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A note on optimal policies for a periodic inventory system with emergency orders

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Abstract

In this article, we develop a dynamic programming model for a periodic review inventory system in which emergency orders can be placed at the start of each period, while regular orders are placed at the beginning of an order cycle (which consists of a number of periods). We assume that the regular and emergency supply lead times differ by one period. We devise a simple algorithm of computing the optimal policy parameters. Thus, the ordering policy developed is easy to implement.

Scope and purpose

Alternative resupply modes are commonly used in inventory systems. For example, a materials manager could choose to replenish the inventory of an item by air if its inventory position gets dangerously low. In this article, we study a periodic review inventory system in which there is an emergency supply mode in addition to a regular supply mode. We develop optimal ordering policies that minimize the total expected discounted cost of procurement, holding, and shortage over a finite planning horizon. These optimal policies are next shown to converge as the planning horizon is extended, if some conditions that are easy to hold are satisfied. Finally, we derive an algorithm which involves solving only a one-order-cycle dynamic program for the optimal policy parameters.

Keywords: Inventory model; Periodic review; Dynamic programming; Optimization

1. Introduction

Alternative resupply modes are commonly used in practice. For example, a retailer could choose to replenish the inventory of an item by a fast resupply mode (e.g., by air) if its inventory position is

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dangerously low. In this article, we study an inventory system in which there are two resupply modes: namely a regular mode and an emergency mode. Orders placed via the emergency mode, compared to orders placed via the regular mode, have a shorter lead time but are subject to higher ordering costs.

Many studies in the literature address this problem. Some studies assume a particular policy form and devise methods for evaluating it, while other studies develop the true optimal policy and solve specific instances of the problem under consideration. While the policy-evaluation studies use broader assumptions and simpler policies, the policy-optimization studies typically have stronger results. This paper treats the problem in a periodic review setting and contributes to the optimization literature.

The earlier research in this area are policy-optimization studies. See Barankin [1], Daniel [2], Neuts [3], Bulinskaya [4], Fukuda [5], Veinott [6], and Wright [7]. They assume that the lead times for regular and emergency replenishment differ by exactly one period (whose length is one or few working days). Whittmore and Saunders [8] extend the analysis by allowing the emergency and regular lead times to be of arbitrary lengths. Unfortunately, the form of the optimal policy they derive is extremely complex. They are able to obtain explicit results only for the case in which the two lead times differ by one period. Later works in this area are policy-evaluation studies. See, e.g., Moinzadeh and Nahmias [9], Moinzadeh and Schmidt [10], and Moinzadeh and Aggarwal [11] for continuous review models.

All of the policy-optimization studies cited above assume that both regular and emergency orders can be placed at the start of each period. In this paper, we also assume that an emergency order can be placed at the start of a period if the inventory position of an item is dangerously low. However, we assume that regular orders are placed at the beginning of an *order cycle* which consists of a number of periods. Possible reasons for this include avoiding large fixed order costs and achieving economies in the coordination and consolidation of orders for different items. The latter is particularly true if many items are purchased from the same source. For example, a retailer may order hundreds of items from a distribution center every two weeks (which then is the length of an order cycle). Also, companies in the import auto industry in Taiwan typically establish their weekly or biweekly ordering of auto parts from abroad. In addition, they place an emergency order if the inventory level of an item falls below a warning point at the start of a working day. The periodic inventory system considered in this article is similar to that depicted in Chiang and Gutierrez [12]. However, Chiang and Gutierrez assume that the regular and emergency supply lead times differ by more than one period but less than the order-cycle length. In this article, we assume that the emergency supply mode has a lead time one period shorter than that of the regular mode. This assumption is used in most of the policy-optimization studies cited above, and may really be true in some real-world situations in which the supplier's warehouse is not too far away from the buyer and emergency orders delivered by a faster transportation mode will arrive earlier than regular orders by one or few working days.

We analyze the problem within the framework of a stochastic dynamic program. We assume that emergency orders have larger variable item costs. It is possible that emergency orders also have a fixed order cost. This paper, like Chiang and Gutierrez [12], treats only a special case, i.e., the cost of placing an emergency order is assumed to be negligible. We will develop optimal regular and emergency ordering policies that minimize the total expected discounted cost of procurement, holding, and shortage over a finite planning horizon. These optimal policies are next shown to converge as the planning horizon is extended, if some conditions that are easy to hold are satisfied. Hence, the ordering rules to which these policies respectively converge are optimal for the infinite horizon model.

The contribution of this paper is twofold. Firstly, we present a dynamic programming model for the finite horizon problem in which there is only one state variable, as opposed to two state variables in the model of Chiang and Gutierrez [12]. We also derive the convergence conditions (for the optimal policies) which are much simpler than those in Chiang and Gutierrez [12]. Secondly, we develop a simple algorithm, which is not available in Chiang and Gutierrez [12], of computing the optimal policy parameters. Thus, the ordering policy developed is easy to implement.

