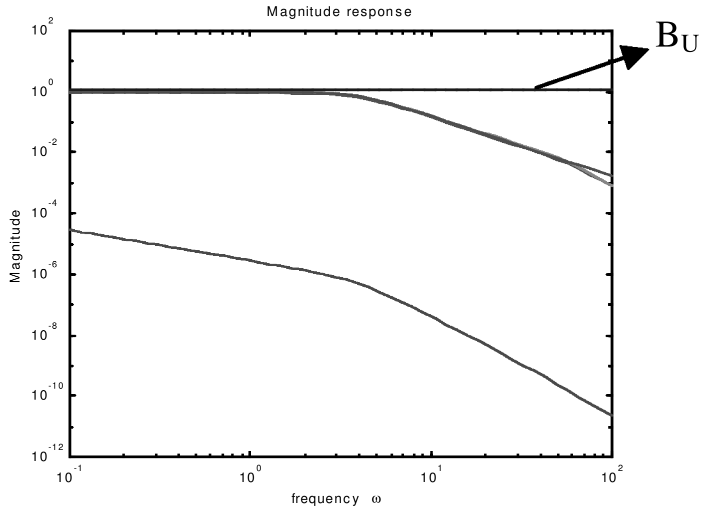

Figure 5. Frequency response of the robust margin for the continuous system.

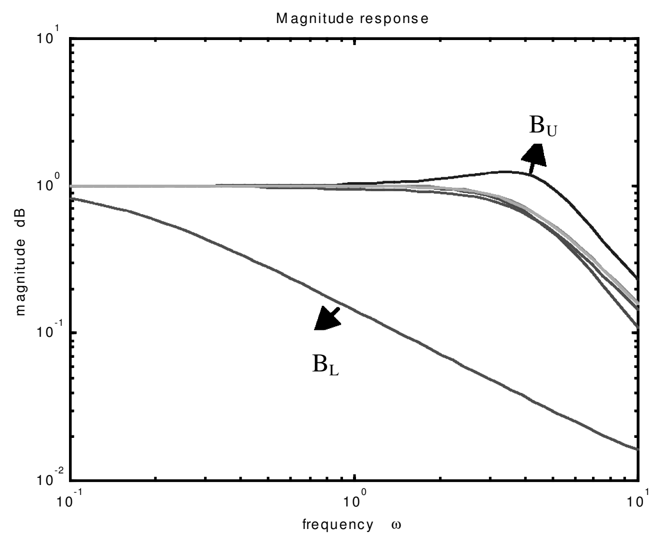

Figure 6. Frequency response (Bode plot) for the continuous system.

Therefore the closed-loop transfer function, which is a function of the sampling time T_s and parametric uncertainties k and a, becomes

$$T_A(z) = \frac{G_A(z)G_h G_p(z)}{1 + G_A(z)G_h G_p(z)} = \frac{\beta(b_0 z^4 + b_1 z^3 + b_2 z^2 + b_3 z + b_4)}{a_0 z^5 + a_1 z^4 + a_2 z^3 + a_3 z^2 + a_4 z + a_5},$$

where

$$\beta = \frac{1155.87 \times 10^6}{250.36} k T_s^2$$

$$b_0 = 2 + 38.3T_s,$$
$$b_1 = 383 + 250.36T_s^2,$$
$$b_2 = -4 + 500.72T_s^2,$$
$$b_3 = -383T_s + 250.336T_s^2,$$
$$b_4 = 2 - 38.3T_s.$$

and

$$a_0 = 12 + (9172.86 + 6a)T_s + (1\,063\,625.66 + 4586.43a)T_s^2 + 531\,807.83aT_s^3$$

$$
\begin{aligned}
a_1 = &-60-(27\,518.58+18a)T_s \\
&+(7\,445\,309.62-4586.34a)T_s^2 \\
&+(25.686\times 10^7+4\,786\,270.47a \\
&+9\,233\,663.525ka)T_s^3 \\
&+(12.843\times 10^7+1\,768\,246\,556.5ka)T_s^4,
\end{aligned}
$$

