Information Processing Letters 77 (2001) 35–41

Information Processing Letters

www.elsevier.com/locate/ipl

The shuffle-cubes and their generalization ☆

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Received 23 February 2000; received in revised form 23 June 2000

Communicated by F.Y.L. Chin

Abstract

In this paper, we first present a new variation of hypercubes, denoted by SQ_n. SQ_n is obtained from Q_n by changing some links. SQ_n is also an n-regular n-connected graph but of diameter about $n/4$. Then, we present a generalization of SQ_n. For any positive integer g, we can construct an n-dimensional generalized shuffle-cube with 2^n vertices which is n-regular and n-connected. However its diameter can be about n/g if we consider g as a constant. © 2001 Elsevier Science B.V. All rights reserved.

Keywords: Hypercubes; Diameter; Connectivity; Interconnection network

1. Introduction

The topology of any interconnection network for parallel and distributed systems can be represented by an undirected graph. For the graph theoretic definitions and notations we follows Harary's book [7]. $G=(V,E)$ is a *graph* if V is a finite set and E is a subset of $\{(a,b) \mid (a,b)$ is an unordered pair of $V\}$. We say that V is the *vertex set* and E is the *edge set*. The *degree* of a vertex x, denoted by $\deg(x)$, is the number of edges incident with x. A k-regular graph is a graph with $\deg(x)=k$ for any vertex $x\in V$. A sequence of vertices $P=\langle x_0,x_1,\ldots,x_k\rangle$ is a path from x_0 to x_k if $(x_{i-1},x_i)\in E$ for $1\leqslant i\leqslant k$ and $x_i\neq x_j$ if $i\neq j$. The *length* of P is k. Let u and v be two vertices of G. The *distance* between u and v, denoted by $d(u,v)$, is the length of the shortest path from u to v. The *diameter* of G, denoted by $D(G)$, is $\max\{d(u,v) \mid u,v\in V\}$. The *connectivity* of G, denoted by $\kappa(G)$, is the minimum number of vertices whose removal leaves the remaining graph disconnected or trivial.

Network topology is a crucial factor for interconnection networks since it determines the performance of a network. However, designing an interconnection network is a multiple-objective optimization problem. Usually, we want to minimize the diameter and to maximize the connectivity. There are a lot of interconnection network topologies proposed in literature. Among these topologies, the n-dimensional hypercube, denoted by Q_n, is one of many popular topologies. It is known that $D(Q_n)=n$ and $\kappa(Q_n)=n$ [11]. However, a hypercube does not make the best use of its hardware. It is possible to fashion networks with lower diameters than that of Q_n and with the same connectivity. For example, the cross cubes [4–6,9], twisted cubes [1,8], and Möbius cubes [3] are derived from Q_n by changing the connection of some hypercube links.

☆ This work was supported in part by the National Science Council of the Republic of China under Contract NSC 89-2213-E-009-013.

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PII: S0020-0190(00)00147-2

All of these topologies have connectivity n and have diameter around $n/2$. Thus, this is an improvement of approximately a factor of 2. A natural question raised is: if there is another way to change the connection of some hypercube links to lower the diameter.

In this paper, we first present a variant of hypercubes, called the shuffle-cubes, SQ_n. SQ_n is obtained from Q_n by changing some links of Q_n. It has connectivity n and has diameter around $n/4$. Then we present a generalization of shuffle-cubes. For any positive integer g, we can construct an n-dimensional generalized shuffle-cube with 2^n vertices which is n-regular and n-connected. Its diameter can be about n/g if we consider g as a constant.