2. A dynamic programming model

Assume that there are two resupply modes available: namely a regular mode and an emergency mode. The unit item costs for the regular and emergency supply modes are c_1 and c_0, respectively, where $c_1 < c_0$. Assume that an order cycle, whose length is exogenously determined (as in Chiang and Gutierrez [12]), consists of m periods. For notational simplicity, the lead times for the regular and emergency modes are assumed to be one and zero periods, respectively. It can be shown that this case can be generalized to situations where the two lead times differ by exactly one period. See Chiang [13] for details. Assume that all demand which is not filled immediately is backlogged. There is a holding cost $h(\cdot)$ based on inventory on hand and a shortage cost $p(\cdot)$ based on backlogged demand. Both the holding cost and shortage cost are charged at the end of each period. Let $\phi(t)$ be the probability density function for demand t during a period with mean μ. Demand is assumed to be non-negative and independently distributed in disjoint time intervals.

Suppose the net inventory (i.e., inventory on hand minus backorder) at the beginning of a period is x; then the expected holding and shortage costs incurred in that period are given by

$$L(x) = \int_0^{x^+} h(x-t)\phi(t)\,\mathrm{d}t + \int_{x^+}^{\infty} p(t-x)\phi(t)\,\mathrm{d}t, \tag{1}$$

where $(\cdot)^+$ denotes $\max\{\cdot, 0\}$. Other functional forms of $L(x)$ are allowed; however, for our analysis we need $L(x)$ to be a convex and differentiable function. Denote $V_{i,j}(x)$ as the expected discounted cost with i order cycles and j periods remaining (where $0 \leqslant j \leqslant m-1$) until the end of the planning horizon when the starting net inventory is x and an optimal ordering policy is used at every review epoch. Then, $V_{i,j}(x)$ satisfies the functional equation

$$V_{i,0}(x) = \min_{x \leqslant r \leqslant R} \{c_0 r + c_1(R-r) + L(r) + \alpha E_t V_{i-1,m-1}(R-t)\} - c_0 x, \tag{2}$$

$$V_{i,j}(x) = \min_{x \leqslant r} \{c_0 r + L(r) + \alpha E_t V_{i,j-1}(r-t)\} - c_0 x, \quad j = m-1, \ldots, 1, \tag{3}$$

where $V_{0,0}(x) \equiv 0$, α $(0 < \alpha < 1)$ is the discount factor, r is the net inventory after a possible emergency order is placed at a review epoch, and R is the inventory position (i.e., net inventory plus inventory on order) after a possible emergency order and a regular order are placed at the beginning of an order cycle (r and R are decision variables). As we see above, $V_{i,0}(x)$, for example, consists of the purchase cost $c_0(r-x)+c_1(R-r)$, one-period expected holding and shortage cost $L(r)$, and the expected discounted cost $\alpha E_t V_{i-1,m-1}(R-t)$ from the next review epoch until the end of the planning horizon.

Define the function $G_{i,j}(r)$ as

$$G_{i,j}(r) = c_0 r + L(r) + \alpha E_t V_{i,j-1}(r-t), \quad j = m-1, \ldots, 1. \tag{4}$$

Assume that $G_{0,1}(r) = c_0 r + L(r)$ attains its minimum (this assumption is satisfied if $\mu < \infty$, $\lim_{x\to\infty} \mathrm{d}h(x)/\mathrm{d}x > 0$, i.e., there is a positive holding cost, and $\lim_{x\to\infty} |\mathrm{d}p(x)/\mathrm{d}x| > c_0$, i.e., the shortage cost per unit is greater than c_0. However, $\lim_{x\to\infty} |\mathrm{d}p(x)/\mathrm{d}x| > c_0$ can be relaxed to $\lim_{x\to\infty} |\mathrm{d}p(x)/\mathrm{d}x| > (1-\alpha)c_0$ if $V_{0,0}(x) \equiv -c_0 x$ for $x < 0$, i.e., any unsatisfied demand at the end of the planning horizon has to be filled at the unit cost of c_0). Also, define $H(r)$ as

$$H(r) = c_0 r - c_1 r + L(r). \tag{5}$$

Denote by Df the first derivative of the function f. Let r_0^* be the value of r that minimizes $H(\cdot)$. If r_0^* is not an unique minimum, we choose the smallest such value (i.e., $\mathrm{D}H(r) < 0$ for $r < r_0^*$ and $\mathrm{D}H(r_0^*) = 0$). Then $V_{i,0}(x)$ can be expressed as

$$\begin{aligned} V_{i,0}(x) &= \min_{x \leqslant r \leqslant R} \left\{ c_1 R + \alpha E_t V_{i-1,m-1}(R-t) + \min_{x \leqslant r \leqslant R} \{c_0 r - c_1 r + L(r)\} \right\} - c_0 x \\ &= \min_{x \leqslant R} \left\{ c_1 R + \alpha E_t V_{i-1,m-1}(R-t) + \min_{x \leqslant r \leqslant R} H(r) \right\} - c_0 x. \end{aligned} \tag{6}$$