$$
\begin{aligned}
a_2 = &\,120+(18\,345.72+12a)T_s \\
&-(27\,654\,007.16-9172.86a)T_s^2 \\
&-(25.686\times 10^7-5\,318\,078.3a)T_s^3 \\
&+(12.843a\times 10^7+1\,768\,246\,565ka)T_s^4 \\
&+1\,155\,870\,000kaT_s^5
\end{aligned}
$$

$$
\begin{aligned}
a_3 = &-120+(18\,345.71+12a)T_s \\
&+(27\,654\,007.16+9172.86a)T_s^2 \\
&-(25.585\times 10^7-5318078.3a \\
&-18\,467\,327.05ka)T_s^3-12.843a\times 10^7T_s^4 \\
&+2\,311\,740\,000kaT_s^5
\end{aligned}
$$

$$
\begin{aligned}
a_4 = &\,60-(27\,518.58+18a)T_s \\
&+(-7\,445\,309.62+4586.43a)T_s^2 \\
&+(25.686\times 10^7+4\,786\,270.47a)T_s^3 \\
&-(12.843a\times 10^7+1\,768\,246\,565ka)T_s^4 \\
&+1\,155\,870\,000kaT_s^5
\end{aligned}
$$

$$
\begin{aligned}
a_5 = &-12+(9172.86-6a)T_s \\
&-(1\,063\,615.66+4586.43a)T_s^2 \\
&+(531\,807.83a+9\,233\,663.525ka)T_s^3 \\
&-176\,824\,656.5kaT_s^4
\end{aligned}
$$

By the Mobius transformation

$$z=(w+1)/(w-1)$$

to transform $T_A(z)$ to $T_A(w)$ and according to the parametric uncertainty and the Khaitonov theorem, we can apply the Routh–Hurwitz criterion to four Kharitonov polynomials and the desired sampling time range to achieve robust stability can be obtained as

$$0<T_s<0.002\,371\,507\,46.$$

Figures 7 and 8 show our results for the stable case ($k=1$, $a=1$ and $T_s=0.002$ s) and the unstable case ($k=1$, $a=1$ and $T_s=0.0028$ s).

The exact discrete equivalent of the plant $G_hG_p(s)$ is described as

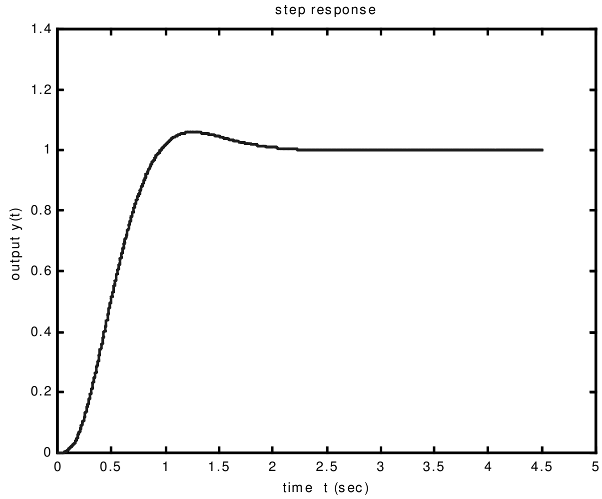

Figure 7. Step response of the stable case.

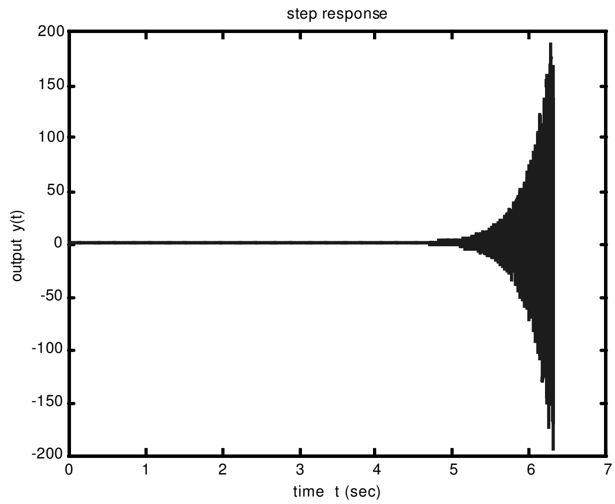

Figure 8. Step respose of the unstable case.