2. Shuffle-cubes

We use n-bit binary strings to represent vertices, for example, $u = u_{n-1}u_{n-2}\ldots u_1u_0$ for $u_i \in \{0,1\}$ and $0 \leqslant i \leqslant n-1$. We use $p_j(u)$ to denote the *j-prefix* of u, i.e., $p_j(u) = u_{n-1}u_{n-2}\ldots u_{n-j}$, and $s_i(u)$ the *i-suffix* of u, i.e., $s_i(u) = u_{i-1}u_{i-2}\ldots u_1u_0$. Let u and v be two vertices. The number of bits that are differing in u and v is called the *Hamming distance* between u and v, denoted by $h(u,v)$. The *n-dimensional hypercube*, Q_n, consists of all of the n-bit binary strings as its vertices and two vertices are adjacent if and only if $h(u,v) = 1$. It is known that Q_n can be recursively constructed from two copies of Q_{n-1}. For this reason, Q_0 is the complete graph K_1 as the basis of the hypercubes. We will use $\oplus$ to denote addition with modulo 2.

To construct shuffle-cubes, we define the following four sets:

$$V_{00} = \{1111, 0001, 0010, 0011\},$$

$$V_{01} = \{0100, 0101, 0110, 0111\},$$

$$V_{10} = \{1000, 1001, 1010, 1011\},$$

$$V_{11} = \{1100, 1101, 1110, 1111\}.$$

For ease of exposition, we limit our discussion to $n = 4k + 2$ for $k \geqslant 0$.

Definition 1. The *n-dimensional shuffle-cube*, SQ_n, is recursively defined as follows: SQ_2 is Q_2. For $n \geqslant 3$, SQ_n consists of 16 subcubes $SQ_{n-4}^{i_1i_2i_3i_4}$, where $i_j \in \{0,1\}$ for $1 \leqslant j \leqslant 4$ and $p_4(u) = i_1i_2i_3i_4$ for all vertices u in $SQ_{n-4}^{i_1i_2i_3i_4}$. The vertices $u = u_{n-1}u_{n-2}\ldots u_1u_0$ and $v = v_{n-1}v_{n-2}\ldots v_1v_0$ in different subcubes of dimension $n-4$ are adjacent in SQ_n if and only if
(1) $s_{n-4}(u) = s_{n-4}(v)$, and
(2) $p_4(u) \oplus p_4(v) \in V_{s_2(u)}$.

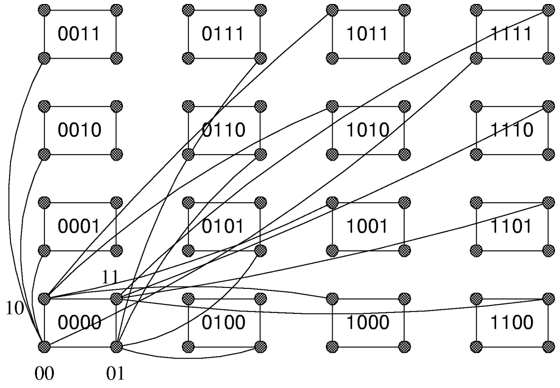

Fig. 1. SQ_6.

For example, the vertex 111101 in SQ_6 is linked to the following vertices in different subcubes of dimension 2: 101101, 101001, 100101 and 100001. We illustrate SQ_6 in Fig. 1 showing only edges incident at vertices in SQ_2^{0000} and omitting others. Obviously, the degree of each vertex of SQ_n is n and the number of vertices (edges, respectively) is the same as that of Q_n.

For $1 \leqslant j \leqslant k$, the jth *4-bit* of u, denoted by u_4^j, is defined as $u_4^j = u_{4j+1}u_{4j}u_{4j-1}u_{4j-2}$. In particular, the 0th *4-bit* of u, u_4^0, is defined as $u_4^0 = u_1u_0$. $u_4^j = v_4^j$ if and only if $u_{4j+i} = v_{4j+i}$ for $-2 \leqslant i \leqslant 1$. Thus, similar to Hamming distance, we define *4-bit Hamming distance* between u and v, denoted by $h_4(u,v)$, as the number of 4-bits u_4^j with $0 \leqslant j \leqslant k$ such that $u_4^j \neq v_4^j$, i.e.,

$$h_4(u,v) = \left|\{j \mid u_4^j \neq v_4^j \text{ for } 0 \leqslant j \leqslant k\}\right|.$$

Using the notion of $h_4(u,v)$, we can redefine SQ_n as follows: The vertex u and the vertex v are linked by an edge if and only if one of the following conditions holds:
(1) $u_4^{j^*} \oplus v_4^{j^*} \in V_{u_4^0}$ for exactly one j^* satisfying $1 \leqslant j^* \leqslant k$ and $u_4^j = v_4^j$ for all $0 \leqslant j \neq j^* \leqslant k$.
(2) $u_4^0 \oplus v_4^0 \in \{01, 10\}$ and $u_4^j = v_4^j$ for all $1 \leqslant j \leqslant k$.