The result below due to Karush [14] (see also Veinott [6]) is used to simplify (6). Let $a \vee b \equiv \max\{a, b\}$.

Lemma 1 (Karush [14]). *Let $f(y)$ be a convex function which is minimized by y^*. Then*

$$\min_{L \leqslant y \leqslant U} f(y) = f^1(L) + f^2(U),$$

where $f^1(L) = f(L \vee y^*)$ is convex non-decreasing in L and $f^2(U) = f(U) - f(U \vee y^*)$ is convex non-increasing in U.

It follows by Lemma 1 that

$$\min_{x \leqslant r \leqslant R} H(r) = H_0(x) + H^{\mathrm{U}}(R), \tag{7}$$

where $H_0(x) = H(x \vee r_0^*)$ is convex non-decreasing in x and $H^{\mathrm{U}}(R) = H(R) - H(R \vee r_0^*)$ is convex non-increasing in R. Note that (i) if $R \leqslant r_0^*$, $H_0(x) + H^{\mathrm{U}}(R) = H(R)$, (ii) if $x \geqslant r_0^*$, $H_0(x) + H^{\mathrm{U}}(R) = H(x)$, and (iii) if $x < r_0^* < R$, $H_0(x) + H^{\mathrm{U}}(R) = H(r_0^*)$. Substituting (7) into (6) yields

$$V_{i,0}(x) = \min_{x \leqslant R} \{c_1 R + \alpha E_t V_{i-1,m-1}(R-t) + H^{\mathrm{U}}(R)\} - c_0 x + H_0(x). \tag{8}$$

Define the function $G_{i,0}(R)$ as

$$G_{i,0}(R) = c_1 R + \alpha E_t V_{i-1,m-1}(R-t) + H^{\mathrm{U}}(R). \tag{9}$$

We next show in Lemma 2 that the cost function $V_{i,j}(x)$ is convex.

Lemma 2. *$V_{i,j}(x)$ for each (i,j) is a convex function.*

Proof. $V_{0,0}(x)$ is convex. Assume that $V_{0,j-1}(x)$ is convex. It follows from (4) that $G_{0,j}(r)$ is a convex function. Thus, $V_{0,j}(x)$ is convex by Proposition B-4 of Heyman and Sobel [15]. This implies from (9) that $G_{1,0}(R)$ is convex. Hence, $V_{1,0}(x)$ is convex again by Proposition B-4 of [15]. Convexity is established by induction for the remaining $V_{i,j}(x)$. □

Let $r_{i,j}$ be the (smallest) value minimizing $G_{i,j}(r)$. Then it follows from (3) and (4) that the optimal policy at a review epoch with i order cycles and j ($j \neq 0$) periods remaining is (i) order up to $r_{i,j}$ if $x < r_{i,j}$ and (ii) do not order if $x \geqslant r_{i,j}$. Let R_i minimize $G_{i,0}(R)$ and $r_{i,0} = \min(r_0^*, R_i)$. Then it can be seen from (8) that the optimal policy at a review epoch with i order cycles remaining is (i) if $x < r_{i,0}$, order amounts $r_{i,0} - x$ and $R_i - r_{i,0}$ at unit costs c_0 and c_1, respectively, (ii) if $r_{i,0} \leqslant x < R_i$, order an amount $R_i - x$ at unit cost c_1, and (iii) if $x \geqslant R_i$, do not order. Notice that if $r_0^* \geqslant R_i$ (thus $r_{i,0} = R_i$), then the regular supply mode is never used at that review epoch with i cycles remaining. We will elaborate on this in the next section.

3. Properties of the optimal policy

In this section, we present important properties about the regular order-up-to levels R_i and emergency order-up-to levels $r_{i,j}$, $j = 0, 1, \ldots, m-1$.