$$G_hG_p(z)=\frac{(-k/a)[(1-aT-e^{-aT})z+(-1+e^{-aT}+aTe^{-aT})]}{(z-1)(z-e^{-aT})}.$$

Let us check our results of each performance requirement for $T_s=0.002$ s.

(i) *Robust stability and robust tracking.* For the four extremes of the uncertain plant, the frequency responses and step responses all fall inside the system specifications as shown in figure 9 (case 1, $k=10$, $a=1$; case 2, $k=10$, $a=10$; case 3, $k=1$, $a=1$; case 4, $k=1$, $a=10$) and figure 10. The B_U and B_L in figure 9 are the same as that in figure 6.

(ii) *Robust margin.* For the four extremes of the uncertain plant, the frequency responses all

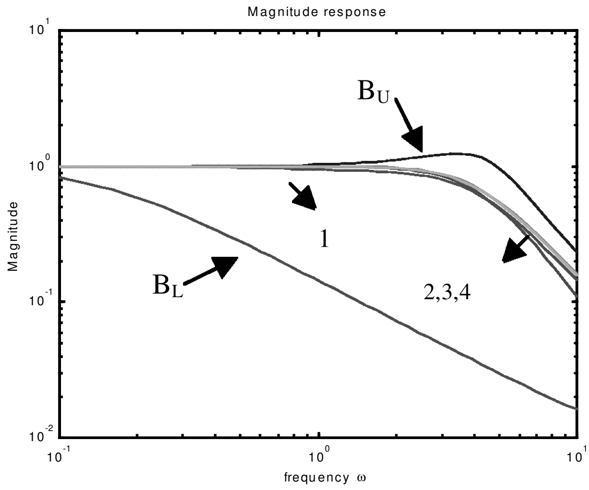

Figure 9. Frequency-domain specification and responses.

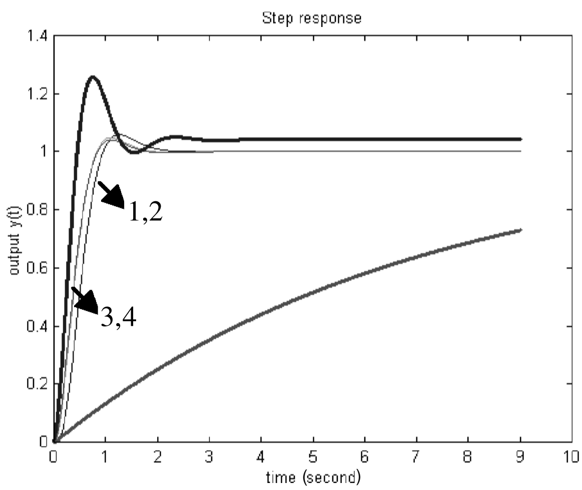

Figure 10. Time-domain specifications and responses.

fall inside the system specification as shown in figure 11.

In comparison with the continuous case as shown in figures 5 and 6, our result, as shown in figures 9 and 11, obtained from the framework proposed in this paper is almost exactly the same as the continuous case. The difficulty in the work of Sidi (1976, 1977), Horowitz and Liao (1984, 1986) and Tsai and Wang (1987) is obviously avoided.

