

Nonequivalence transformation of λ -matrix eigenproblems and model embedding approach to model tuning

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SUMMARY

In this paper we present some theory for a non-equivalence transformation of matrix eigenvalues for λ -matrix polynomials. Application of this transformation to eigenvalue embedding for model tuning on a realistic industry problem is illustrated. The new approach has several advantages such as flexibility, efficiency, and structure-preservation. A numerical experiment on a pseudosimulation data set from The Boeing Company is reported. Copyright © 2001 John Wiley & Sons, Ltd.

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1. INTRODUCTION

The use of mathematical models for simulation is common in industry. However, as the demand for greater performance and cost reductions increase, the demand for increased fidelity in mathematical models also increases. More accurate models allow for a reduction in time consuming and expensive physical tests. (Testing is still done, but more insightful tests can be performed and less tests on trying to determine what the real model is have to be performed.) The process of obtaining an accurate model is not a simple one-step process. It is built over time and experience, both with previous models as well as the current model and its ability to predict performance.

Given an initial model and some collected data, the process of comparing the data to the model and modifying the model is called *model updating*. A recurring, and key, component to model updating is the identification of parameters in the model that could be in error. We refer to this problem as *model tuning*. This process is often currently done by an analyst's intuition. It is a slow, tedious process which has few tools for the analyst and often results in

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a modified model that has been improved, although it may no longer make physical sense. It is also a process that few analysts enjoy repeating. As a result the insight they gain is often difficult to recover or left for the next analyst to rediscover.

In this paper we present a means for the model analyst to identify potential parameters to modify in the updating process, (i.e. model tuning). One means for identifying such parameters that has been advanced is to compute the sensitivity of all of the model parameters to the particular objective function of interest. Although this makes much intuitive sense, the sensitivity that we are really interested in is the sensitivities at the actual solution and of course we do not know the actual solution. Thus the sensitivities computed in this manner are only reasonable guesses. Moreover, choosing those parameters with large sensitivities is only choosing parameters that will need to be changed the least in order to obtain an updated model. These parameters may in fact not be in error and only being modified to account for incorrect modelling of other parameters. Although the approach we put forward in this paper cannot declare that the identified parameters are in error, it does provide an alternative way to look at model tuning. The approach presented in this paper quickly produces a set of parameters to modify and does so using a solution to the problem. Of course, it may not be the solution desired by the engineer. This leads to one shortfall of the method namely that of being able to easily add constraints on the solution computed.

In the method presented in this paper we are trying to identify what elements in the model matrices appear to be the most in error. The goal is for the final model to match some characteristics of the observed model. This approach has the additional advantage of being applicable for models in generalized co-ordinates. The method proceeds by first forming a small model (realized model) that has been built from the observed data via, e.g. system identification [1], and embedded into the analytic model so that some characteristic of the model matches the actual behaviour of the system. This procedure has been done in Reference [2] but at the expense of destroying the structure of the mathematical model and rendering it difficult to interpret physically. This paper presents a means by which the structure of the model is preserved.

Throughout this paper we consider our model to be the state-space model of the form

$$\dot{x}(t) = Ax(t) + Bu(t) \quad (1)$$

$$y(t) = Cx(t) + Du(t) \quad (2)$$

where A, B, C, D are matrices of orders $n \times n$, $n \times p$, $q \times n$, and $q \times p$, respectively, and $x(t)$ is the state vector, $u(t)$ the input vector, and $y(t)$ the output vector. The components of $u(t)$ satisfy

$$\int_0^{\infty} |u_i(t)| e^{-\sigma t} dt < \infty, \quad i = 1, \dots, p$$

for some $\sigma < \infty$.

From the observed data we construct a model of much lower order of a similar form as above, i.e.

$$\dot{x}_r(t) = A_r x_r(t) + B_r u_r(t) \quad (3)$$

$$y_r(t) = C_r x_r(t) + D_r u_r(t) \quad (4)$$

We denote the model with a subscript r to indicate that it is a model realized from the observed data. The accuracy of the models is assessed by comparing the observed and predicted frequency responses.

One of the commonly used techniques is to transform the time-domain model into frequency-domain model. Considerable information can be deduced by examining frequency response functions. Taking the Laplace transforms of the first-order matrix differential equation (1) and output measurement equation (2) with zero initial condition $x(0) = 0$ yields

$$\begin{aligned} s\hat{x}(s) &= A\hat{x}(s) + B\hat{u}(s) \\ \hat{y}(s) &= C\hat{x}(s) + D\hat{u}(s) \end{aligned}$$

Hence

$$\begin{aligned} \hat{y}(s) &= [C(sI - A)^{-1}B + D]\hat{u}(s) = \left(\frac{C \operatorname{adj}(sI - A)B}{\det(sI - A)} + D \right) \hat{u}(s) \\ &\equiv \frac{p(s)}{q(s)} \hat{u}(s) \equiv G(s)\hat{u}(s) \end{aligned} \tag{5}$$

The matrix $[C(sI - A)^{-1}B + D]$ is called the transfer function between the input $\hat{u}(s)$ and output $\hat{y}(s)$. The roots of the determinant $\det(sI - A)$ are called the poles of the system and are the eigenvalues of A . The eigenvalues and eigenvectors of A provide important information to the prediction of system behavior.

In this paper we investigate a three-stage embedding process for model tuning. The first is a matching of eigenvalues in the analytic system matrix A of (1) and the realized system matrix A_r of (3) from observed data. The second step is simply pole assignment. The third is back-transforming so that a modified system matrix \tilde{A} of order $n \times n$ is obtained and the model is again stated in terms of the same states, inputs, and outputs. (Matrices B, C, D are also modified accordingly.) The resulting back-transformed system is then compared with the original system to identify those parameters most likely to be modified to achieve a more accurate model. A non-equivalence transformation [3] technique is developed for the eigenvalue embedding in the second step so that the modified matrix \tilde{A} has all the desired eigenvalues of A_r and has exactly the same structure as the original matrix A . With this approach it becomes possible to select and preserve some desired eigenmodes and remove those unwanted.