We first introduce some preliminary observations that will be useful to establish the results of this section. It follows from (3) and (4) that $V_{i,j}(x)$, $j \neq 0$, can be expressed as

$$V_{i,j}(x) = G_{i,j}(x \vee r_{i,j}) - c_0 x, \quad j = m-1, \ldots, 1. \tag{10}$$

Similarly, from (8) and (9), $V_{i,0}(x)$ can be rewritten as

$$V_{i,0}(x) = G_{i,0}(x \vee R_i) - c_0 x + H_0(x). \tag{11}$$

Thus, $\mathrm{D}V_{i,j}(x) = -c_0$ for $x \leqslant r_{i,j}$, $j = m-1, \ldots, 1$, and $\mathrm{D}V_{i,0}(x) = -c_0 + \mathrm{D}H_0(x)$ for $x \leqslant R_i$ (which implies, due to $r_{i,0} = \min(r_0^*, R_i)$, that $\mathrm{D}V_{i,0}(x) = -c_0$ for $x \leqslant r_{i,0}$). Also, $\mathrm{D}V_{i,0}(x) \geqslant -c_0 + \mathrm{D}H_0(x) \geqslant -c_0$ and $\mathrm{D}V_{i,j}(x) \geqslant -c_0$, $j = m-1, \ldots, 1$, for all x.

We show in the following theorem that emergency order-up-to levels are non-decreasing within an order cycle as the planning horizon is extended.

Theorem 1. $r_{i,m-1} \geqslant r_{i,m-2} \geqslant \cdots \geqslant r_{i,1}$.

Proof. We show that $r_{i,2} \geqslant r_{i,1}$. The remaining inequalities can be established similarly. To show $r_{i,2} \geqslant r_{i,1}$, we show that $\mathrm{D}G_{i,2}(r) < 0$ for $r < r_{i,1}$. It follows from (4) that $\mathrm{D}G_{i,1}(r) = c_0 + \mathrm{D}L(r) +$

$\alpha E_t \mathrm{D}V_{i,0}(r-t) < 0$ for $r < r_{i,1}$. Also, $\mathrm{D}G_{i,2}(r) = c_0 + \mathrm{D}L(r) + \alpha E_t \mathrm{D}V_{i,1}(r-t) = c_0 + \mathrm{D}L(r) - \alpha c_0$ for $r < r_{i,1}$. Hence, $\mathrm{D}G_{i,2}(r) < -\alpha[c_0 + E_t \mathrm{D}V_{i,0}(r-t)] \leqslant 0$, for $r < r_{i,1}$. □

We next show that if $c_1 < \alpha c_0$, the regular order-up-to level at the beginning of an order cycle is greater than or equal to the emergency order-up-to level at the next review epoch.

Lemma 3. *If $c_1 < \alpha c_0$, then $R_{i+1} \geqslant r_{i,m-1}$ for all i.*

Proof. We show that if $c_1 < \alpha c_0$, $\mathrm{D}G_{i+1,0}(R) < 0$ for $R < r_{i,m-1}$, implying that $R_{i+1} \geqslant r_{i,m-1}$. It follows from (9) that for $R < r_{i,m-1}$, $\mathrm{D}G_{i+1,0}(R) = \mathrm{D}H^{\mathrm{U}}(R) + c_1 + \alpha E_t \mathrm{D}V_{i,m-1}(R-t) = \mathrm{D}H^{\mathrm{U}}(R) + c_1 - \alpha c_0$. As $\mathrm{D}H^{\mathrm{U}}(\cdot) \leqslant 0$, if $c_1 < \alpha c_0$, $\mathrm{D}G_{i+1,0}(R) < 0$ for $R < r_{i,m-1}$. □

The condition $c_1 < \alpha c_0$ is usually true; otherwise, the regular supply mode never will be used. This is shown in the following theorem.

Theorem 2. *If $c_1 \geqslant \alpha c_0$, then $r_0^* \geqslant R_i$ for all i.*

Proof. We show that if $c_1 \geqslant \alpha c_0$, $\mathrm{D}G_{i,0}(R) \geqslant 0$ for $R \geqslant r_0^*$ and thus $r_0^* \geqslant R_i$. It follows from (9) that $\mathrm{D}G_{i,0}(R) = \mathrm{D}H^{\mathrm{U}}(R) + c_1 + \alpha E_t \mathrm{D}V_{i-1,m-1}(R-t)$. As $\mathrm{D}H^{\mathrm{U}}(R) = 0$ for $R \geqslant r_0^*$, $\mathrm{D}G_{i,0}(R) = c_1 + \alpha E_t \mathrm{D}V_{i-1,m-1}(R-t) \geqslant c_1 - \alpha c_0 \geqslant 0$ for $R \geqslant r_0^*$. □

The most important result of this section is stated in Theorem 3, which shows that if two consecutive regular order-up-to levels are equal to each other, and greater than or equal to all intermediate emergency order-up-to levels, then both the regular and emergency order-up-to levels have converged.

