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Second-Order Delta–Sigma Modulation with Interfered Reference

Yu-Chung Huang and Wei-Shinn Wey

***Abstract*—A delta–sigma $(\Delta\Sigma)$ modulator has been traditionally analyzed by assuming its reference to be constant, but practically the reference may be interfered and thus vary with time. For an interfered reference modulator, the performance of quantization noise is degraded by quantization noise leakage due to interfered feedback. In this paper, a systematic study for observing the behavior of a second-order modulator with an interfered references is presented, based on a linear modeling, spectral analysis, and behavioral simulations. An analytical form of the output of a $\Delta\Sigma$ modulator with an interfered feedback is obtained and compared with behavioral simulation. Due to the agreement between the theoretical calculation and the behavioral simulation results, it is concluded that the quantization noise leakage should be considered for describing the behavior of the $\Delta\Sigma$ modulators more precisely.**

***Index Terms*—Analog-to-digital converter (ADC), delta–sigma modulation, interfered reference, quantization noise, quantization noise leakage, sigma–delta modulation, varying reference.**

I. INTRODUCTION

Delta–sigma $(\Delta\Sigma)$ modulators have become increasingly important in mixed-mode signal processing ICs. Conventionally, $\Delta\Sigma$ modulators are analyzed by assuming their reference inputs to be constant. In practice, the reference input may be interfered by deterministic signals (e.g., pickup of radio frequency, power lines, etc.) and/or by random noise (e.g., flicker, thermal noise, etc.), thus varying with time. Taking account of the reference input, the signal-transfer characteristic of a $\Delta\Sigma$ modulator becomes a ratiometric function. Hence, interference occurring on the reference will modulate with the signal on the input due to the ratiometric operation. Based on this ratiometric concept, the transfer characteristic of the modulator has been analyzed while the reference interfered by deterministic signals [1] or kT/C noise [2]. However, because $\Delta\Sigma$ modulators use feedback to lock onto a band-limited input, interfered feedback incurs quantization leakage to the band of interest. Hence, not only the signal-transfer characteristics but also the quantization noise spectrum will be affected by the interfered reference. Therefore, exploring how the quantization noise spectrum is influenced is relevant to the study of $\Delta\Sigma$ modulators.

In this paper, some aspects of the behavior of a second-order $\Delta\Sigma$ modulator with an interfered reference are studied. The main reason why $\Delta\Sigma$ modulators are difficult to analyze rigorously is the existence of a 1-bit analog-to-digital converter (ADC) in a feedback loop introducing strong nonlinearity. The most popular approach to analyze $\Delta\Sigma$ modulators is to assume the quantization noise to be a signal-independent white random signal [3]. This model replaces an inherent nonlinear modulator by a stochastic linear system, thereby permitting the use of linear systems methods to analyze $\Delta\Sigma$ modulators. Although this model cannot perfectly describe loop stability [4] and pattern noise [5], it predicts the in-band noise surprisingly well [6]–[8]. However, this model really cannot describe the interfered reference case since it assumes that the reference is constant. To capture the behavior of an interfered reference modulator, a modified model based on the same uncorrelated white noise assumption is proposed, which is presented in Section II. Based on this model, the transfer function of the interfered reference modulator can be derived. In Section III, assuming a stochastic noise interference, the quantization noise spectral density and cumulative power density are obtained by taking Fourier transformation for the autocorrelation function of the modulator's output. In Section IV, theoretical spectral densities are compared with simulated ones. These results provide a good match between the calculated and simulated power spectral densities. In Section V, some comments are made that relate a deterministic signal interfering to the modulator performance. Finally, conclusions are presented in Section VI.

II. ANALYTICAL MODEL

A conventional linear model of a 1-bit ADC in a $\Delta\Sigma$ modulator is shown in Fig. 1(a), which considers the quantization process as an additive white noise source e and the output step size as Δ. An analog input x is assumed to be in the no-overload range of $\pm\Delta$. Note here that the reference signal is normalized to unity for convenience since it is constant.