We organize this paper as follows. The general theory of non-equivalence transformation for analytic matrix polynomials is developed in the next section, then we demonstrate the model embedding procedure in Section 3 by applying the theory to a practical application arising in the aerospace industry. Numerical results on a pseudosimulation data set, provided by The Boeing Company, is shown in Section 4. Concluding remarks are in Section 5.

2. NON-EQUIVALENCE TRANSFORMATION OF λ -MATRIX EIGENPROBLEMS

The purpose of this paper is to develop an eigenvalue embedding approach method for the model tuning problem. To achieve this goal, we reduce the problem to the following basic

question: “How can a λ -matrix polynomial be transformed to another λ -matrix polynomial such that most of eigenvalues are preserved except for a finite number of eigenvalues which are replaced by some prescribed values?” Precisely speaking, let us consider an $n \times n$ matrix polynomial in λ ,

$$P(\lambda) = \lambda^m A_m + \cdots + \lambda A_1 + A_0 \quad (6)$$

where A_0, A_1, \dots, A_m are $n \times n$ matrices. If there exist a scalar μ and a non-zero n -vector y such that $P(\mu)y = 0$, then μ is an eigenvalue of $P(\lambda)$ and y is the associated eigenvector. Let us denote $\sigma(P(\lambda))$ the set of eigenvalues related to $P(\lambda)$. Now, suppose that μ is the eigenvalue to be replaced by a new value, say η , then the basic question is to find a λ -matrix polynomial

$$\tilde{P}(\lambda) = \lambda^m \tilde{A}_m + \cdots + \lambda \tilde{A}_1 + \tilde{A}_0$$

such that $\sigma(\tilde{P}(\lambda)) = (\sigma(P(\lambda)) \setminus \{\mu\}) \cup \{\eta\}$. Before presenting our method, we first briefly review some related works. Instead of looking at λ -matrix polynomial, we consider a more general case, λ -matrix $Q(\lambda)$ [4–6]. With the help of a special factorization of $Q(\lambda)$ developed by Dewilde and Vandewalle [7], one can show that for any *isolated* eigenvalue μ of $Q(\lambda)$, i.e. $\det(Q(\mu)) = 0$ and the multiplicity of μ is equal to one, there exists a rational transfer function $c(\lambda)$ of degree 1 such that μ is no longer an eigenvalue of $\tilde{Q}(\lambda) = Q(\lambda) \cdot c(\lambda)$, i.e. $\det(\tilde{Q}(\mu)) \neq 0$. Obviously, this result can be used to solve the zero assignment problem and therefore provide an answer to our aforementioned basic question. However, in [7], to obtain the elementary factor $c(\lambda)$, not only we have to know the Laurent expansion of $Q(\lambda)$ at μ but also we need to check certain additional vector conditions for some enlarged matrices. To avoid dealing with those tedious matters, Van Dooren [8] proposed a rather elegant method to recursively remove unwanted poles and zeros of the transfer function given by (5). The main idea of this method is based on a QR-like algorithm where the relative transfer function is reduced to a block upper triangular matrix. But, if we want to take into account of the special structure of matrices associated with the transfer function, then the QR-like method would not be able to give us the right structure. In fact, Fritchman and Pierce [2] have implemented a QR-like method to solve the similar problem. However, the new matrix derived in Reference [2] failed to fit in any physical model.

All methods described above are designed to handle the λ -matrix $Q(\lambda)$. In this paper, we are only interested in the λ -matrix polynomial $P(\lambda)$. Since we have the better knowledge about the λ -matrix polynomial, it is anticipated that we can design a more efficient method to solve the basic question than the method used in Reference [7]. For a λ -matrix polynomial $P(\lambda)$, Guo *et al.* [3] developed a so-called non-equivalence transformation method to deflate the unwanted eigenvalues of $P(\lambda)$. The non-equivalence transformation method can be regarded as a special case of Dewilde and Vandewalle’s results [7]. By the non-equivalence transformations, we can find the elementary factor $c(\lambda)$ explicitly without checking any additional vector conditions nor computing the Laurent expansion or Taylor expansion of $P(\lambda)$. Furthermore, the non-equivalence transformation can be exploited to deflate a *non-simple* multiple eigenvalue μ , namely, the multiplicity of μ is greater than the dimension of the null space of $P(\mu)$, denoted by $\mathcal{N}(P(\mu))$, as well as to preserve the special structures of the matrices associated with the transfer function. We shall address these points at the end of this section and the next section.

We now generalize the results in Reference [3] and develop non-equivalence transformations to solve the basic question.

Let $P(\lambda)$ be a real λ -matrix polynomial as in (6). For an isolated real eigenpair (μ_1, y_1) , we have $P(\mu_1)y_1 = 0$. Now, we consider the transformed matrix function $\tilde{P}(\lambda)$ defined as

$$\tilde{P}(\lambda) = P(\lambda) \left[I_n - \frac{\varepsilon_1}{\lambda - \mu_1} y_1 z_1^T \right] \tag{7}$$

with $\varepsilon_1 \in \mathbb{R}$ and $z_1 \in \mathbb{R}^n$ such that $z_1^T y_1 = 1$. Here I_n denotes the $n \times n$ identity matrix. Since the Taylor expansion of $P(\lambda)$ gives

$$\begin{aligned} \tilde{P}(\lambda) &= P(\lambda) - \varepsilon_1 \left(\sum_{k=0}^m \frac{P^{(k)}(\mu_1)}{k!} (\lambda - \mu_1)^{k-1} \right) y_1 z_1^T \\ &= P(\lambda) - \varepsilon_1 \left(P'(\mu_1) + \frac{\lambda - \mu_1}{2!} P''(\mu_1) + \dots + \frac{(\lambda - \mu_1)^{m-1}}{m!} P^{(m)}(\mu_1) \right) y_1 z_1^T \end{aligned}$$

it follows that $\tilde{P}(\lambda)$ is a matrix polynomial in λ . With some algebraic manipulations, $\tilde{P}(\lambda)$ can be written as

$$\tilde{P}(\lambda) = \lambda^m \tilde{A}_m + \lambda^{m-1} \tilde{A}_{m-1} + \dots + \lambda \tilde{A}_1 + \tilde{A}_0 \tag{8}$$

where

$$\begin{aligned} \tilde{A}_m &= A_m \\ \tilde{A}_{m-1} &= A_{m-1} - \varepsilon_1 A_m y_1 z_1^T \\ &\vdots \\ \tilde{A}_1 &= A_1 - \varepsilon_1 (\mu_1^{m-2} A_m + \mu_1^{m-3} A_{m-1} + \dots + A_2) y_1 z_1^T \\ \tilde{A}_0 &= A_0 - \varepsilon_1 (\mu_1^{m-1} A_m + \mu_1^{m-2} A_{m-1} + \dots + \mu_1 A_2 + A_1) y_1 z_1^T \end{aligned} \tag{9}$$