For example, the ten neighbors of 1011000010 in SQ_{10} are given by <u>0011</u>000010, <u>0010</u>000010,

$\underline{0001}000010$, $\underline{0000}000010$, $1011\underline{1000}10$, $1011\underline{1001}10$, $1011\underline{1010}10$, $1011\underline{1011}10$, $10110000\underline{0}0$, and $101100001\underline{1}$. In other words, the vertex u is adjacent to the vertex v only if $h_4(u,v)=1$. The converse is not necessarily true. For example, $u=0000000000$ is not adjacent to $v=0000000011$ though $h_4(u,v)=1$. Thus, $d(u,v)\geqslant h_4(u,v)$ for any two vertices u,v of SQ_n.

3. Properties of shuffle-cubes

In this paper, we only discuss on the connectivity and the diameter of SQ_n.

Theorem 1. *SQ_n is n-connected.*

Proof. We prove this theorem by induction. Since $SQ_2=Q_2$, SQ_2 is 2-connected. Since SQ_n is n-regular, it suffices to show that after removing arbitrary f vertices from SQ_n for $1\leqslant f\leqslant n-1$, the remaining graph is still connected. Let F be an arbitrary set of f vertices.

Now consider $n=6$. By definition, SQ_6 consists of 16 SQ_2 subcubes. We decompose SQ_6 into two subgraphs H_1 and H_2, where H_1 consists of those SQ_2 subcubes containing vertices in F, and H_2 consists of the remaining SQ_2 subcubes. It is observed that H_2 is connected, and that H_1-F is not necessarily connected. We distinguish the following two cases:

Case 1.1. Each SQ_2 subcube has at most one vertex in F. It follows that each subcube of SQ_6-F is still connected and has at least three vertices. Furthermore, H_1 contains at most five subcubes since $|F|\leqslant 5$. Let Q' be a subcube in H_1-F. Since Q' contains three vertices, it has twelve edges connected with eleven or twelve other subcubes in SQ_6-F. Since there are at most five subcubes in H_1-F, Q' is connected to some subcubes in H_2. Since H_2 is connected and each subcube in H_1-F is also connected to H_2, SQ_6-F is connected.

Case 1.2. There is a subcube containing at least two vertices of F. Let v be a vertex in H_1-F. Then v is connected to four other subcubes. Since $|F|\leqslant 5$, H_1 contains at most four subcubes and therefore, v is connected to a subcube in H_2. Since H_2 is connected, it follows that each vertex in H_1-F is connected to some vertices in H_2. Therefore, SQ_6-F is connected.

Hence, SQ_6 is 6-connected.

We assume that SQ_{4k-2} is $(4k-2)$-connected for $k\geqslant 2$. Now consider SQ_n for $n=4k+2$ and $k\geqslant 2$, and there are at most $4k+1$ vertices in F. Each subcube of SQ_{4k+2} is an SQ_{4k-2}. We distinguish the following two cases for F:

