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The decay number and the maximum genus of diameter 2 graphs

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Abstract

Let $\zeta(G)$ (resp. $\xi(G)$) be the minimum number of components (resp. odd size components) of a co-tree of a connected graph G. For every 2-connected graph G of diameter 2, it is known that $m(G)\geqslant 2n(G)-5$ and $\xi(G)\leqslant\zeta(G)\leqslant 4$. These results define three classes of extremal graphs. In this paper, we prove that they are the same, with the exception of loops added to vertices.

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1. Introduction

Throughout this paper, a graph may have multiple edges or loops. It is said to be simple if it contains neither multiple edges nor loops.

Let G be a graph with $n(G)=|V(G)|$ vertices and $m(G)=|E(G)|$ edges. Murty [2] (see also [1]) proved the following result.

Theorem 1.1 (Murty [2]). *If G is a 2-connected graph of diameter 2, then*

$$m(G)\geqslant 2n(G)-5.$$

Let A be a subset of $E(G)$ and let $G-A$ denote the spanning subgraph obtained from G by deleting all edges in A. Let $c(G-A)$ be the number of components

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of $G - A$. The Betti number (or cycle rank) of $G - A$ is defined by $\beta(G - A) = m(G) - |A| - n(G) + c(G - A)$.

For G connected, let T be a spanning tree of G. Denote by $\xi(G - T)$ the number of components with an odd number of edges of the co-tree $G - T$. Let $\xi(G)$ be the minimum value of $\xi(G - T)$ over all co-trees of G. The invariant $\xi(G)$, called the Betti deficiency of G, was first introduced in Ref. [10] to calculate the maximum genus $\gamma_M(G)$ of G by the formula $\gamma_M(G) = (\beta(G) - \xi(G))/2$.

Motivated by this result, Škoviera [7] defined the decay number of G, $\zeta(G)$, to be the minimum value of $c(G-T)$ over all co-trees of G. Clearly, $\zeta(G) = 2n(G) - m(G) - 1 + \min\{\beta(G - T)\}$. It follows that

Theorem 1.2 (Škoviera [8]). *If G is a connected graph, then $\zeta(G) \geqslant 2n(G) - m(G) - 1$ and equality holds if and only if G admits an acyclic co-tree.*

For 2-connected graph G of diameter 2, Škoviera [8] gave a tight upper bound on $\zeta(G)$, and hence $\xi(G)$.

Theorem 1.3 (Škoviera [8]). *If G is a 2-connected graph of diameter 2, then*

$$\xi(G) \leqslant \zeta(G) \leqslant 4.$$

It is interesting to note that the preceding bound, together with Theorem 1.2, yields an another proof of Theorem 1.1.

For general G, Nebeský [5] discovered a formula to calculate $\zeta(G)$.

Theorem 1.4 (Nebesky [5]). *For any connected graph G,*

$$1 + \zeta(G) = \max\{2c(G - A) - |A| \mid A \subseteq E(G)\}.$$

As above, one notes that the preceding formula, together with Theorem 1.3, yields another proof of Theorem 1.1 (take $A = E(G)$ in Theorem 1.4).

This paper concerns the extremal graphs of Theorems 1.1 and 1.3. We will prove that they are the same, with the exception of loops added to vertices.

2. Extremal 2-connected graphs of diameter 2

A 2-connected graph G of diameter 2 is called *extremal* if and only if $m(G) = 2n(G) - 5$. By Theorem 1.1, such a graph is simple.

Remark 2.1. Let G be a connected graph with $m(G) = 2n(G) - 5$. Then the diameter of G is at least 2. Moreover, it has at least 4 vertices and its minimum degree is at most 3.

Note here the following result proved in [6] (see also [3]).

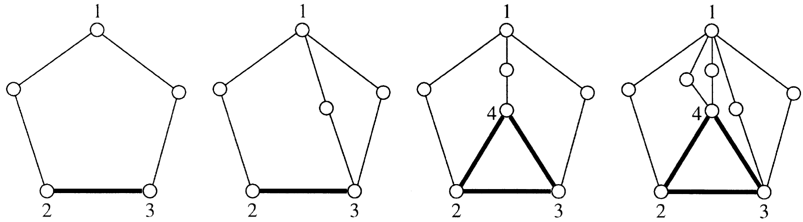

Fig. 1.

Theorem 2.2 (Palumbiny [6]). *Let G be a simple graph of diameter* 2 *with minimum degree at least* 3. *Then* $m(G)=2n(G)-5$ *if and only if G is the Petersen graph.*

Murty [3] characterized extremal 2-connected graphs of diameter 2. His result can be stated as follows.

Theorem 2.3 (Murty [3]). *The following statements are equivalent for a graph G.*

(1) *G is an extremal* 2*-connected graph of diameter* 2.
(2) *G is either the Petersen graph or is constructed by connecting all vertices of* K_2 *or* K_3 *to a new vertex by paths of length* 2.

Examples: In Fig. 1, K_2 and K_3 are in heavy lines, 1 is the new vertex.