However, for an interfered reference modulator, this conventional model fails to capture the behavior related to the reference. In order to catch this behavior, the reference signal is represented explicitly by a variable w instead of unity, and the two possible output states of y are defined as ± 1 instead of $\pm\Delta/2$, as shown in Fig. 1(b). Without normalization (to the reference), analog feedback signal v has two possible levels of $\pm w$, and the 1-bit ADC input x is in the range between $\pm 2w$. Consequently, it can be found that the additive white noise model should be led by a factor of $1/w$. Thus, the digital output y can be given by

$$y = \frac{x}{w} + \varepsilon \tag{1}$$

featuring a ratiometric function plus a white noise. If the white quantization error has equal probability of lying anywhere in the range ± 1

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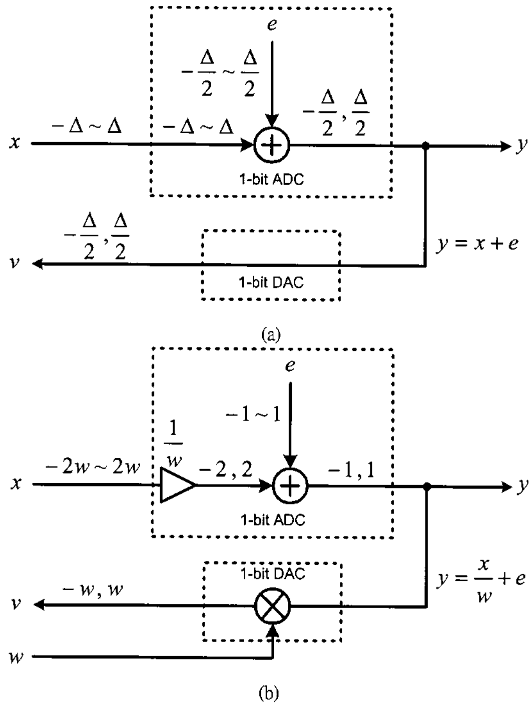

Fig. 1. (a) Traditional linear model and (b) modified model.

and is sampled at frequency $f_S = 1/\tau$, then the autocorrelation function can be given by [9]

$$E\{\varepsilon_n \varepsilon_k\} = \frac{2}{3}\,\tau\,\delta_{n-k} \tag{2}$$

where δ_{n-k} is the Kronecker delta

$$\delta_{n-k} \equiv \begin{cases} 1, & \text{for } n = k \\ 0, & \text{for } n \neq k. \end{cases} \tag{3}$$

Based on this model, a sampled-data equivalent circuit of a second-order $\Delta\Sigma$ modulator with an interfered reference can be illustrated in Fig. 2. The output y_k can be given by

$$y_k = r_k + n_k = \frac{x_{k-1}}{w_k} + (\varepsilon_k - 2b_{1,k}\varepsilon_{k-1} + b_{2,k}\varepsilon_{k-2}) \tag{4}$$

where

$$b_{1,k} = \frac{w_{k-1}}{w_k} \quad \text{and} \quad b_{2,k} = \frac{w_{k-2}}{w_k} \tag{5}$$

which shows a ratiometric function r_k and a noise equation n_k. Thus, we have

$$n_k = \varepsilon_k - 2b_{1,k}\varepsilon_{k-1} + b_{2,k}\varepsilon_{k-2}. \tag{6}$$

As viewed from quantization noise ε_k, the output noise n_k depending on w_k is time variant. In a conventional case with a constant reference, $b_{1,k}$ and $b_{2,k}$ both are unity, and thus the output noise n_k reduces to a well-known second difference equation as follows [3]:

$$n_k = \varepsilon_k - 2\varepsilon_{k-1} + \varepsilon_{k-2}. \tag{7}$$

Comparing (6) with (7), it can be found that the varying reference makes the zeros of the noise transfer function be no longer at dc, and thus the signal-to-noise ratio (SNR) of the modulator is degraded by the quantization noise leakage in the band of interest. The more rapidly the reference varies, the more crucial the SNR degradation becomes. Moreover, an analytical solution to describe the quantization noise power spectral density can be found by its autocorrelation function, which will be presented in next section.