Theorem 3. *If $R_{i+1} = R_i \geqslant r_{i,m-1}$, then $r_{n+1,1} = r_{i,1}, r_{n+1,2} = r_{i,2}, \ldots, r_{n+1,m-1} = r_{i,m-1}$, and $R_{n+2} = R_{i+1}$, for $n \geqslant i$.*

Proof. We show that if $R_{i+1} = R_i \geqslant r_{i,m-1}$, then $r_{i+1,1} = r_{i,1}$, $r_{i+1,2} = r_{i,2}, \ldots, r_{i+1,m-1} = r_{i,m-1}$, and $R_{i+2} = R_{i+1}$, and thus similarly, the equalities hold for all $n \geqslant i+1$. If $R_{i+1} = R_i \geqslant r_{i,m-1}$, $R_{i+1} = R_i \geqslant r_{i,1}$ by Theorem 1. Notice that $\mathrm{D}V_{i+1,0}(x) = \mathrm{D}V_{i,0}(x)$ for $x \leqslant R_{i+1} = R_i$. Also, $\mathrm{D}G_{i,1}(r) = c_0 + \mathrm{D}L(r) + \alpha E_t \mathrm{D}V_{i,0}(r-t)$ and $\mathrm{D}G_{i+1,1}(r) = c_0 + \mathrm{D}L(r) + \alpha E_t \mathrm{D}V_{i+1,0}(r-t)$. Hence, $\mathrm{D}G_{i,1}(r) = \mathrm{D}G_{i+1,1}(r)$ for $r \leqslant R_{i+1} = R_i$. As $r_{i,1}$ minimizes $G_{i,1}(r)$, it follows that it also minimizes $G_{i+1,1}(r)$, i.e., $r_{i+1,1} = r_{i,1}$. In addition, for $x > r_{i+1,1} = r_{i,1}$, $\mathrm{D}V_{i+1,1}(x) = \mathrm{D}L(x) + \alpha E_t \mathrm{D}V_{i+1,0}(x-t)$ and $\mathrm{D}V_{i,1}(x) = \mathrm{D}L(x) + \alpha E_t \mathrm{D}V_{i,0}(x-t)$. Thus $\mathrm{D}V_{i+1,1}(x) = \mathrm{D}V_{i,1}(x)$ for $x \in (r_{i,1}, R_i]$. Also, $\mathrm{D}V_{i+1,1}(x) = \mathrm{D}V_{i,1}(x) = -c_0$ for $x \leqslant r_{i,1}$. As a result, $\mathrm{D}V_{i+1,1}(x) = \mathrm{D}V_{i,1}(x)$ for $x \leqslant R_i$. By the same logic, we can show that $r_{i+1,j} = r_{i,j}$ for $j = 2, \ldots, m-1$ and $\mathrm{D}V_{i+1,j}(x) = \mathrm{D}V_{i,j}(x), j = 2, \ldots, m-1$, for $x \leqslant R_i = R_{i+1}$. Moreover, $\mathrm{D}G_{i+1,0}(R) = \mathrm{D}H^{\mathrm{U}}(R) + c_1 + \alpha E_t \mathrm{D}V_{i,m-1}(R-t)$ and $\mathrm{D}G_{i+2,0}(R) = \mathrm{D}H^{\mathrm{U}}(R) + c_1 + \alpha E_t \mathrm{D}V_{i+1,m-1}(R-t)$. As $\mathrm{D}V_{i+1,m-1}(x) = \mathrm{D}V_{i,m-1}(x)$ for $x \leqslant R_i = R_{i+1}$, $\mathrm{D}G_{i+2,0}(R) = \mathrm{D}G_{i+1,0}(R)$ for $R \leqslant R_{i+1}$. As R_{i+1} minimizes $G_{i+1,0}(R)$, it follows that it also minimizes $G_{i+2,0}(R)$, i.e., $R_{i+2} = R_{i+1}$. □

We see from Theorem 3 that if for some i, $R_{i+1} = R_i \geqslant r_{i,m-1}$, then the sequences $\{R_n\}$ and $\{r_{n,j}\}$, $j = 1, 2, \ldots, m-1$, converge, respectively, to $R^* = R_{i+1}$ and $r_j^* = r_{i,j}$, $j = 1, \ldots, m-1$. The condition of $R_{i+1} = R_i \geqslant r_{i,m-1}$ for some i usually holds if $c_1 < \alpha c_0$, as implied by Lemma 3. Consequently, R^*, r_j^* for $j = 1, \ldots, m-1$, and $\min\{r_0^*, R^*\}$ are, respectively, the optimal regular and emergency order-up-to levels for the infinite horizon model.