3. Two-connected graphs of diameter 2 and decay number 4

In this section, we proceed to characterize 2-connected graphs G of diameter 2 satisfying the equality $\zeta(G)=4$.

Before stating this characterization, it will be convenient to introduce the following concept.

Definition 1. Let G be a connected graph. We say that a subset A of $E(G)$ is ζ-minimal if $1+\zeta(G)=2c(G-A)-|A|$ and, for every $B\subset A$, $1+\zeta(G)>2c(G-B)-|B|$.

The following remark will prove useful subsequently.

Remark 3.1. Let G be a connected graph and let A be a ζ-minimal subset of $E(G)$. Then each component of $G-A$ is an induced subgraph of G and any two different components are joined by at most one edge in A.

We are now prepared to describe the family of 2-connected graphs G of diameter 2 with $\zeta(G)=4$.

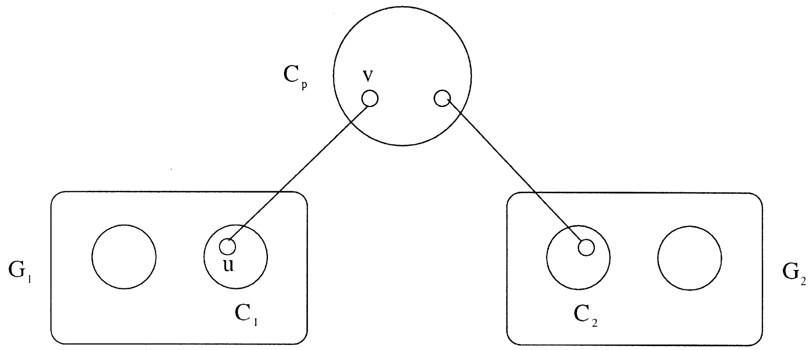

Fig. 2.

Theorem 3.2. *Let G be a* 2*-connected graph of diameter* 2. *Then* $\zeta(G)=4$ *if and only if G is an extremal* 2*-connected graph of diameter* 2 *with loops added to vertices.*

Proof. First let $B(G)$ be the set of loops of G and suppose that $G - B(G)$ is an extremal 2-connected graph of diameter 2. Then the removal of $A = E(G) - B(G)$ from G results in a graph with $n(G)$ components. Since $|A| = 2n(G) - 5$, Theorem 1.4 gives $\zeta(G) \geqslant 4$. By Theorem 1.3, equality must hold and the sufficiency of the condition is proved.

Conversely, let G be a 2-connected graph of diameter 2 with $\zeta(G) = 4$. Let A be a ζ-minimal subset of $E(G)$. Let $\{C_1, C_2, \ldots, C_p\}$ be the set of components of $G - A$. Then $|A| = 2p - (1 + \zeta(G)) = 2p - 5$. It follows that the loopless graph obtained from G by contracting each C_i to a single vertex verifies Remark 2.1. So $p \geqslant 4$ and there are two components that are not joined by an edge in A. We are going to show that $p = n(G)$, or equivalently, every C_i has only one vertex. Suppose, on the contrary, there is a component C_k with $|V(C_k)| \geqslant 2$. Since G is 2-connected, there must be two disjoint edges joining C_k to the remainder of the graph. Let a and b be their endvertices in $V(C_k)$. Now contract each component C_i $(i \neq k)$ to a single vertex, then identify any vertex in $V(C_k) - \{a\}$ with b (loops are deleted and multiple adjacencies between a and b are replaced by a single edge). Let H be the resulting graph. Then H has diameter 2 since it is not complete. We now show that H is 2-connected. Assume not. Then H has a cutvertex, say C_p, which is adjacent to any C_i. This implies that the removal of C_p from G results in a graph with at least 2 components G_1 and G_2 (Fig. 2).

It follows that there are two disjoint subsets I and J of $\{1, 2, \ldots, p-1\}$ such that G_1 and G_2 are spanned, respectively, by $\bigcup_{i \in I} C_i$ and $\bigcup_{j \in J} C_j$. Let, say C_1, be any component contained in G_1. By Remark 3.1, let $u \in V(C_1)$ and $v \in V(C_p)$ be the endvertices of the unique edge between C_1 and C_p. Now there is in G_2 a component, let C_2, which is not joined to v by an edge; for otherwise, v would be a cutvertex of G (recall that there is an unique edge between C_p and any C_j). So, in G, every vertex of C_2 is at distance $\geqslant 3$ from u, contradicting the fact that G has diameter 2. In conclusion, H is a 2-connected diameter 2 graph. But this contradicts Theorem 1.1 because $m(H)=|A|+1=(2p-5)+1=2(p+1)-6=2n(H)-6$ and the theorem is proved. □

On the other hand, the above result can be also proved by an extension of Theorem 1.1 which was obtained by Tsai [9]. Before stating this method, we need the following definition.

Definition 2. Let G be a connected graph. We say that a subset A of $E(G)$ is *E-minimal* if any two different components of $G-A$ are joined by at most one edge in G. Furthermore, we denote by G/A the graph obtained from G by contracting each component of $G-A$ into a vertex.

The following remark is an extension of Theorem 1.1.