The following theorem shows that when $\varepsilon_1 = \eta_1 - \mu_1$ for some desired scalar $\eta_1 \in \mathbb{R}$, the non-equivalence transformation (7) preserves all the eigenvalues of $P(\lambda)$ except μ_1 , which is replaced by η_1 , and (η_1, y_1) is an isolated eigenpair of $\tilde{P}(\lambda)$.

Theorem 2.1. *Let (μ_1, y_1) be an isolated eigenpair of a matrix polynomial $P(\lambda)$ as in (6). Then the transformed matrix polynomial $\tilde{P}(\lambda)$ defined by (7) with $\varepsilon_1 = \eta_1 - \mu_1$ for some $\eta_1 \in \mathbb{R}$ has the same eigenvalues as those of $P(\lambda)$ except μ_1 which is replaced by η_1 . Moreover, (η_1, y_1) is an eigenpair of $\tilde{P}(\lambda)$.*

Proof

By using the identity

$$\det(I_n + RS) = \det(I_m + SR) \tag{10}$$

from fundamental matrix theory, one can show that

$$\begin{aligned}\det \tilde{P}(\lambda) &= \det P(\lambda) \det \left[I_n - \frac{\varepsilon_1}{\lambda - \mu_1} y_1 z_1^T \right] \\ &= \det P(\lambda) \det \left(1 - \frac{\varepsilon_1}{\lambda - \mu_1} z_1^T y_1 \right) \\ &= \det P(\lambda) \left(\frac{\lambda - \eta_1}{\lambda - \mu_1} \right)\end{aligned}$$

Hence each eigenvalue of $P(\lambda)$ is an eigenvalue of $\tilde{P}(\lambda)$ except μ_1 which is replaced by η_1 . By substituting (η_1, y_1) into (7), one can verify that

$$\tilde{P}(\eta_1)y_1 = P(\eta_1) \left(I_n - \frac{\eta_1 - \mu_1}{\eta_1 - \mu_1} y_1 z_1^T \right) y_1 = P(\eta_1)y_1 - P(\eta_1)y_1 z_1^T y_1 = 0$$

Hence (η_1, y_1) is an eigenpair of $\tilde{P}(\lambda)$. ■

The theorem shows that when the eigenvalue μ_1 is replaced by η_1 , the corresponding eigenvector y_1 is preserved. For any unchanged eigenvalue μ_2 , the corresponding eigenvector is, however, modified by a rank-one updating. This fact is concluded in the following corollary.

Corollary 2.1. *Suppose $\mu_2 \neq \mu_1$ and (μ_2, y_2) is an eigenpair of $P(\lambda)$. Let*

$$\tilde{y}_2 = y_2 - y_1 \frac{\eta_1 - \mu_1}{\eta_1 - \mu_2} (z_1^T y_2) \quad (11)$$

Then (μ_2, \tilde{y}_2) is an eigenpair of $\tilde{P}(\lambda)$.

One can rewrite (11) and relate \tilde{y}_2 to y_2 by $\tilde{y}_2 = T y_2$ with

$$T \equiv I_n - y_1 \frac{\eta_1 - \mu_1}{\eta_1 - \mu_2} z_1^T$$

Then T is non-singular since by applying (10) one can show that

$$\det(T) = \det \left(I_n - y_1 \frac{\eta_1 - \mu_1}{\eta_1 - \mu_2} z_1^T \right) = \det \left(1 - \frac{\eta_1 - \mu_1}{\eta_1 - \mu_2} z_1^T y_1 \right) = \frac{\mu_1 - \mu_2}{\eta_1 - \mu_2} \neq 0$$

Analogous theory can be developed for complex conjugate eigenvalues. Suppose that μ_1 and $\bar{\mu}_1$ are non-zero complex conjugate eigenvalues of $P(\lambda)$, and are to be replaced by η_1 and $\bar{\eta}_1$, respectively. Write $\mu_1 = \alpha_1 + i\beta_1$ and $\eta_1 = \tilde{\alpha}_1 + i\tilde{\beta}_1$. Let

$$E_1 = \Omega_1 - \Lambda_1 \quad (12)$$

where

$$\Lambda_1 = \begin{bmatrix} \alpha_1 & \beta_1 \\ -\beta_1 & \alpha_1 \end{bmatrix}, \quad \Omega_1 = \begin{bmatrix} \tilde{\alpha}_1 & \tilde{\beta}_1 \\ -\tilde{\beta}_1 & \tilde{\alpha}_1 \end{bmatrix} \quad (13)$$

The complex eigenpair of a matrix polynomial $P(\lambda)$ can be defined as follows.