Remark: From the above analysis, we know that the final stable range of the sampling period is only a sufficient condition, since the over-bounding process is used to determine the bounds. □

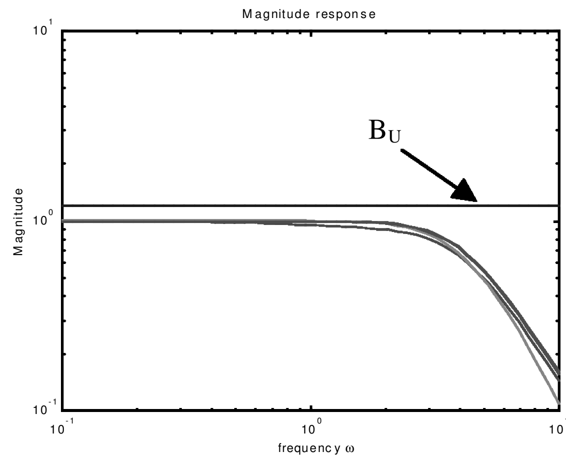

Figure 11. Robust margin of four extremes of the plant uncertainty.

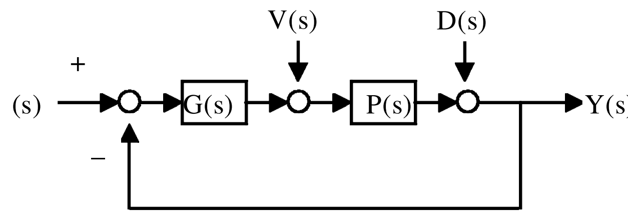

Figure 12. Block diagram for example 2.

6.2. *Example 2* (*Borghesani et al. 1994*)

Consider a continuous-time SISO negative unit feedback system as shown in figure 12. The plant $G_p(s)$ has a parametric uncertainty model with three free parameters:

$$G_p(s) = \frac{k}{(s+a)(s+b)},$$

$$k \in [1, 10], \quad a \in [1, 5], \quad b \in [20, 30].$$

The performance specifications are to design a controller $G(s)$ such that it achieves the following:

(i) robust stability;

(ii) robust margin (via closed-loop magnitude peaks)

$$\left| \frac{G(j\omega)G_p(j\omega)}{1 + G(j\omega)G_p(j\omega)} \right| < 1.2, \quad \omega > 0;$$

(iii) robust output disturbance rejection

$$\left| \frac{Y(j\omega)}{D(j\omega)} \right| < 0.02 \left| \frac{(j\omega)^3 + 64(j\omega)^2 + 748(j\omega) + 2400}{(j\omega)^2 + 14.4(j\omega) + 169} \right|,$$

$$\omega < 10;$$

(iv) robust input disturbance rejection

$$\left|\frac{Y(j\omega)}{V(j\omega)}\right| < 0.01, \quad \omega < 50.$$

The objective is first to design a controller $G(s)$ to meet all requirements by using QFT methodology. Then we convert $G(s)$ into digital equivalent $G_A(z)$ and $G_p(s)$ into $G_h G_p(z)$ by the approximate Z transform and redesign the system as shown in figure 3. Finally we can find the range of stable sampling times so that the design performance can be achieved.

Design procedure:

Step 1. *QFT design for continuous systems*. Following the above discussion of QFT design framework, the nominal plant is chosen as $k = 1$, $a = 5$ and $b = 30$, the achieved loop transmission $L(s) = G(s)P(s)$ is shown in figure 13, and the controller is found to be

$$G(s) = \frac{379(1 + s/42)}{(1 + s/165)}.$$

The frequency responses of the final results for eight extremes of the uncertain plant are described in following figures:

(i) robust margin (via closed-loop magnitude peaks), which equals $20\log(1.2)$;
(ii) robust output disturbance rejection;
(iii) robust input disturbance rejection.