Remark 3.3 (Tsai [9]). Let G be a 2-connected graph of diameter 2 and let A be an E-minimal subset of $E(G)$. Then

$$m(G/A)\geqslant 2n(G/A)-5+i(G/A),$$

where $i(G/A)$ is the number of components in $G-A$ containing at least two vertices of G.

Remark 3.4. By Remark 3.1, any ζ-minimal subset of $E(G)$ is also an E-minimal subset of $E(G)$. To prove the necessity of Theorem 3.2, one can let A be an ζ-minimal subset of $E(G)$. Then $\zeta(G)=2n(G/A)-1-m(G/A)=4$. This implies $i(G/A)=0$ and $m(G/A)=2n(G/A)-5$. Hence Theorem 3.2 can be obtained by Remark 3.3.

4. Two-connected graphs of diameter 2 and Betti deficiency 4

We conclude this paper with a characterization of 2-connected graphs G of diameter 2 satisfying the equality $\xi(G)=4$. Notice here a formula discovered by Nebeský [4] to calculate $\xi(G)$.

Theorem 4.1 (Nebesky [4]). *For any connected graph G,* $1+\xi(G)=\max\{c(G-A)+o(G-A)-|A| \mid A\subseteq E(G)\}$, *where* $o(G-A)$ *denotes the number of components of odd Betti number.*

As above, the following fact may prove useful.

Remark 4.2. Let G be a connected graph and let A be a ξ-minimal subset of $E(G)$ (i.e. $1+\xi(G)=c(G-A)+o(G-A)-|A|$ and, for every $B\subset A$, $1+\xi(G)>$

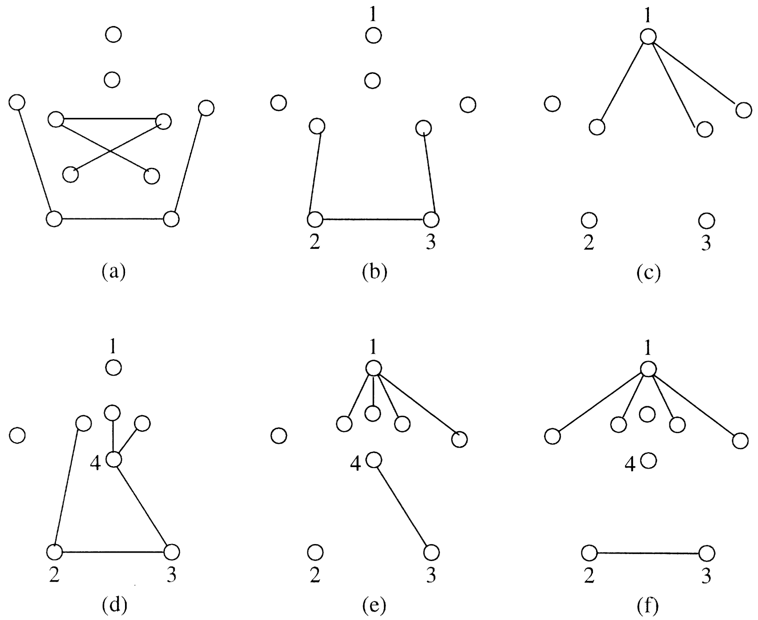

Fig. 3.

$c(G-B)+o(G-B)-|B|)$. Then each component of $G-A$ is an induced subgraph of odd Betti number (i.e. $c(G-A)=o(G-A)$) and any two different components are joined by at most one edge in A.

Now, we state

Theorem 4.3. *Let G be a 2-connected graph of diameter 2. Then $\xi(G)=4$ if and only if G is an extremal 2-connected graph of diameter 2 at each vertex of which an odd number of loops are added.*

Proof. A proof can be readily supplied by imitating that of Theorem 3.2. Here, let us apply Theorems 2.3 and 3.2 to present a short one of the non-trivial part of the statement.

Let G be a 2-connected graph of diameter 2 with $\xi(G)=4$. Then $\zeta(G)=4$ by Theorem 1.3. Hence, we know from Theorem 3.2 that G arises from an extremal 2-connected graph H of diameter 2 by adding loops to vertices. We now show that if some vertex of G has an even number of loops then $\xi(G)\leqslant 3$. To see this we shall show, in fact, that for any vertex x of an extremal 2-connected graph H of diameter 2, there is a co-tree K with $\zeta(K)=4$ such that the component containing x has only one vertex. Using Theorem 2.3, this follows immediately from the constructions illustrated in Fig. 3 below.

(a) a co-tree of the Petersen graph.

(b), (c) co-trees of the graph built up from K_2 by connecting 2 and 3 to 1 by two 2-paths.

(d), (e), (f) co-trees of the graph built up from K_3 by connecting 2 and 4 to 1 by two 2-paths, 3 to 1 by one 2-path.

Remark 4.4. By Remark 4.2, any ξ-minimal subset of $E(G)$ is also an E-minimal subset of $E(G)$. To prove the necessity of Theorem 3.2, one can let A be a ξ-minimal subset of $E(G)$. Hence Theorem 4.3 can also be obtained by Remark 3.3 as Remark 3.4.

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