Definition 2.1. Suppose $y_1 = \text{Re}(y_1) + i\text{Im}(y_1) \in \mathbb{C}^n$ and $y_1 \neq 0$. Let

$$Y_1 = [\text{Re}(y_1), \text{Im}(y_1)] \quad (14)$$

Then (Λ_1, Y_1) is called an eigenpair of $P(\lambda)$ if

$$A_m Y_1 \Lambda_1^m + A_{m-1} Y_1 \Lambda_1^{m-1} + \cdots + A_1 Y_1 \Lambda_1 + A_0 Y_1 = 0 \quad (15)$$

The non-equivalence transformation for the complex case is a generalization of (9). The transformed λ -matrix polynomial $\tilde{P}(\lambda)$ (8) has coefficients as follows:

$$\begin{aligned} \tilde{A}_m &= A_m \\ \tilde{A}_{m-1} &= A_{m-1} - A_m Y_1 E_1 Z_1^T \\ &\vdots \\ \tilde{A}_1 &= A_1 - (A_m Y_1 \Lambda_1^{m-2} + A_{m-1} Y_1 \Lambda_1^{m-3} + \cdots + A_2 Y_1) E_1 Z_1^T \\ \tilde{A}_0 &= A_0 - (A_m Y_1 \Lambda_1^{m-1} + A_{m-1} Y_1 \Lambda_1^{m-2} + \cdots + A_2 Y_1 \Lambda_1 + A_1 Y_1) E_1 Z_1^T \end{aligned} \quad (16)$$

where $Z_1 \in \mathbb{R}^{n \times 2}$ and $Z_1^T Y_1 = I_2$.

The following two theorems are generalizations of Theorem 2.1 and Corollary 2.1, respectively, to the complex eigenpair case.

Theorem 2.2. Let (Λ_1, Y_1) be an eigenpair of $P(\lambda)$. Then (Ω_1, Y_1) is an eigenpair of $\tilde{P}(\lambda)$ after the non-equivalence transformation (16).

Proof

With (12), (15), (16), and $Z_1^T Y_1 = I_2$, one can verify that

$$\begin{aligned} &\tilde{A}_m Y_1 \Omega_1^m + \tilde{A}_{m-1} Y_1 \Omega_1^{m-1} + \cdots + \tilde{A}_1 Y_1 \Omega_1 + \tilde{A}_0 Y_1 \\ &= A_m Y_1 \Omega_1^m + A_{m-1} Y_1 \Omega_1^{m-1} - A_m Y_1 (\Omega_1 - \Lambda_1) \Omega_1^{m-1} \\ &\quad + A_{m-2} Y_1 \Omega_1^{m-2} - (A_m Y_1 \Lambda_1 + A_{m-1} Y_1) (\Omega_1 - \Lambda_1) \Omega_1^{m-2} + \cdots \\ &\quad + A_1 Y_1 \Omega_1 - (A_m Y_1 \Lambda_1^{m-2} + A_{m-1} Y_1 \Lambda_1^{m-3} + \cdots + A_2 Y_1) (\Omega_1 - \Lambda_1) \Omega_1 \\ &\quad + A_0 Y_1 - (A_m Y_1 \Lambda_1^{m-1} + A_{m-1} Y_1 \Lambda_1^{m-2} + \cdots + A_2 Y_1 \Lambda_1 + A_1 Y_1) (\Omega_1 - \Lambda_1) \\ &= A_m Y_1 \Lambda_1^m + A_{m-1} Y_1 \Lambda_1^{m-1} + \cdots + A_1 Y_1 \Lambda_1 + A_0 Y_1 \\ &= 0. \end{aligned}$$

Hence (Ω_1, Y_1) is an eigenpair of $\tilde{P}(\lambda)$. ■

Theorem 2.3. Suppose that (Λ_2, Y_2) , $\Lambda_2 \neq \Lambda_1$ and $\Lambda_2 \neq \Omega_1$, is any eigenpair of $P(\lambda)$. Let

$$\tilde{Y}_2 = Y_2 + Y_1 L \quad (17)$$

where L is the unique solution of the Sylvester equation

$$\Omega_1 L - L \Lambda_2 = -E_1 (Z_1^T Y_2) \quad (18)$$

Then (Λ_2, \tilde{Y}_2) is an eigenpair of the transformed matrix polynomial $\tilde{P}(\lambda)$.

Proof

The proof is straightforward. Using (16) and (17) one has

$$\begin{aligned}
& \tilde{A}_m \tilde{Y}_2 \Lambda_2^m + \tilde{A}_{m-1} \tilde{Y}_2 \Lambda_2^{m-1} + \cdots + \tilde{A}_1 \tilde{Y}_2 \Lambda_2 + \tilde{A}_0 \tilde{Y}_2 \\
&= A_m Y_2 \Lambda_2^m + A_{m-1} Y_2 \Lambda_2^{m-1} + \cdots + A_1 Y_2 \Lambda_2 + A_0 Y_2 \\
&\quad - [A_m Y_1 (E_1 Z_1^T Y_2 \Lambda_2^{m-1} + \Lambda_1 E_1 Z_1^T Y_2 \Lambda_2^{m-2} + \cdots + \Lambda_1^{m-2} E_1 Z_1^T Y_2 \Lambda_2 + \Lambda_1^{m-1} E_1 Z_1^T Y_2) \\
&\quad + A_{m-1} Y_1 (E_1 Z_1^T Y_2 \Lambda_2^{m-2} + \Lambda_1 E_1 Z_1^T Y_2 \Lambda_2^{m-3} + \cdots + \Lambda_1^{m-3} E_1 Z_1^T Y_2 \Lambda_2 + \Lambda_1^{m-2} E_1 Z_1^T Y_2) \\
&\quad + \cdots + A_2 Y_1 (E_1 Z_1^T Y_2 \Lambda_2 + \Lambda_1 E_1 Z_1^T Y_2) + A_1 Y_1 E_1 Z_1^T Y_2] \\
&\quad + A_m Y_1 L \Lambda_2^m + A_{m-1} Y_1 L \Lambda_2^{m-1} + \cdots + A_1 Y_1 L \Lambda_2 + A_0 Y_1 L \\
&\quad - [A_m Y_1 (E_1 L \Lambda_2^{m-1} + \Lambda_1 E_1 L \Lambda_2^{m-2} + \cdots + \Lambda_1^{m-2} E_1 L \Lambda_2 + \Lambda_1^{m-1} E_1 L) \\
&\quad + A_{m-1} Y_1 (E_1 L \Lambda_2^{m-2} + \Lambda_1 E_1 L \Lambda_2^{m-3} + \cdots + \Lambda_1^{m-3} E_1 L \Lambda_2 + \Lambda_1^{m-2} E_1 L) \\
&\quad + \cdots + A_2 Y_1 (E_1 L \Lambda_2 + \Lambda_1 E_1 L) + A_1 Y_1 E_1 L]
\end{aligned}$$