To ensure this modulator's being in the no-overload region, the instantaneous reference times a factor α has to be always greater than the absolute value of instantaneous signal so that

$$\alpha w_k > |x_{k-1}| \tag{8}$$

where α is less than 1.0 for the second-order modulator [10]. In practice, the reference signal can be represented by a positive constant voltage with an interfering signal η_k so that

$$w_k = 1 + \eta_k \tag{9}$$

where the constant voltage is normalized to unity for simplicity. Note here that η_k is an interfering signal and has to be restricted to ensure w_k's satisfying (8). Thus, we also have

$$|\eta_k| < 1. \tag{10}$$

III. Spectral Analysis

Spectral analysis of $\Delta\Sigma$ modulators can be formulated in the framework of quasi-stationary process as considered as in [11] and [12]. The classification of quasi-stationary processes forms a general class of deterministic and random processes for which the first- and second-order moments are well defined and to which traditional system autocorrelation and spectral analysis can be applied [13]. This class includes stationary as well as asymptotically mean stationary random processes [14].

A discrete time process ε_k is said to be quasi-stationary if there is a constant C such that

$$\begin{aligned} &E\{\varepsilon_k\} \le C; && \text{for all } k \\ &\overline{E}\{\varepsilon_k\} = \lim_{N\to\infty} \frac{1}{N} E\{\varepsilon_k\} \text{ exists} \\ &|R_\varepsilon(n,k)| \le C; && \text{for all } n,\, k \text{ where } R_\varepsilon(n,k) \equiv E\{\varepsilon_n \varepsilon_k\} \end{aligned} \tag{11}$$

and if for each k the limit

$$\overline{E}\{\varepsilon_n \varepsilon_k\} = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} R_\varepsilon(n,k) \equiv R_\varepsilon(k) \tag{12}$$

exists. The power spectrum of ε is defined as the discrete Fourier transform of an autocorrelation function

$$S_\varepsilon(n, e^{j\omega}) = \sum_{k=-\infty}^{\infty} R_\varepsilon(n,k) e^{-j\omega k} \tag{13}$$

which may depend on time n. If ε is a quasi-stationary process, then for each k the limit

$$S_\varepsilon(e^{j\omega}) \equiv \overline{S_\varepsilon(n, e^{j\omega})} = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{\infty} \sum_{k=-\infty}^{\infty} R_\varepsilon(n,k) e^{-j\omega k} \tag{14}$$

exists. We can now compute the noise spectrum of the second-order modulator with its reference interfered by a signal.

From (4) and assuming that ε_n, x_n, and η_n are uncorrelated with each other, we get

$$E\{y_n y_{n+k}\} = E\{r_n r_{n+k}\} + E\{n_n n_{n+k}\}. \tag{15}$$

Considering the noise transfer characteristic, we have

$$\begin{aligned} &E\{n_n n_{n+k}\} \\ &= \tfrac{2}{3}\tau \begin{bmatrix} (1 + 4E\{b_{1,n+k} b_{1,n}\} + E\{b_{2,n+k} b_{2,n}\})\ \delta_k \\ -2\,(E\{b_{1,n}\} + E\{b_{1,n+k} b_{2,n}\})\ \delta_{k+1} \\ -2\,(E\{b_{1,n}\} + E\{b_{1,n} b_{2,n+k}\})\ \delta_{k-1} \\ +E\{b_{2,n}\}(\delta_{k+2} + \delta_{k-2}) \end{bmatrix} \\ &= R_n(n, n+k). \end{aligned} \tag{16}$$

Plugging (16) into (13), we obtain

$$\begin{aligned} &S_n(n, e^{j\omega}) \\ &= \tfrac{2}{3}\tau \begin{bmatrix} (1 + 4E\{b_{1,n}^2\} + E\{b_{2,n}^2\}) \\ -4\,(E\{b_{1,n}\} + E\{b_{1,n} b_{2,n+1}\}) \cos(\omega) \\ +2E\{b_{2,n}\} \cos(2\omega) \end{bmatrix}. \end{aligned} \tag{17}$$

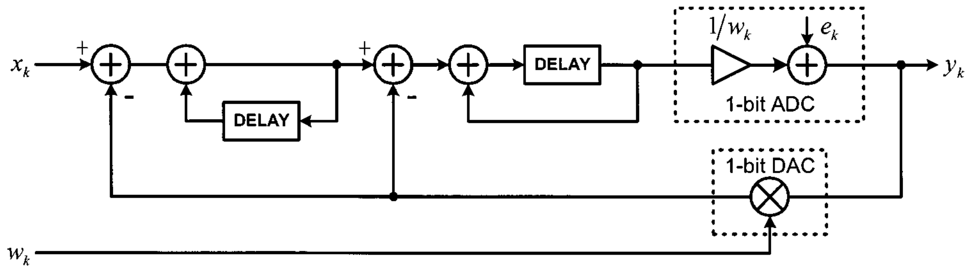

Fig. 2. Discrete-time equivalent circuit of a second-order $\Delta\Sigma$ modulator.