Case 2.1. Each subcube contains at most $4k-3$ vertices in F. By the induction hypothesis, each subcube is still connected. Consider two arbitrary subcubes $SQ_{4k-2}^{i_1i_2i_3i_4}$ and $SQ_{4k-2}^{j_1j_2j_3j_4}$. The edges (u,v) between $SQ_{4k-2}^{i_1i_2i_3i_4}$ and $SQ_{4k-2}^{j_1j_2j_3j_4}$ satisfy $s_{4k-2}(u)=s_{4k-2}(v)$, and $p_4(u)\oplus p_4(v)=i_1i_2i_3i_4\oplus j_1j_2j_3j_4\in V_{s_2(u)}$. Therefore, the number of edges in SQ_n between $SQ_{4k-2}^{i_1i_2i_3i_4}$ and $SQ_{4k-2}^{j_1j_2j_3j_4}$ is 2^{4k-4} which is greater than $|F|$. Consequently, each subcube $SQ_{4k-2}^{i_1i_2i_3i_4}-F$ is connected to every subcube $SQ_{4k-2}^{j_1j_2j_3j_4}-F$. Furthermore, it follows from the induction hypothesis that each subcube $SQ_{4k-2}^{i_1i_2i_3i_4}-F$ is connected. Hence $SQ_{4k+2}-F$ is connected.

Case 2.2. There is a subcube containing at least $4k-2$ vertices in F. It follows that H_1 contains at most four subcubes. The proof is similar to Case 1.2 for SQ_6.

Hence, SQ_{4k+2} is $4k+2$ connected. And the theorem follows. □

Lemma 1. *$D(SQ_n)\geqslant\lceil n/4\rceil+3$ if $n=4k+2$ with $k\geqslant 4$.*

Proof. Let P be any path of SQ_n from u to v. We can view P as a sequence of 4-bits changing from u to v. Let $u=u_{n-1}u_{n-2}\ldots u_1u_0=u_4^ku_4^{k-1}\ldots u_4^0$ with $u_4^0=00$, $u_4^1=1100$, $u_4^2=1000$, $u_4^3=0100$, and $u_4^j=0001$ if $4\leqslant j\leqslant k$. Let $v=v_{n-1}v_{n-2}\ldots v_1v_0$ with $v_j=0$ for $0\leqslant j<n$.

Note that $0001, 0100, 1000$, and 1100 are only in V_{00}, V_{01}, V_{10}, and V_{11}, respectively. We can change any 4-bit $0001, 0100, 1000$, or 1100 into 0000 in one step only if the 0th 4-bit is $00, 01, 10$, or 11, respectively. Thus, $d(u,v)\geqslant\lceil n/4\rceil+3$. Hence $D(SQ_n)\geqslant\lceil n/4\rceil+3$. □

Next, we propose a routing algorithm on SQ_n. Let u and v be two vertices of SQ_n. We use $h_4^*(u,v)$

to denote the number of u_4^j for $1 \leqslant j \leqslant k$ such that $u_4^j \neq v_4^j$.

Route1(u, v)

(1) If $u = v$, then accept the message.

(2) Find a neighbor w of u such that $h_4^*(w, v) = h_4^*(u, v) - 1$ if w exists. Then route into w.

(3) If there is no neighbor w of u such that $h_4^*(w, v) = h_4^*(u, v) - 1$, then route into the neighbor w of u that changes $u_1 u_0$ in a cyclic manner with respect to 00, 01, 11, 10. For example, $w = p_{n-2}(u)00$ if $u_1 u_0 = 10$.

Example 1. Let $u = 0001000101001000110000$ and $v = 0000000000000000000011$ be two vertices of SQ_{20}. The path obtained from **Route1**(u, v) is

0001000101001000110000,

0000000101001000110000,

0000000001001000110000,

0000000001001000110001,

0000000000001000110001,

0000000000001000110011,

0000000000001000000011,

0000000000001000000010,

0000000000000000000010,

0000000000000000000000,

0000000000000000000001,

0000000000000000000011.

We note that this path is not the shortest path.

Applying the above algorithm to any two vertices u and v on SQ_n, it is observed that we may apply step (3) at most three times to obtain a vertex w such that $h_4^*(w, v) = 0$. Hence the algorithm will find a path, not necessarily the shortest path, of length at most $h_4^*(u, v) + 6$ that joins u to v. Therefore, $D(SQ_n) \leqslant \lceil n/4 \rceil + 5$. We will discuss the exact value of $D(SQ_n)$ after we introduce the concept of generalized shuffle-cubes.