Since (Λ_2, Y_2) is an eigenpair of $P(\lambda)$, $A_m Y_2 \Lambda_2^m + A_{m-1} Y_2 \Lambda_2^{m-1} + \cdots + A_1 Y_2 \Lambda_2 + A_0 Y_2 = 0$. If L solves the Sylvester equation (18), then, with substitution of (12), the above identity can be simplified to

$$(A_m Y_1 \Lambda_1^m + A_{m-1} Y_1 \Lambda_1^{m-1} + \cdots + A_1 Y_1 \Lambda_1 + A_0 Y_1) L$$

Therefore $\tilde{A}_m \tilde{Y}_2 \Lambda_2^m + \tilde{A}_{m-1} \tilde{Y}_2 \Lambda_2^{m-1} + \cdots + \tilde{A}_1 \tilde{Y}_2 \Lambda_2 + \tilde{A}_0 \tilde{Y}_2 = 0$, and the theorem is proved. \blacksquare

We have shown that transformation (7) can be used to replace an isolated unwanted eigenvalue μ_1 by a eigenvalue η_1 . Next, we consider replacing an unwanted eigenvalue with multiplicity greater than one.

Let $\mu_1 \in \mathbb{R}$ be an unwanted simple eigenvalue of $P(\lambda)$ with multiplicity $p_1 > 1$ which will be replaced by η_1 . A multiple eigenvalue is *simple*, if the multiplicity of μ_1 is equal to the dimension of $\mathcal{N}(P(\mu_1))$. We can easily generalize Theorems 2.2 and 2.3 to obtain a new λ -matrix polynomial $\tilde{P}(\lambda)$ such that η_1 is a simple multiple eigenvalue of $\tilde{P}(\lambda)$ with multiplicity p_1 which replaces the unwanted eigenvalue μ_1 . Let $Y_1 \in \mathbb{R}^{n \times p_1}$ be a basis for $\mathcal{N}(P(\mu_1))$ and $Z_1 \in \mathbb{R}^{n \times p_1}$ satisfy $Z_1^T Y_1 = I_{p_1}$. Similar to (7) $\tilde{P}(\lambda)$ can be written as

$$\tilde{P}(\lambda) = P(\lambda) \cdot \left(I - \frac{\varepsilon_1}{\lambda - \mu_1} Y_1 Z_1^T \right)$$

with $\varepsilon_1 = \eta_1 - \mu_1$. If we set $\Lambda_1 = \mu_1 I_{p_1}$, $\Omega_1 = \eta_1 I_{p_1}$ and $E_1 = \Omega_1 - \Lambda_1$ as in (12), then we can also obtain a λ -matrix polynomial $\tilde{P}(\lambda)$ in explicit matrix forms of (16) such that the unwanted simple eigenvalue μ_1 is completely deflated and is replaced by η_1 .

Note that we can also use the same argument as above to extend the results in Theorems 2.2 and 2.3 into the case of simple multiple eigenvalues with complex conjugate. We omit the proofs here in detail.

For the general case when the unwanted eigenvalue μ_1 of $P(\lambda)$ is multiple but *non-simple*, we shall develop a non-linear non-equivalence transformation for replacing μ_1 by η_1 . For

simplicity, let μ_1 be a multiple eigenvalue with partial multiplicities $p_1 > 1$ and $q_1 > 1$, that is, the characteristic polynomial of $P(\lambda)$ has non-linear elementary divisors $(\lambda - \mu_1)^{p_1}$ and $(\lambda - \mu_1)^{q_1}$, respectively corresponding to μ_1 , and let y_1, \hat{y}_1 be the associated linearly independent eigenvectors. Suppose that the matrix Z_1 satisfies $Z_1^T Y_1 = I_2$, where $Y_1 = [y_1, \hat{y}_1]$. Then one can apply the non-linear non-equivalence transformation

$$\tilde{P}(\lambda) = P(\lambda) \left(I_n - Y_1 \begin{bmatrix} \frac{\varepsilon_1}{(\lambda - \mu_1)^{p_1}} & 0 \\ 0 & \frac{\delta_1}{(\lambda - \mu_1)^{q_1}} \end{bmatrix} Z_1^T \right) \tag{19}$$

to replace the *non-simple* multiple eigenvalue μ_1 by η_1 , where

$$\begin{aligned} \varepsilon_1 = & (\eta_1 - \mu_1)[(\lambda - \mu_1)^{p_1-1} + (\lambda - \mu_1)^{p_1-2}(\lambda - \eta_1)^1 + (\lambda - \mu_1)^{p_1-3}(\lambda - \eta_1)^2 \\ & + \dots + (\lambda - \mu_1)^1(\lambda - \eta_1)^{p_1-2} + (\lambda - \eta_1)^{p_1-1}] \end{aligned}$$

and

$$\begin{aligned} \delta_1 = & (\eta_1 - \mu_1)[(\lambda - \mu_1)^{q_1-1} + (\lambda - \mu_1)^{q_1-2}(\lambda - \eta_1)^1 + (\lambda - \mu_1)^{q_1-3}(\lambda - \eta_1)^2 \\ & + \dots + (\lambda - \mu_1)^1(\lambda - \eta_1)^{q_1-2} + (\lambda - \eta_1)^{q_1-1}] \end{aligned}$$

Clearly $\tilde{P}(\lambda)$ is still a λ -matrix polynomial of degree m . By taking determinant on the both sides of Equation (19) one can easily check that $\tilde{P}(\lambda)$ has the same eigenvalues as those of $P(\lambda)$ except that μ_1 is replaced by η_1 with multiplicity $p_1 + q_1$.