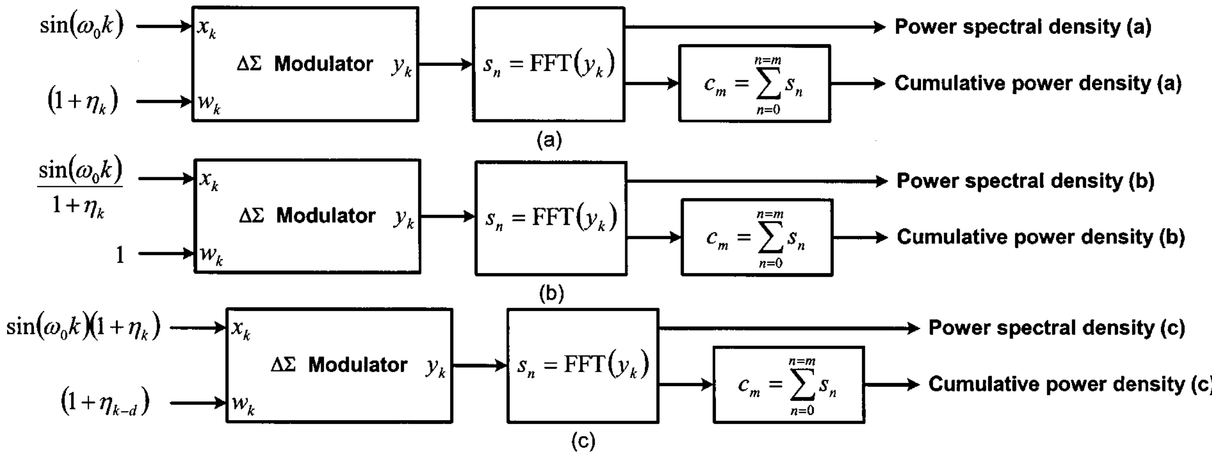

Fig. 3. Simulation diagrams for observing (a) quantization noise leakage due to interfered feedback as well as interfered signal transfer characteristics, (b) interfered signal transfer characteristics regardless of interfered feedback induced quantization noise leakage, and (c) interfered noise transfer characteristics with compensated signal input.

It can be assumed that η_n is a zero-mean stochastic signal with root-mean-square (rms) power of σ_η^2 and an instantaneous value far less than the reference voltage such that $|\eta_n| \ll 1$. Thus, we have

$$\begin{aligned} E\{\eta_n\} &= E\{\eta_{n-i}\} = 0, \\ E\{\eta_n^2\} &= \sigma_\eta^2 \qquad \text{for } n = 1, 2, 3, \ldots, \quad \text{and } i < n. \end{aligned} \tag{18}$$

By using this property, it can be further obtained that

$$E\{\eta_n^{2l-1}\} = 0 \qquad \text{for } l = 1, 2, 3, \ldots . \tag{19}$$

Then we can take

$$\begin{aligned} E\{b_{1,n}\} &= E\{b_{2,n}\} = 1 + \sigma_\eta^2 \\ E\{b_{1,n}^2\} &= E\{b_{2,n}^2\} = 1 + 4\sigma_\eta^2 \\ E\{b_{1,n} b_{2,n+1}\} &= 1 + 3\sigma_\eta^2 \end{aligned} \tag{20}$$

(in the Appendix). All of them depend only on the rms power, regardless of which kind of stochastic process η_n. Hence, substituting (20) into (17), the noise spectrum of the second-order modulator with reference interfered by a random noise is given by

$$\begin{aligned} S_n(e^{j\omega}) = \tfrac{2}{3}\,\tau\,\big(6 + 20\sigma_\eta^2 - 8(1 + 2\sigma_\eta^2)\cos(\omega) \\ + 2(1 + \sigma_\eta^2)\cos(2\omega)\big) \end{aligned} \tag{21}$$

which depends on the rms power of the random process η_n. Assuming a signal in frequency band $0 \le f < f_0$, the oversampling ratio is defined as

$$OSR = \frac{f_S}{2f_0} = \frac{1}{2f_0\tau} = \frac{\pi}{\omega_0}. \tag{22}$$

For a sufficiently high oversampling ratio, we have

$$S_n(e^{j\omega}) \approx \tfrac{2}{3}\,\tau\left(\omega^4 + \sigma_\eta^2\left(6 + 4\omega^2 + \tfrac{2}{3}\,\omega^4\right)\right), \quad \text{where } \frac{\omega}{\pi} \ll 1. \tag{23}$$