Step 2. *Redesign digital control systems*. The discrete equivalent approximate Z transform of the analogue controller $G(s)$ using higher-order integrators is obtained as

$$G_A(z) = \frac{(62\,535/42)[(21T_s + 1)z + (21T_s - 1)]}{(82.5T_s + 1)z + (82.5T_s - 1)}.$$

The approximate Z transform of the plant $G_p(s)$ is expressed as

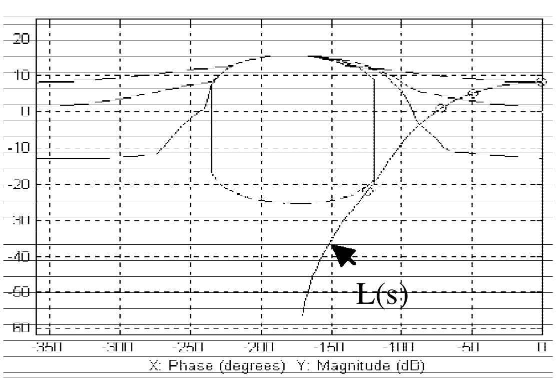

Figure 13. Bounds on the NC and *L*(*s*).

$$G_h G_p(z) = (1 - z^{-1}) \times \left.\frac{G_p(s)}{s}\right|_{s^{-k} = (T_s/2)^k [R_k(z^{-1})/(1 - z^{-1})^k]} \frac{1}{T_s}$$
$$= \frac{6kT_s^2(z+1)}{[12 + 6(a+b)T_s + abT_s^2] \times z^2(-24 + 10abT_s^2)z + [12 - 6(a+b)T_s + abT_s^2]}.$$

Therefore the closed-loop transfer function, which is a function of the sampling time T_s and parametric uncertainties k, a and b, becomes

$$T_A(z) = \frac{G_A(z)G_hG_p(z)}{1 + G_A(z)G_hG_p(z)} = \frac{\beta(b_0z^2 + b_1z + b_2)}{a_0z^3 + a_1z^2 + a_2z + a_3},$$

where

$$\beta = 6kT_s^2\alpha, \quad \alpha = \frac{62535}{42},$$

$$b_0 = 126k\alpha T_s^3 + 6k\alpha T_s^2,$$
$$b_1 = 252k\alpha T_s^3,$$
$$b_2 = 12k\alpha T_s^3 - 6k\alpha T_s^2,$$

and

$$a_0 = 12 + [990 + 6(a+b)]T_s + [495(a+b) + ab]T_s^2 + 82.5abT_s^3,$$
$$a_1 = -36 - [990 + 6(a+b)]T_s + [495(a+b) + 9ab + 6k\alpha]T_s^2 + (907.5ab + 126k\alpha)T_s^3,$$
$$a_2 = 36 - [990 + 6(a+b)]T_s - [495(a+b) - 9ab]T_s^2 + (907.5ab + 252k\alpha)T_s^3,$$
$$a_3 = -12 + [990 + 6(a+b)]T_s - [495(a+b) - ab - 6k\alpha]T_s^2 + (82.5ab + 126k\alpha)T_s^3.$$

By the Mobius transformation

$$z = (w+1)/(w-1)$$

to transform $T_A(z)$ to $T_A(w)$ and according to the parametric uncertainty and the Khaitonov theorem, we can apply the Routh–Hurwitz criterion to four Kharitonov polynomials and the desired sampling time range to achieve robust stability and performance specifications can be obtained as

$$0 < T_s < 0.003\,129.$$

The exact discrete equivalent of the plant $G_h G_p(s)$ is described as

$$G_h G_p(z) = \frac{\begin{array}{l} k[b(1+e^{-bT_s}) - a(1+e^{-aT_s}) \\ \quad +(a-b)(e^{-aT_s}+e^{-bT_s})]z \\ \quad +[(b-a)e^{-(a+b)T_s} + a\,e^{-aT_s} - b\,e^{-bT_s}] \end{array}}{ab(b-a)(z-e^{-bT_s})(z-e^{-aT_s})}.$$

Let us check our results of each performance requirement for $T_s = 0.003$ s.

(i) *Robust stability*. For the eight extremes of the uncertain plant, the step responses are shown in figure 17.

(ii) *Robust margin*. For the eight extremes of the uncertain plant, the frequency responses all fall inside the system specification as shown in figure 18.

(iii) *Robust output disturbance rejection*. For the eight extremes of the uncertain plant, the robust output disturbance rejection all fall inside the system specification as shown in figure 19.