To determine the partial multiplicities p_1 and q_1 of μ_1 , and the associated eigenvectors y_1, \hat{y}_1 , Van Dooren/Dewilde [9] and Lin [10] proposed some efficient numerical algorithms to find the eigenstructure (Jordan canonical form) of a λ -matrix polynomial $P(\lambda)$. The partial multiplicities p_1 and q_1 are exactly the size of elementary Jordan blocks corresponding to the eigenvalue μ_1 of the augment eigensystem (see, e.g. Section 3.4 in Reference [11]):

$$\begin{bmatrix} 0 & I & & & \\ & 0 & I & & \\ & & \ddots & \ddots & \\ & & & 0 & I \\ A_1 & A_2 & \dots & & A_{m-1} \end{bmatrix} \begin{bmatrix} y \\ \lambda y \\ \vdots \\ \vdots \\ \lambda^{m-1} y \end{bmatrix} = \lambda \begin{bmatrix} I & & & & \\ & I & & & \\ & & \ddots & & \\ & & & I & \\ & & & & -A_m \end{bmatrix} \begin{bmatrix} y \\ \lambda y \\ \vdots \\ \vdots \\ \lambda^{m-1} y \end{bmatrix}$$

Similarly, we can also use formula (19) to replace $\bar{\mu}_1$ by $\bar{\eta}_1$ provided that μ_1 is a non-simple complex conjugate eigenvalue. Unfortunately, to construct a formula as in (16) preserving the real arithmetic for the case of non-simple complex eigenvalues is still lacking.

3. EIGENVALUE EMBEDDING AND MODEL TUNING

In the aerospace industry, DYLOFLEX [12] is a common tool for constructing aircraft structures models for dynamic loads analysis. These models come in the form

$$M\ddot{q} + (D_1 + \zeta(s)D_2)\dot{q} + (K_1 + \zeta(s)K_2)q = H(t, s) \tag{20}$$

where M is the inertia matrix, D_1 and K_1 are the structural damping and stiffness matrices respectively, D_2 and K_2 are the aerodynamic damping and stiffness matrices, respectively, $\zeta(s)$ is the Wagner lift-growth buildup function [13], $H(t,s)$ is the generalized forces and gust inputs, q is an n -vector of generalized coordinates, and s is the Laplace operator. For this study we have chosen $\zeta(s)$ to be of the form

$$\zeta(s) = \frac{\rho(s - \omega) + \rho}{s - \omega}$$

with constants $\rho \neq 0$ and $\omega \neq 0$. For this Wagner lift-growth buildup function, (20) can be rewritten as a first-order realization

$$\dot{x} = Ax + Bu(t,s)$$

where

$$A = \begin{bmatrix} M_1 & M_2 & M_3 & M_4 \\ I & 0 & 0 & 0 \\ 0 & \rho I & \omega I & 0 \\ \rho I & 0 & 0 & \omega I \end{bmatrix} \in \mathbb{R}^{4n \times 4n} \quad (21)$$

$$\begin{aligned} M_1 &= -M^{-1}(D_1 + \rho D_2) \\ M_2 &= -M^{-1}(K_1 + \rho K_2) \\ M_3 &= -M^{-1}K_2 \\ M_4 &= -M^{-1}D_2 \end{aligned} \quad (22)$$

$$x = \begin{bmatrix} x_1 \\ q \\ x_3 \\ x_4 \end{bmatrix}$$

and $Bu(t,s)$ represents $H(t,s)$.

The problem is then to replace some eigenvalues of A with the desired eigenvalues, that is, embed some desired eigenvalues (maybe eigenvectors also) into A , and reconstruct a new matrix \tilde{A} of the form (21).

A possible approach [2] is to first reduce the full-order model to real Schur form so that the eigenvalues to be replaced occur in its leading principal minor. More precisely, one computes

$$Q^T A Q = \begin{bmatrix} R_{11} & R_{12} \\ 0 & R_{22} \end{bmatrix}$$

so that the eigenvalues of R_{11} have been matched with those of the realized system matrix A_r . Then one performs the diagonalization reduction by a similarity transformation

$$\begin{bmatrix} I & K \\ 0 & I \end{bmatrix} \begin{bmatrix} R_{11} & R_{12} \\ 0 & R_{22} \end{bmatrix} \begin{bmatrix} I & -K \\ 0 & I \end{bmatrix} = \begin{bmatrix} R_{11} & 0 \\ 0 & R_{22} \end{bmatrix}$$

where K solves the Sylvester equation

$$R_{11}K - KR_{22} = R_{12}.$$

In order to make the embedding consistent one must also transform the R_{11} block into a form \tilde{R}_{11} , e.g., block diagonal, that in some sense matches the form of the realized system matrix. Next the eigenvalues from observed data are embedded into (replace those of) \tilde{R}_{11} . Then all transformations are applied in reverse order to the block diagonal matrix

$$\begin{bmatrix} \tilde{R}_{11} & 0 \\ 0 & R_{22} \end{bmatrix}$$

to obtain the modified system matrix \tilde{A} .

This approach is straightforward, however, the major disadvantage is that the special sparsity structure of the system matrix as in (21) is usually destroyed through the back-transformation and cannot be recovered. Also rather large growth in the magnitude of the elements in \tilde{A} has been experienced.

Another possible approach [14] is to firstly transform (20) to a quadratic eigenvalue system

$$[\lambda^2 M + \lambda(D_1 + \zeta(s)D_2) + (K_1 + \zeta(s)K_2)]v = 0$$

Then let

$$\hat{A} = \begin{bmatrix} 0 & I \\ -M^{-1}(K_1 + \zeta(s)K_2) & -M^{-1}(D_1 + \zeta(s)D_2) \end{bmatrix} \quad (23)$$

and the method proposed in Reference [2] can be used to find \hat{f} such that

$$\tilde{A} = \hat{A} - \hat{b}\hat{f}^T \quad (24)$$

has the desired spectrum, where \hat{b} is some suitable vector. However, from DYLOFLEX [12], realization (21) rather than (23) can better reveal the characteristics of physical model. Furthermore, if the transformation (24) is applied to (21), then it can not guarantee to preserve the structure of (21) in the final back-transforming stage.