4. Generalized shuffle-cubes

In this section, we generalize the shuffle-cubes into *generalized shuffle-cubes*. For any positive integer l, we use $S(l)$ to denote the set of all binary strings of length l and we use $S^*(l)$ to denote $S(l) - \{\underbrace{00\cdots0}_{l}\}$.

Let b and g be any positive integers satisfying $2^b \geqslant (2^g - 1)/g$. For each $i_1 i_2 \cdots i_b \in S(b)$, we associate it with a subset $A_{i_1 i_2 \cdots i_b}$ of $S^*(g)$ with the following properties:

(1) $|A_{i_1 i_2 \cdots i_b}| = g$, and

(2) $\bigcup_{i_1 i_2 \cdots i_b \in S(b)} A_{i_1 i_2 \cdots i_b} = S^*(g)$.

We say the family $A = \{A_{i_1 i_2 \cdots i_b} \mid i_1 i_2 \cdots i_b \in S(b)\}$ with the above properties is a *normal* (g, b) *family*. For example, $\{A_{00}, A_{01}, A_{10}, A_{11}\}$ is the normal $(4, 2)$ family where $A_{00}, A_{01}, A_{10}, A_{11}$ are defined in Section 2.

Definition 2. Let B be any b-regular graph with vertex set $S(b)$ and A be any normal (g, b) family. Then we can recursively define the n-dimensional generalized shuffle-cube $GSQ(n, A, B)$ for any $n = kg + b$ for $k \geqslant 0$ with its vertex set to be $S(n)$ as follows:

(1) If $n = b$, $GSQ(n, A, B)$ is B.

(2) If $n = kg + b$ for $k \geqslant 1$, any two vertices u and v in $GSQ(n, A, B)$ are adjacent if and only if
   - (a) $s_{n-g}(u)$ and $s_{n-g}(v)$ are adjacent in $GSQ(n - g, A, B)$, and $p_g(u) = p_g(v)$; or
   - (b) $s_{n-g}(u) = s_{n-g}(v)$ and $p_g(u) \oplus p_g(v) \in A_{u_{b-1} u_{b-2} \cdots u_0}$.

For example, Q_n is the $GSQ(n, A, B)$ where $A = \{A_0\}$ is a normal $(1, 0)$ family with $A_0 = \{1\}$ and $B = Q_0$; and SQ_n is the $GSQ(n, A, B)$ where $A = \{A_{00}, A_{01}, A_{10}, A_{11}\}$ is a normal $(4, 2)$ family and B is Q_2.

Assume that $GSQ(n, A, B)$ be a generalized shuffle-cube. Obviously, $GSQ(n, A, B)$ is an n-regular graph with 2^n vertices. Let u and v be vertices in $GSQ(n, A, B)$. For $1 \leqslant j \leqslant k$, the jth *g-bit* of u, denoted by u_g^j, is $u_g^j = u_{gj+b-1} u_{gj+b-2} \cdots u_{gj+b-g}$. In particular, the 0th g-bit of u is $u_g^0 = u_{b-1} u_{b-2} \cdots u_0$. The *g-bit Hamming distance* between u and v, denoted by $h_g(u, v)$, is the number of g-bits u_g^j with $0 \leqslant j \leqslant k$ such that $u_g^j \neq v_g^j$, i.e., $h_g(u, v) = |\{j \mid u_g^j \neq$

v_g^j for $0 \leqslant j \leqslant k\}|$. We also use $h_g^*(u,v)$ to denote the number of u_g^j for $1 \leqslant j \leqslant k$ such that $u_g^j \neq v_g^j$.

Applying similar arguments to Theorem 1, we have the following theorem.

Theorem 2. *$\kappa(GSQ(n,A,B)) = n$ if $\kappa(B) = b$.*

To discuss the diameter of a generalized shuffle-cube $GSQ(n,A,B)$, we assume that B has some Hamiltonian properties. Let G be a graph. A sequence of vertices $C = \langle x_0, x_1, \ldots, x_k\rangle$ in a graph G is a *cycle* if $k \geqslant 3$, $(x_{i-1}, x_i) \in E$ for $1 \leqslant i \leqslant k$, $x_0 = x_k$, and $x_i \neq x_j$ for $0 \leqslant i \neq j < k$. A *Hamiltonian path* (*cycle*) is a path (cycle) that spans all the vertices of G. We say that G is *Hamiltonian* if G has a Hamiltonian cycle.