The noise power in the signal band can be approximately given by

$$\begin{aligned} n_0^2 &= \int_0^{f_0} S_n(f)df \\ &\approx \frac{\pi^4}{15}\,\frac{1}{\mathrm{OSR}^5} + \sigma_\eta^2\left[\frac{2}{\mathrm{OSR}} + \frac{4}{9}\,\frac{\pi^2}{\mathrm{OSR}^3} + \frac{2}{45}\,\frac{\pi^4}{OSR^5}\right]. \end{aligned} \tag{24}$$

It appears to be the power of the second-order shaped noise plus quantization noise leakage. The leakage power is proportional to the power of the interfering noise and depends on the oversampling ratio.

IV. Simulation Results

In this section, simulation results of the second-order $\Delta\Sigma$ modulator with the reference interfered by a zero-mean stochastic process are presented. The simulation results obtained are compared with the analytical results of the previous section.

The output of $\Delta\Sigma$ modulators can be represented by a combination of a signal function and a noise function as (4). It is useful to investigate the influences of the interfered reference on signal function, on noise function, and on both of them. The simulation diagrams for investigating these influences are illustrated by Fig. 3. Based on the diagrams, the cumulative power density as well as the power spectral density of the three different cases can be obtained by simulation. Note that the cumulative power density, the integral of the power spectral, of the modulator's output represents the in-band noise power.

The simulation diagram of a $\Delta\Sigma$ modulator with reference interfered by a signal is shown as Fig. 3(a). The signal and reference inputs are, respectively, assumed by

$$\begin{aligned} x_k &= \sin(\omega_0 k) \\ w_k &= 1 + \eta_{k-1} \end{aligned} \tag{25}$$

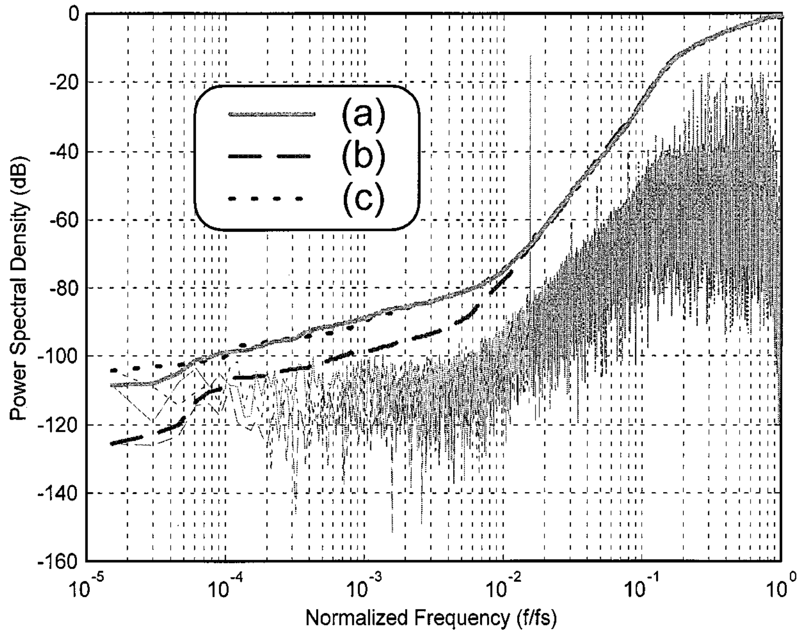

Fig. 4. Output spectrums of the simulation results of Fig. 3(a)–(c) while the interfering process is uniformly distributed and has an rms power of 1/1000.