In this section we show that the problem can be solved by applying the non-equivalence transformation technique developed in the previous section and the matrix structure of the realization (21) can be preserved.

First we give the following theorem which relates the eigenvalues of the $4n \times 4n$ matrix A of (21) to a $3n \times 3n$ companion matrix, and consequently a cubic λ -matrix polynomial.

Theorem 3.1. *Let A be the $4n \times 4n$ matrix A of (21) and A_c the following $3n \times 3n$ companion matrix*

$$A_c = \begin{bmatrix} A_2 & A_1 & A_0 \\ I & 0 & 0 \\ 0 & I & 0 \end{bmatrix} \quad (25)$$

where

$$\begin{aligned} A_0 &= -\omega M_2 + \rho M_3 \\ A_1 &= -\omega M_1 + M_2 + \rho M_4 \\ A_2 &= M_1 + \omega I \end{aligned} \quad (26)$$

Then there exists an invertible matrix Q such that

$$Q^{-1}AQ = \left[\begin{array}{c|c} A_c & M_4 \\ \hline 0 & \omega I \end{array} \right] \quad (27)$$

Proof

Let

$$Q = \begin{bmatrix} I & -\omega I & 0 & 0 \\ 0 & I & -\omega I & 0 \\ 0 & 0 & \rho I & 0 \\ 0 & \rho I & 0 & I \end{bmatrix}. \quad (28)$$

Then Q is invertible and

$$Q^{-1} = \begin{bmatrix} I & \omega I & \frac{\omega^2}{\rho} I & 0 \\ 0 & I & \frac{\omega}{\rho} I & 0 \\ 0 & 0 & \frac{1}{\rho} I & 0 \\ 0 & -\rho I & -\omega I & I \end{bmatrix} \quad (29)$$

It is easy to verify that

$$Q^{-1}AQ = \left[\begin{array}{ccc|c} \omega I + M_1 & M_2 + \rho M_4 - \omega M_1 & \rho M_3 - \omega M_2 & M_4 \\ I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ \hline 0 & 0 & 0 & \omega I \end{array} \right] = \left[\begin{array}{c|c} A_c & M_4 \\ \hline 0 & \omega I \end{array} \right] \quad \blacksquare$$

Corollary 3.1. *Let*

$$P(\lambda) \equiv \lambda^3 A_3 + \lambda^2 A_2 + \lambda A_1 + A_0 \quad (30)$$

where $A_3 = -I$. Then $P(\lambda)$ and A have the same eigenvalues except for n occurrences of the eigenvalue ω .

One can further analyse the eigenpairs of A and give explicit formulations for the eigenvectors. Suppose that (λ, x) , where x is a $4n$ -vector, is an eigenpair of A . Then $Ax = \lambda x$ can be written as

$$\begin{bmatrix} M_1 & M_2 & M_3 & M_4 \\ I & 0 & 0 & 0 \\ 0 & \rho I & \omega I & 0 \\ \rho I & 0 & 0 & \omega I \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

where $x_i, i = 1, \dots, 4$, are n -vectors, or equivalently,

$$\begin{aligned} x_1 &= \lambda x_2 \\ \rho x_2 + \omega x_3 &= \lambda x_3 \\ \rho x_1 + \omega x_4 &= \lambda x_4 \\ M_1 x_1 + M_2 x_2 + M_3 x_3 + M_4 x_4 &= \lambda x_1 \end{aligned} \quad (31)$$

If $\lambda = 0$, then (31) implies $x_1 = x_4 = 0$, and

$$\begin{aligned} \rho x_2 + \omega x_3 &= 0 \\ M_2 x_2 + M_3 x_3 &= 0 \end{aligned}$$

That is, $x_3 = -(\rho/\omega)x_2$ and $(M_2 - (\rho/\omega)M_3)x_2 = 0$. Note that $x_2 \neq 0$ in this case. Therefore $\lambda = 0$ is an eigenvalue of A if and only if $M_2 - (\rho/\omega)M_3$ is singular and

$$\begin{bmatrix} 0 \\ x_2 \\ -\frac{\rho}{\omega}x_2 \\ 0 \end{bmatrix}$$

where $x_2 \in \mathcal{N}(M_2 - (\rho/\omega)M_3)$, is the corresponding eigenvector.

If $\lambda = \omega \neq 0$, (31) is reduced to $x_1 = x_2 = 0$, and $M_3 x_3 + M_4 x_4 = 0$. Hence

$$\begin{bmatrix} x_3 \\ x_4 \end{bmatrix} \in \mathcal{N}([M_3 \quad M_4])$$

Since the nullity of $[M_3, M_4]$ is greater than or equal to n , ω must be an eigenvalue of A with multiplicity greater than or equal to n .

Now assume that $\lambda \neq 0$ and ω . From (3.12) one has

$$x_2 = \frac{1}{\lambda}x_1, \quad x_3 = \frac{\rho}{\lambda(\lambda - \omega)}x_1, \quad x_4 = \frac{\rho}{\lambda - \omega}x_1$$

and

$$\left(M_1 + \frac{1}{\lambda}M_2 + \frac{\rho}{\lambda(\lambda - \omega)}M_3 + \frac{\rho}{\lambda - \omega}M_4 \right) x_1 = \lambda x_1$$

Consequently,

$$[-\lambda^3 I + \lambda^2(M_1 + \omega I) + \lambda(-\omega M_1 + M_2 + \rho M_4) + (-\omega M_2 + \rho M_3)]x_1 = 0$$

which is $P(\lambda)x_1 = 0$ with $P(\lambda)$ defined in (30). Therefore if (λ, x_1) is an eigenpair of $P(\lambda)$, then

$$\begin{bmatrix} x_1 \\ \frac{1}{\lambda}x_1 \\ \frac{\rho}{\lambda(\lambda - \omega)}x_1 \\ \frac{\rho}{\lambda - \omega}x_1 \end{bmatrix}$$

is the eigenvector of A corresponding to the eigenvalue λ .