To illustrate, consider the example (referred to as the base case thereafter): $m = 10, c_1 = \$10, c_0 = \$15$, $\mu = 2$ (with Poisson demand), $h(x) = 0.01x$ and $p(x) = 20x$ (both for $x \geqslant 0$) (this choice of the holding and shortage cost functions implies that holding and shortage are charged at \$0.01 and \$20 per unit, respectively), and $\alpha = 0.999$. After solving, we find that $r_0^* = 3, r_{0,1} = 1, r_{0,2} = 3, r_{0,3} = 4, r_{0,4} = 5, r_{0,5} = 6, r_{0,6} = \cdots = r_{0,9} = 7, R_1 = 19$; $r_{1,1} = 4$, $r_{1,2} = r_{1,3} = 6, r_{1,4} = \cdots = r_{1,9} = 7, R_2 = 32$; and $r_{i,1} = 4, r_{i,2} = r_{i,3} = 6, r_{i,4} = \cdots = r_{i,9} = 7$ for $i \geqslant 2$, and $R_i = 32$ for $i \geqslant 3$. Thus, we see that the sequences $\{R_n\}$, $\{r_{n,1}\}$, $\{r_{n,2}\}$, $\{r_{n,3}\}$, and $\{r_{n,j}\}$, $j = 4, 5, \ldots, 9$, converge, respectively, to $R^* = 32, r_1^* = 4, r_2^* = r_3^* = 6$, and $r_4^* = \cdots = r_9^* = 7$ after just two order cycles.

4. Discussion

We briefly compare the results derived in Section 3 to those in Chiang and Gutierrez [12]. As mentioned in Section 1, both this paper and Chiang and Gutierrez [12] study the same periodic inventory system, except that this paper assumes that the regular and emergency lead times differ by one period, while Chiang and Gutierrez [12] assumes that the two lead times differ by more than one period (but less than the order-cycle length).

The non-decreasingness property of emergency order-up-to levels within an order cycle in Theorem 1 and the convergence conditions for the optimal policy parameters in Theorem 3 are similar to those in Chiang and Gutierrez [12]. However, the convergence conditions in Theorem 3 are simpler and easier to hold than their counterpart in Chiang and Gutierrez [12].

In addition, the emergency order-up-to levels derived in Chiang and Gutierrez [12] are a function of inventory on order (if there is any at a review epoch), while they are not in this paper. Moreover, the optimal regular order-up-to level R^* derived in this paper is a fixed level, while it is a variable level in Chiang and Gutierrez [12] (depending on the inventory position after a possible emergency order is placed at the beginning of an order cycle). As a result, the optimal policy developed in this paper consists of only a few parameters (i.e., r_j^*, $j = 0, \ldots, m-1$, and R^*), while it contains a lot many in Chiang and Gutierrez [12].

Take the base case in Section 3 for example: $m = 10, c_1 = \$10, c_0 = \$15$, $\mu = 2$ (Poisson demand), $h(x) = 0.01x$ and $p(x) = 20x$ (for $x \geqslant 0$), and $\alpha = 0.999$. If the emergency and regular supply lead times are one and two periods, respectively (note in this case that expressions (2) and (3) would have $L(r)$ replaced by $\alpha E_t L(r-t)$ [13]), we obtain after solving that $r_0^* = 5, r_1^* = 7$, $r_2^* = 8, r_3^* = 9, r_4^* = 10, r_5^* = \cdots = r_9^* = 11$, and $R^* = 35$. On the other hand, if the two lead times are one and six periods, respectively, it is found in Chiang and Gutierrez [12] that $r_0^* = r_1^* = \cdots = r_4^* = 11$ (r_j^* for $j = 5, \ldots, 9$ are a function of inventory on order), and the optimal regular order-up-to level is between 41 and 45. As we see, the emergency order-up-to levels in the former situation are smaller than their counterpart levels in the latter. This is because the more we are close to the time of the arrival of a regular order, the less inventory we need to carry on hand (against possible stockouts) and thus the smaller the emergency order-up-to levels. For example, if we are at the beginning of an order cycle, there are only two periods till the time of the arrival of a regular order in the former situation, while there are six periods in the latter; hence, $r_0^* = 5$ in the former while $r_0^* = 11$ in the latter. Also, the optimal regular order-up-to level is lower when the difference between the two lead times is only one period. The reason for this is simple. As we know, the regular order-up-to level should be large enough to cover demand over an order cycle plus the regular lead time since it will take such time for the next regular order to arrive. Hence, a larger regular lead time (other things being equal) will yield a higher order-up-to level.

5. A simple method

In this section, we develop a simple method for computing R^* and r_j^*, $j = 1, \ldots, m-1$. While solving $V_{i,j}(x)$ until $R_{i+1} = R_i \geqslant r_{i,m-1}$ is satisfied may take little time, this simple method obtains R^* and r_j^*'s by solving only a m-stage dynamic program.