(iv) *Robust input disturbance rejection*. For the eight extremes of the uncertain plant, the robust input disturbance rejection all fall inside of the system specification as shown in figure 20.

In comparison with the continuous case as shown in figures 14–16, our results shown in figures 18–20, obtained from the framework proposed in this paper, are almost exactly the same as the continuous case. The difficulty in the work of Sidi (1976, 1977), Horowitz

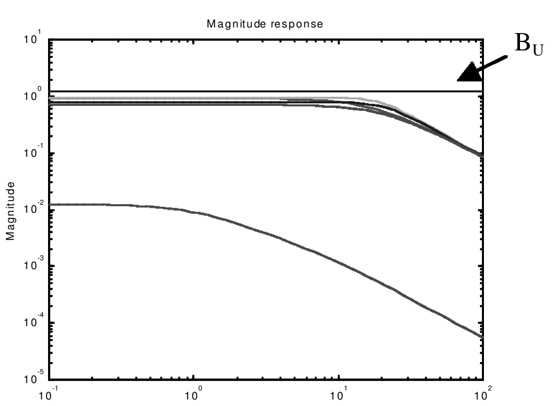

Figure 14. Frequency response of therobust margin for continuous system.

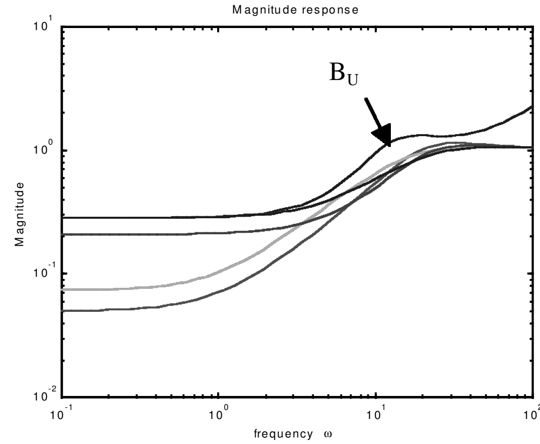

Figure 15. Output disturbance rejection of the continuous case.

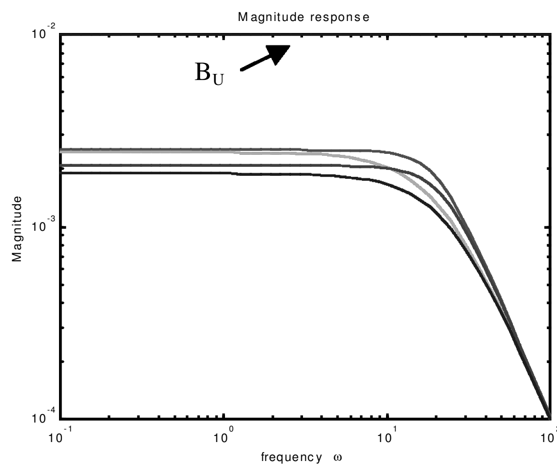

Figure 16. Input disturbance rejection of the continuous case.

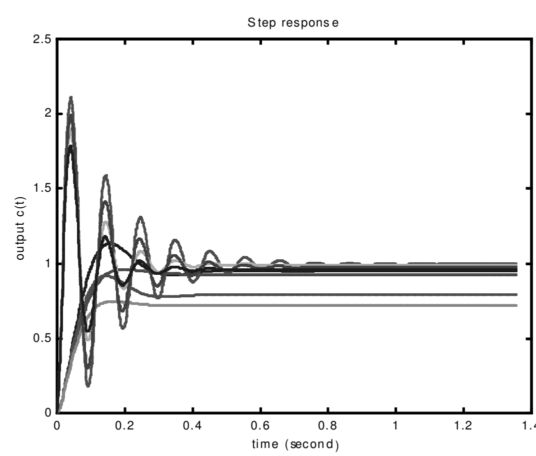

Figure 17. Step responses of eight extremes of the plant uncertainty.