A graph G is *Hamiltonian connected* if there exists a Hamiltonian path from u to v for any two different vertices u and v in G. However, it is known that any bipartite graph with at least three vertices is not Hamiltonian connected. A bipartite graph with bipartition (X,Y) is *Hamiltonian laceable* if there exists a Hamiltonian path from u to v for any two different vertices u and v that are in different parts, i.e., one in X and one in Y. For example, Q_n is Hamiltonian laceable [10].

Suppose that B is Hamiltonian. Let $C = \langle x_0, x_1, \ldots, x_k = x_0\rangle$ is a Hamiltonian cycle of B. The cycle $\langle 00, 01, 11, 10, 00\rangle$, for example, is a Hamiltonian cycle of Q_2. We generalize the routing algorithm **Route1**(u,v) for $GSQ(n,A,B)$ as follows:

Route2(u,v)
(1) If $u = v$, then accept the message.
(2) Find a neighbor w of u such that $h_g^*(w,v) = h_g^*(u,v) - 1$ if w exists. Then route into w.
(3) If $h_g^*(u,v) > 0$ and there is no neighbor w of u such that $h_g^*(w,v) = h_g^*(u,v) - 1$, then route into the neighbor w of u that changes u_g^0 in a cyclic manner with respect to C.
(4) If $h_g^*(u,v) = 0$, find a neighbor z of $s_b(u)$ in B such that the distance between z and $s_b(v)$ is the distance between $s_b(u)$ and $s_b(v)$ minus one. Then route into $p_{n-b}(u)z$.

So we have the following theorem.

Theorem 3. *$D(GSQ(n,A,B)) \leqslant (n-b)/g + 2^b - 1 + D(B)$ if B is Hamiltonian.*

The upper bound for the $D(GSQ(n,A,B))$ can be further reduced if B is Hamiltonian connected or Hamiltonian laceable. Assume that B is Hamiltonian laceable. To route u to v, we first compute a vertex sequence $Z(u,v)$ of $S(b)$ as follows: If $s_b(u)$ and $s_b(v)$ are in different parts, set $Z(u,v)$ to be any Hamiltonian path from $s_b(u)$ to $s_b(v)$. If $s_b(u)$ and $s_b(v)$ are in the same part, find a neighborhood $s_b(z)$ of $s_b(v)$ in B, let P be a Hamiltonian path from $s_b(u)$ to $s_b(z)$, and set $Z(u,v)$ to be the vertex sequence $\langle P, s_b(v)\rangle$. Then the path of $GSQ(n,A,B)$ from u to v can be determined by the following algorithm:

Route3(u,v)
(1) If $u = v$, then accept the message.
(2) Find a neighbor w of u such that $h_g^*(w,v) = h_g^*(u,v) - 1$ if w exists. Then route into w.
(3) If there is no neighbor w of u such that $h_g^*(w,v) = h_g^*(u,v) - 1$, then route into the neighbor w of u that changes u_g^0 in the order of $Z(u,v)$.

Example 2. As we point out before, SQ_n is a generalized shuffle-cube $GSQ(n,A,B)$ with $B = Q_2$. It is known that Q_2 is Hamiltonian laceable. Let

$u = 0001000101001000110000$

and

$v = 0000000000000000000011$

be two vertices of SQ_{20}. Obviously, 00 and 11 are in the same part and 10 is a neighbor of 11. Hence $\langle 00, 01, 11, 10\rangle$ is a Hamiltonian path from 00 to 10 in Q_2. Thus, we can set $Z(u,v)$ as $\langle 00, 01, 11, 10, 11\rangle$. The path obtained from **Route3**(u,v) is

0001000101001000110000,

$\underline{0000}000101001000110000$,

$0000\underline{0000}01001000110000$,

$000000000100100011000\underline{1}$,

$00000000\underline{0000}1000110001$,

$00000000000010001100\underline{1}1$,

$0000000000001000\underline{0000}11$,

$000000000000100000001\underline{0}$,

$000000000000\underline{0000}000010$,

$000000000000000000001\underline{1}$.