Assume that there exists some i such that $R_{i+1} = R_i \geqslant r_{i,m-1}$, and thus $R^* = R_{i+1}$ and $r_j^* = r_{i,j}$, $j = 1, \ldots, m-1$. Let $Q_{i,0}(x) = G_{i,0}(x \vee R_i) = G_{i,0}(x \vee R^*)$, which is convex non-decreasing and equal to $G_{i,0}(R^*)$ for $x \leqslant R^*$. Thus, it follows from (11) that $V_{i,0}(x)$ can be expressed as

$$V_{i,0}(x) = Q_{i,0}(x) - c_0 x + H_0(x). \tag{12}$$

Then,

$$V_{i,1}(x) = \min_{x \leqslant r} \{c_0 r + L(r) + \alpha E_t V_{i,0}(r-t)\} - c_0 x$$

$$= \min_{x \leqslant r} \{c_0 r + L(r) + \alpha E_t Q_{i,0}(r-t) + \alpha E_t H_0(r-t) - \alpha c_0 r + \alpha c_0 \mu\} - c_0 x.$$

Let

$$J_1(r) = (1-\alpha)c_0 r + L(r) + \alpha E_t H_0(r-t). \tag{13}$$

Thus $G_{i,1}(r) = J_1(r) + \alpha E_t Q_{i,0}(r-t) + \alpha c_0 \mu$. For $r \leqslant R^*$, $\mathrm{D}G_{i,1}(r) = \mathrm{D}J_1(r)$, since $\mathrm{D}Q_{i,0}(x) = 0$ for $r \leqslant R^*$. As $r_1^* = r_{i,1} \leqslant r_{i,m-1} \leqslant R_i = R^*, r_1^*$ can be obtained by solving $\mathrm{D}J_1(r) = 0$. Let $H_1(x) = J_1(x \vee r_1^*)$ and $Q_{i,1}(x) = \alpha E_t Q_{i,0}((x \vee r_1^*) - t)$, which are both convex and non-decreasing. Then,

$$V_{i,1}(x) = G_{i,1}(x \vee r_1^*) - c_0 x = Q_{i,1}(x) + \alpha c_0 \mu - c_0 x + H_1(x).$$

Note that for $x \leqslant R^*$, $Q_{i,1}(x) = \alpha E_t Q_{i,0}((x \vee r_1^*) - t) = \alpha E_t G_{i,0}(((x \vee r_1^*) - t) \vee R^*) = \alpha G_{i,0}(R^*)$. Similarly, we can repeat the above logic and show that if we let

$$J_j(r) = (1-\alpha)c_0 r + L(r) + \alpha E_t H_{j-1}(r-t), \quad j = 2, \ldots, m-1 \tag{14}$$

which is minimized by $r_j^* = r_{i,j}$ and define

$$H_j(x) = J_j(x \vee r_j^*), \tag{15}$$

$$Q_{i,j}(x) = \alpha E_t Q_{i,j-1}((x \vee r_j^*) - t) \tag{16}$$

which are both convex and non-decreasing, then

$$V_{i,j}(x) = Q_{i,j}(x) + \sum_{k=1}^{j} \alpha^k c_0 \mu - c_0 x + H_j(x), \quad j = 1, \dots, m-1, \tag{17}$$

where $Q_{i,j}(x) = \alpha^j G_{i,0}(R^*)$ for $x \leqslant R^*$.

We finally show in this section that R^* can be obtained by solving $\mathrm{D}J_0(R) = 0$, where $J_0(R)$ is defined as

$$J_0(R) = (c_1 - \alpha c_0)R + H^{\mathrm{U}}(R) + \alpha E_t H_{m-1}(R-t). \tag{18}$$

It follows from (16) and (17) that $V_{i+1,0}(x)$ can be written as

$$V_{i+1,0}(x) = \min_{x \leqslant R} \{H^{\mathrm{U}}(R) + c_1 R + \alpha E_t V_{i,m-1}(R-t)\} - c_0 x + H_0(x)$$

$$= \min_{x \leqslant R} \left\{ J_0(R) + \alpha E_t Q_{i,m-1}(R-t) + \sum_{k=1}^{m} \alpha^k c_0 \mu \right\} - c_0 x + H_0(x),$$

where $G_{i+1,0}(R) = J_0(R) + \alpha E_t Q_{i,m-1}(R-t) + \sum_{k=1}^{m} \alpha^k c_0 \mu$ is minimized by the value $R_{i+1} = R^*$, i.e., $\mathrm{D}G_{i+1,0}(R^*) = \mathrm{D}J_0(R^*) + \alpha E_t \mathrm{D}Q_{i,m-1}(R^* - t) = 0$. However, $\alpha E_t \mathrm{D}Q_{i,m-1}(R^* - t) = 0$ since $Q_{i,m-1}(x) = \alpha^{m-1} G_{i,0}(R^*)$ for $x \leqslant R^*$. Thus, it follows that $\mathrm{D}J_0(R^*) = 0$.