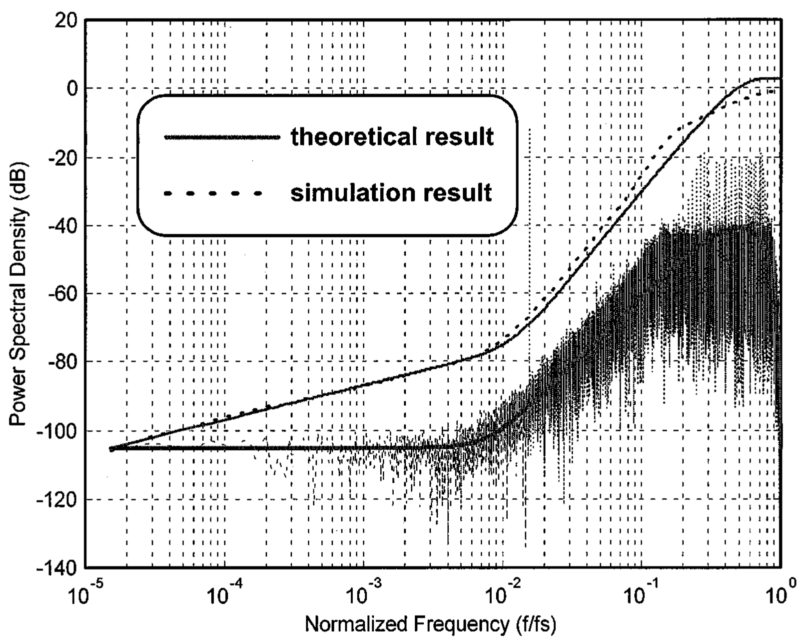

Fig. 5. Simulated and theoretical calculated power spectral density while the interfering process is uniformly distributed with rms power of 1/1000.

where η_k is a zero-mean stochastic process and should be restricted to satisfy (8). In this case, the interfered reference has influence on both signal function and noise function.

In traditional analysis based on the ratiometric concept, the effect of interfered reference on a signal function has been considered [1], [2], but the quantization noise leakage incurred by the interfered feedback was ignored. This case can be modeled by assuming

$$\begin{aligned} x_k &= \frac{\sin(\omega_0 k)}{1+\eta_k} \\ w_k &= 1 \end{aligned} \tag{26}$$

as shown in Fig. 3(b).

As deduced above, not only the signal function but also the quantization noise equation is affected by the interfered reference, resulting in quantization noise leakage to the band of interest, as expressed by (21). In order to compare this theoretical prediction directly with a simulation result, the signal input should be multiplied by the reference input, as shown in Fig. 3(c). Then, from (4), the output function y_k can be given by

$$\begin{aligned} y_k &= \frac{x_{k-1}}{w_k} + (\varepsilon_k - 2b_{1,k}\varepsilon_{k-1} + b_{2,k}\varepsilon_{k-2}) \\ &= \sin(\omega_0(k-1)) + (\varepsilon_k - 2b_{1,k}\varepsilon_{k-1} + b_{2,k}\varepsilon_{k-2}). \end{aligned} \tag{27}$$

The signal function is reduced to a sinusoidal signal whose Fourier transform will appears to be a single spectral line. Thus, how the quantization noise spectrum is affected by the interfered reference can be observed directly and clearly from the output spectrum.

Output spectra of the simulation results are shown in Fig. 4, where the zero-mean interfering process η_k has uniform distribution with power of 0.001 and the signal frequency is $f_s/64$. It can be found