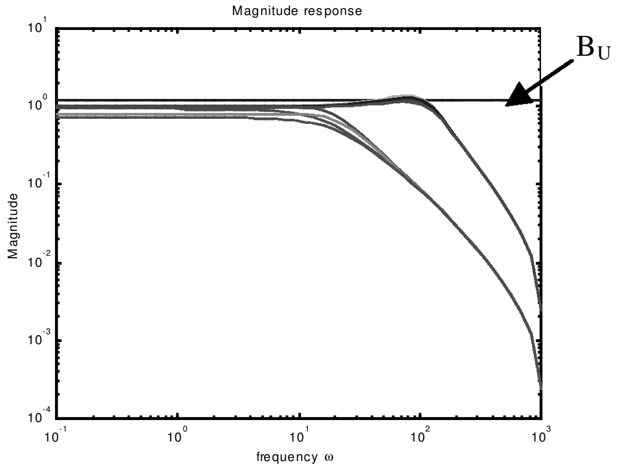

Figure 18. Frequency response of eight extremes of the plant uncertainty.

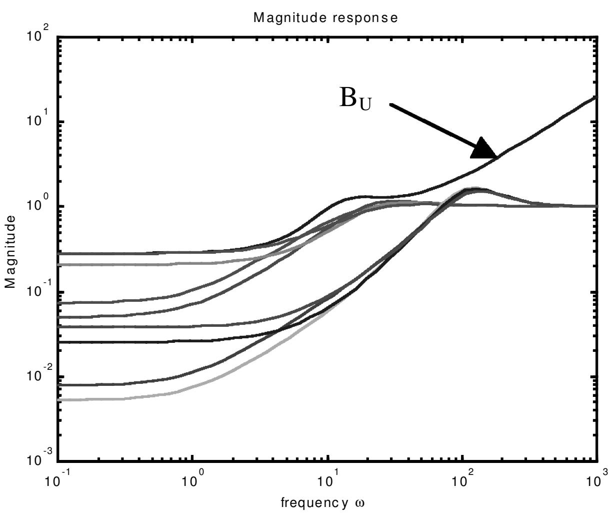

Figure 19. Output disturbance rejection of eight extremes of the plant uncertainty.

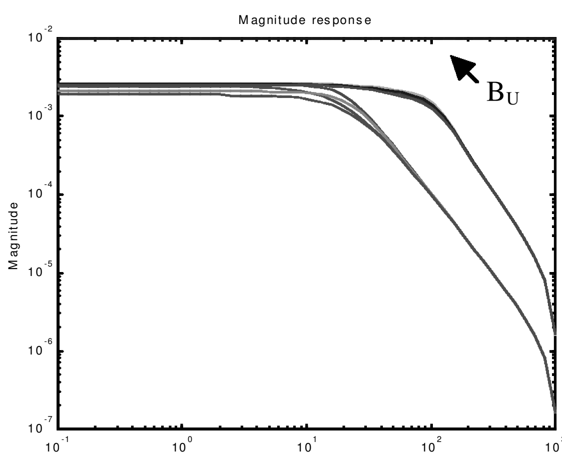

Figure 20. Input disturbance rejection of eight extremes of the plant uncertainty.

and Liao (1984, 1986) and Tsai and Wang (1987) is obviously bypassed and therefore avoided.

7. Conclusions

In this paper, a simple but effective framework for quantitative feedback design of a sampled-data system is proposed. There are limitations in the QFT design of non-minimum-phase feedback systems, since in the w domain the uncertain plant $P(w)$ has one non-minimum-phase zero located at $w = 1$ if the continuous transfer function $P(s)$ is of an order higher than one at high frequencies. Our advocated design methodology consists of only algebraic manipulations to implement the digital controller using the approximate Z transform of the uncertain plant so that the system performance can be achieved and other conventional difficulties in QFT sampled-data design can be avoided. Performance of the redesigned digital system depends on the controlled process and the sampling time T_s. Two numerical examples are used to illustrate fully our new design methodology.

Acknowledgements

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