We note that this path is shorter than the path obtained in Example 1.

It is observed that we should apply step (3) exactly $2^b - 1$ times to obtain a vertex w such that either $w = v$ or w is a neighbor of v. Thus, we have the following theorem.

Theorem 4. *$(n - b)/g \leqslant D(GSQ(n, A, B)) \leqslant (n - b)/g + 2^b$ if B is Hamiltonian laceable.*

Assume that B is Hamiltonian connected. To avoid trivial case, we assume that $b \geqslant 1$. To route from u to v in $GSQ(n, A, B)$, we compute a vertex sequence $Z(u, v)$ of $S(b)$ as follows: If $s_b(u) \neq s_b(v)$, set $Z(u, v)$ to be any Hamiltonian path from $s_b(u)$ to $s_b(v)$. If $s_b(u) = s_b(v)$, find a neighborhood $s_b(z)$ of $s_b(v)$ in B, let P be a Hamiltonian path from $s_b(u)$ to $s_b(z)$, and set $Z(u, v)$ to be the vertex sequence $\langle P, s_b(v)\rangle$. We can also apply **Route3**(u, v) to obtain a path from u to v. So we have the following theorem.

Theorem 5. *$D(GSQ(n, A, B)) \leqslant (n - b)/g + 2^b$ if B is Hamiltonian connected.*

Now, we can determine the diameter of SQ_n for $n = 4k + 2$.

Theorem 6. *Assume $n = 4k + 2$. Then $D(SQ_n)$ is 2 if $n = 2$; 4 if $n = 6$; or $\lceil n/4\rceil + 3$ if $n \geqslant 10$.*

Proof. Using breadth first search, we can easily determine $D(SQ_2) = 2$, $D(SQ_6) = 4$, $D(SQ_{10}) = 6$, and $D(SQ_{14}) = 7$. Combining Lemma 1 and Theorem 4, we can conclude that $D(SQ_n) = \lceil n/4\rceil + 3$ if $n \geqslant 18$. The theorem is proved. □

5. Conclusion

In this paper, we present a variation of hypercubes, called shuffle-cubes, and their generalization. All the present known variations of hypercubes of dimension n are n regular graphs with connectivity n and of diameter around $n/2$. The shuffle-cube of dimension n, SQ_n, has the same parameters of these topological properties except the diameter is around $n/4$. Furthermore, for any positive integer g, we can construct an n-dimensional generalized shuffle-cube with 2^n vertices which is n-regular and n-connected. Its diameter can be about n/g if we consider g as a constant.

We can also choose g as $\log n$ and choose b as $\log n - \log\log n$. Obviously, $2^b \geqslant (2^g - 1)/g$. So we can find a (g, b) family $\hat{A}$ and construct an n-dimensional generalized shuffle-cube $GSQ(n, \hat{A}, Q_b)$. By Theorem 4,

$$D\big(GSQ(n, \hat{A}, Q_b)\big) \leqslant 2\frac{n}{\log n}.$$

Let $N = 2^n$ be the number of nodes of $GSQ(n, \hat{A}, Q_b)$. Thus, the diameter of $GSQ(n, \hat{A}, Q_b)$ is

$$\mathrm{O}\left(\frac{\log N}{\log\log N}\right).$$

The star graphs, S_n, is another family of famous interconnection networks [2]. S_n is an $(n - 1)$-regular graphs with $N = n!$ vertices and of diameter $\lfloor 3(n - 1)/2\rfloor$. Thus, the diameter of S_n is also $\mathrm{O}(\frac{\log N}{\log\log N})$ which is of the same order as that of $GSQ(n, \hat{A}, Q_b)$.

Acknowledgements

The authors are very grateful to the anonymous referees for their thorough review of the paper and concrete and helpful suggestions.

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