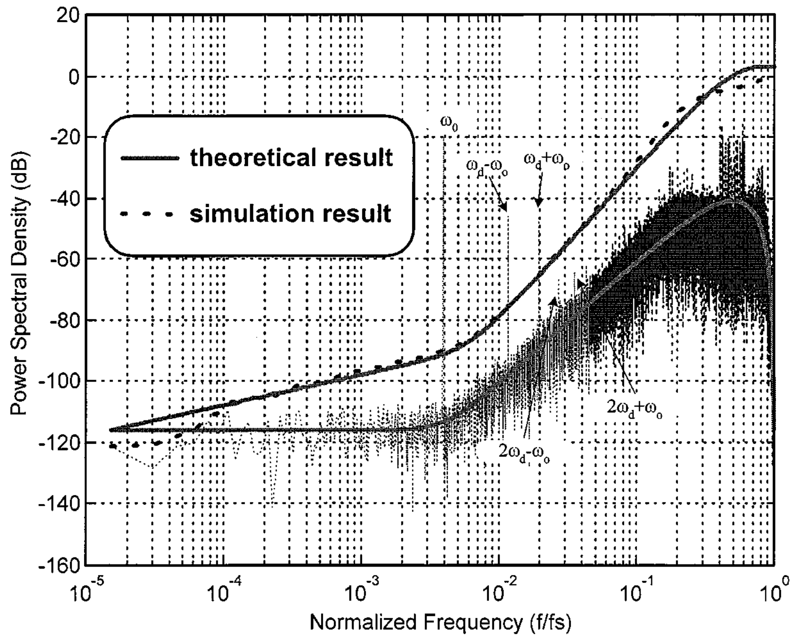

Fig. 6. Simulated and theoretical calculated power spectral density while the interfering signal is a sinusoidal wave.

that the cumulative power spectrum of ignoring the quantization noise leakage, case (b), is approximately 15 dB lower than that of the practical interfered case (a). Besides, the output spectrum of only considering noise leakage is plotted by curve (c), which is much closer to curve (a) than curve (b). Therefore, we can conclude that to describe the behavior of the modulator more precisely, the quantization noise leakage should be taken into account for a modulator with an interfered reference.

Furthermore, the output spectrum of the $\Delta\Sigma$ modulator with its reference interfered by different stochastic processes having different rms power is obtained by behavioral simulation and compared with theoretical results, as shown in Fig. 5. The interfering process is normal (Gaussian) distribution with rms power of 1/1000. This result provides a good match between the calculated and simulated power spectral densities. With this approach, the analytical description matches the simulation results with a difference of just a few decibels for all signal levels, oversampling ratios, and interfering stochastic processes.

V. Deterministic Signal Interfering

In practice, we are also interested to know the performance impact of the modulator with reference interfered by a deterministic signal. A sinusoidal interfering signal is taken here as an example to observe this phenomenon. The signal and reference inputs are, respectively, assumed by

$$\begin{aligned} x_k &= A\sin(\omega_0 k) \\ w_k &= 1+\gamma\sin(\omega_d k) \end{aligned} \tag{28}$$

where $\gamma\sin(\omega_d k)$ is the interfering signal. From (4), the impact on the linearity of this modulator can be found by

$$r_k = \frac{x_{k-1}}{w_k} = \frac{A\sin(\omega_0 k)}{1+\gamma\sin(\omega_d k)}. \tag{29}$$

It can be extended that

$$\begin{aligned} r_k &= A\sin(\omega_0 k) - A\gamma\sin(\omega_0 k)\sin(\omega_d k) \\ &\quad + A\gamma^2\sin(\omega_0 k)\sin^2(\omega_d k) - \cdots \\ &= A\left(1+\frac{\gamma^2}{2}+\cdots\right)\sin(\omega_0 k) \\ &\quad + \frac{A\gamma}{2}\left[\cos\left((\omega_d+\omega_0)k\right) - \cos\left((\omega_d-\omega_0)k\right)\right] \\ &\quad - \frac{A\gamma^2}{4}\left[\sin\left((2\omega_d+\omega_0)k\right) - \sin\left((2\omega_d-\omega_0)k\right)\right] + \cdots \end{aligned} \tag{30}$$

where the first term is the signal tone including aliasing components and the second and third terms are the signal tones modulated by the interfering signal and by its harmonics. Assuming that $A=\gamma=0.1$ and $\omega_d = 4\omega_0 = 2\pi/64$, the simulated results are shown in Fig. 6. It can be found that the simulated amplitude of the signal and harmonics tones agrees with the calculated results as (30).

Furthermore, the noise spectrum of this interfering can be derived by plugging (28) into (17) and (14). As with the stochastic case, this deterministic interfering also incurs quantization leakage to the band of interest. However, in contrast to that case, the values of $\overline{E}\{b_{i,n}\}$, $\overline{E}\{b_{i,n}^2\}$, and $\overline{E}\{b_{1,n}b_{2,n}\}$ depend not only on signal power but also on its waveform and are obtained by numerical computation here since they are difficult to obtain by hand calculation. Finally, if $\overline{E}\{b_{i,n}\}$, $\overline{E}\{b_{i,n}^2\}$, and $\overline{E}\{b_{1,n}b_{2,n}\}$ exist, there is a good match between the analytical and simulated power spectral densities even if the interfering signal is deterministic, as shown in Fig. 6.