6. Algorithm

In this section, we present an algorithm which summarizes the simple method described in Section 5 for computing the optimal policy parameters r_j^*, $j = 0, \dots, m-1$, and R^*. We also investigate the sensitivity of several important parameters towards the optimal solution.

Note that all the functions involved are convex when computing r_j^*, $j = 0, \dots, m-1$, and R^* in Section 5. Hence, the following algorithm can be easily implemented.

Step 1: Compute r_0^* by using (5), and let $H_0(x) = H(x \vee r_0^*)$.

Step 2: Compute r_1^* by differentiating $J_1(r)$ in (13) and solving $\mathrm{D}J_1(r) = 0$, and let $H_1(x) = J_1(x \vee r_1^*)$. Similarly, compute r_j^* for $j = 2, \dots, m-1$ by alternatively using $J_j(r)$ and $H_j(x)$, $j = 2, \dots, m-1$, which are defined in (14) and (15), respectively.

Step 3: Finally, compute R^* by differentiating $J_0(R)$ in (18) and solving $\mathrm{D}J_0(R) = 0$.

This algorithm involves solving only a one-order-cycle dynamic program with m stages.

Consider the base case in Section 3: $m = 10$, $c_1 = \$10$, $c_0 = \$15$, $\mu = 2$ (with Poisson demand), $h(x) = 0.01x$ and $p(x) = 20x$ (for $x \geqslant 0$) (holding and shortage are charged at \$.01 and \$20 per unit, respectively), and $\alpha = 0.999$. Using the above algorithm, we obtain the same emergency and regular order-up-to levels r_j^*'s and R^* (i.e., $r_0^* = 3$, $r_1^* = 4$, $r_2^* = r_3^* = 6$, $r_4^* = \cdots = r_9^* = 7$, and $R^* = 32$).

We next investigate the sensitivity of several important parameters towards the optimal solution. First, as the emergency unit cost c_0 increases (other things being equal), the emergency order-up-to levels r_j^*'s are likely to decrease and the regular order-up-to level R^* is likely to increase, reflecting the fact that we employ the emergency supply mode less often and tend to order more quantity via the regular mode. For example, compared to the base case, if $c_0 = \$12.5$, then $r_0^* = 4$, $r_1^* = 5$,

$r_2^* = 6$, $r_3^* = \cdots = r_9^* = 7$, and $R^* = 31$; and if $c_0 = \$20.0$, then $r_0^* = 2$, $r_1^* = 4$, $r_2^* = 5$, $r_3^* = r_4^* = 6$, $r_5^* = \cdots = r_9^* = 7$, and $R^* = 33$.

Second, as the shortage cost per unit increases (other things being equal), r_j^*'s are likely to increase (and thus R^* may also increase), implying that we need to provide more inventory against possible stockouts. For example, compared to the base case, if $p(x) = 10x$, then $r_0^* = 2$, $r_1^* = 4$, $r_2^* = 5$, $r_3^* = r_4^* = 6$, $r_5^* = \cdots = r_9^* = 7$, and $R^* = 32$; and if $p(x) = 40x$, then $r_0^* = 4$, $r_1^* = 5$, $r_2^* = 6$, $r_3^* = r_4^* = 7$, $r_5^* = \cdots = r_9^* = 8$, and $R^* = 33$.

Finally, as the holding cost per unit increases (other things being equal), R^* is likely to decrease (r_j^*'s may also decrease), since it is economical to carry less inventory on hand. For example, compared to the base case, if $h(x) = 0.005x$, then $r_0^* = 3$, $r_1^* = 4$, $r_2^* = r_3^* = 6$, $r_4^* = r_5^* = 7$, $r_6^* = \cdots = r_9^* = 8$, and $R^* = 33$; and if $h(x) = 0.02x$, then $r_0^* = 3$, $r_1^* = 4$, $r_2^* = 5$, $r_3^* = 6$, $r_4^* = \cdots = r_9^* = 7$, and $R^* = 31$.

7. Conclusion

In this paper, we develop a dynamic programming model for an inventory system in which while emergency orders can be placed at the beginning of each period, regular orders are placed at the beginning of order cycles. Such inventory systems are found in the import auto industry in Taiwan. We assume that the regular and emergency channel lead times differ by one period. We develop optimal ordering policies as well as a simple algorithm for computing the optimal policy parameters. We hope that these results will help material managers decide how to use the emergency supply mode when this alternative option is available.

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