VI. Conclusion

A systematic study of the $\Delta\Sigma$ modulator with an interfered reference has been carried out. It reveals that the interfered feedback incurs quantization noise leakage to the band of interest and degrades the quantization noise performance. An analytical model has been proposed to obtain the output function of the modulator with an interfered feedback. The output function shows that the interfered reference makes the zeros of the noise equation be no longer at dc, and thus the SNR of the $\Delta\Sigma$ modulator is degraded. Based on the quasi-stationary approximation, the quantization noise spectrum and the in-band quantization noise power have been derived by taking the Fourier transform of the autocorrelation function of the interfered output function. For a stochastic interfering, the in-band power of quantization noise leakage is proportional to the power of the interfering noise and also depends on the oversampling ratio. If the interfering signal is deterministic, the impact on linearity of the interfered modulator can be evaluated by a ratiometric function. The theoretical results have been approved by the comparison between the calculated and simulated results. Note that the approach developed for the second-order system could be extended to general high-order modulators using similar techniques. It can be found that the quantization noise leakage should be taken into account to describe the behavior of the modulator more precisely while the modulator's reference is interfered by a signal. Finally, the performance impact of the modulator with an interfered reference can be quickly evaluated, and the simulation items about the reference interfering could be decreased or even omitted.

APPENDIX

From (5) and (8), the expected value of $b_{1,n}$ and $b_{2,n}$ can be written as

$$E\{b_{i,n}\} = E\left\{\frac{1+\eta_{n-i}}{1+\eta_n}\right\} \qquad \text{for } i = 1,\, 2. \tag{A.1}$$

Since $|\eta_n| < 1$, the Taylor expansion of this equation exists; then we have

$$\begin{aligned}&= (1+E\{\eta_{n-i}\})\left(1+\sum_{l=1}^{\infty}(-1)^l E\{\eta_n^l\}\right)\\ &= 1+\sum_{l=1}^{\infty}(-1)^l E\{\eta_n^l\} + E\{\eta_{n-i}\}\left(1+\sum_{l=1}^{\infty}(-1)^l E\{\eta_n^l\}\right).\end{aligned}$$

Using (18) and (19), we can be take

$$\begin{aligned}&= 1-\sum_{l=1}^{\infty}E\{\eta_n^{2l-1}\}+\sum_{l=1}^{\infty}E\{\eta_n^{2l}\}\\ &= 1+\sum_{l=1}^{\infty}E\{\eta_n^{2l}\} = 1+\sum_{l=1}^{\infty}(2l-1)!!\sigma_\eta^{2l}\\ &\approx 1+\sigma_\eta^2\end{aligned} \tag{A.2}$$

Similarly, the expected value of $b_{i,n}^2$ can be written by

$$E\{b_{i,n}^2\} = E\left\{\frac{1+2\eta_{n-i}+\eta_{n-i}^2}{1+2\eta_n+\eta_n^2}\right\}. \tag{A.3}$$

To ensure the Taylor expansion of this equation existing, we assume that

$$|\eta_n| < \sqrt{2}-1, \qquad \text{for all } n. \tag{A.4}$$

Hence, we have

$$\begin{aligned}&\left(1+E\{2\eta_{n-i}+\eta_{n-i}^2\}\right)\left(1+\sum_{l=1}^{\infty}(-1)^l E\left\{[2\eta_n+\eta_n^2]^l\right\}\right)\\ &\quad= 1+E\{\eta_n^2\}+\sum_{l=0}^{\infty}(-1)^l\sum_{m=0}^{l}\binom{l}{m}2^m E\{\eta_n^{2l-m}\}\\ &\quad\approx 1+4\sigma_\eta^2.\end{aligned} \tag{A.5}$$

Furthermore, the expected value of $b_{1,n}b_{2,n+1}$ can be given by

$$E\{b_{1,n}b_{2,n+1}\} \approx 1+3\sigma_\eta^2. \tag{A.6}$$

Note that $E\{b_{i,n}\}$, $E\{b_{i,n}^2\}$, and $E\{b_{1,n}b_{2,n+1}\}$ all depend only on the mean-square value of η regardless of the kind of random process.

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