To summarize the discussion above, it is evident that instead of dealing with the $4n \times 4n$ matrix A it is preferable to work on the $3n \times 3n$ matrix A_c or equivalently the cubic matrix polynomial $P(\lambda)$. Hence to replace some eigenvalues of A , we embed the eigenvalues of A_r into A_c by using the non-equivalence transformation technique. The modified but

structure-preserved matrix \tilde{A} can be obtained with the back-transformation (29). The detailed procedure is described as follows:

Let $\tilde{P}(\lambda) = \lambda^3 \tilde{A}_3 + \lambda^2 \tilde{A}_2 + \lambda \tilde{A}_1 + \tilde{A}_0$ be the transformed matrix polynomial. If (μ_1, y_1) is a real eigenpair of A and μ_1 is replaced by η_1 , then by (23) the coefficient matrices become

$$\begin{aligned}\tilde{A}_3 &= A_3 = -I \\ \tilde{A}_2 &= A_2 - \varepsilon_1 A_3 y_1 z_1^T \\ \tilde{A}_1 &= A_1 - \varepsilon_1 (\mu_1 A_3 + A_2) y_1 z_1^T \\ \tilde{A}_0 &= A_0 - \varepsilon_1 (\mu_1^2 A_3 + \mu_1 A_2 + A_1) y_1 z_1^T\end{aligned}$$

where $\varepsilon_1 = \eta_1 - \mu_1$ and z_1 is chosen such that $z_1^T y_1 = 1$. If a complex conjugate pair $\alpha_1 + i\beta_1$ and $\alpha_1 - i\beta_1$ are replaced by $\tilde{\alpha}_1 + i\tilde{\beta}_1$ and $\tilde{\alpha}_1 - i\tilde{\beta}_1$, then, followed from (16),

$$\begin{aligned}\tilde{A}_3 &= A_3 = -I \\ \tilde{A}_2 &= A_2 - A_3 Y_1 E_1 Z_1^T \\ \tilde{A}_1 &= A_1 - (A_3 Y_1 \Lambda_1 + A_2 Y_1) E_1 Z_1^T \\ \tilde{A}_0 &= A_0 - (A_3 Y_1 \Lambda_1^2 + A_2 Y_1 \Lambda_1 + A_1 Y_1) E_1 Z_1^T\end{aligned}$$

where E_1, Λ_1, Ω_1 , and Y_1 are as in (12)–(14), and Z_1 is chosen such that $Z_1^T Y_1 = I_2$.

With the modified matrices $\tilde{A}_i, i=0, 1, 2$, which result in a new companion matrix \tilde{A}_c , the new system matrix A can be constructed by applying the similarity transformation (27) in reverse order, that is,

$$\tilde{A} = Q \left[\begin{array}{c|c} \tilde{A}_c & M_4 \\ \hline 0 & \omega I \end{array} \right] Q^{-1} = \begin{bmatrix} \tilde{M}_1 & \tilde{M}_2 & \tilde{M}_3 & \tilde{M}_4 \\ I & 0 & 0 & 0 \\ 0 & \rho I & \omega I & 0 \\ \rho I & 0 & 0 & \omega I \end{bmatrix} \quad (32)$$

where

$$\begin{aligned}\tilde{M}_1 &= \tilde{A}_2 - \omega I \\ \tilde{M}_2 &= \tilde{A}_1 + \omega \tilde{M}_1 - \rho \tilde{M}_4 \\ \tilde{M}_3 &= \frac{1}{\rho} (\tilde{A}_0 + \omega \tilde{M}_2) \\ \tilde{M}_4 &= M_4\end{aligned} \quad (33)$$

Therefore the structure of the system matrix is successfully preserved and furthermore the computational cost of this back-transformation is negligible.

In practice it will be more useful and meaningful to provide the update formulations to the damping and stiffness matrices for model tuning on the differential equation (21). With the modified $\tilde{M}_i, i=1, \dots, 4$, and (23), the explicit formulations are given as

$$\begin{aligned}\tilde{D}_2 &= -M \tilde{M}_4 = -M M_4 \\ \tilde{D}_1 &= -M \tilde{M}_1 - \rho \tilde{D}_2 \\ \tilde{K}_2 &= -M \tilde{M}_3 \\ \tilde{K}_1 &= -M \tilde{M}_2 - \rho \tilde{K}_2\end{aligned} \quad (34)$$

Note that M_4 and consequently D_2 are not modified by the embedding process.

With this non-equivalence transformation, the eigenvectors will be modified when some eigenvalues are replaced (Theorem 2.3). It may be required to preserve some desired eigenmodes because of physical constraints. This can be achieved by choosing an appropriate Z_1 in the non-equivalence transformation formulation (16). Let (Λ_1, Y_1) be an eigenpair of $P(\lambda)$ and Λ_1 be replaced by Ω_1 . Suppose that the columns of V are eigenvectors of $P(\lambda)$ to be preserved. It may not be possible to keep all, but we can select as many vectors from V , and denoted by V_1 , such that columns of $[Y_1, V_1]$ are linearly independent. Then Z_1 can be obtained from the QR factorization of $[Y_1, V_1]$ such that

$$Z_1^T [Y_1, V_1] = [I, 0]$$

Let Γ_1 be the block diagonal matrix with 1×1 or 2×2 blocks consisting of eigenvalues corresponding to the eigenvectors in V_1 . Then one can verify that (Γ_1, V_1) is an eigenpair of $\tilde{P}(\lambda)$ by showing

$$\tilde{A}_3 V_1 \Gamma_1^3 + \tilde{A}_2 V_1 \Gamma_1^2 + \tilde{A}_1 V_1 \Gamma_1 + \tilde{A}_0 V_1 = A_3 V_1 \Gamma_1^3 + A_2 V_1 \Gamma_1^2 + A_1 V_1 \Gamma_1 + A_0 V_1 = 0$$

Therefore the desired eigenmodes are kept.

Sometimes one may want to embed not only the eigenvalues but also eigenvectors. Let (Λ_1, Y_1) and (Λ_2, Y_2) be eigenpairs of $P(\lambda)$. Suppose Λ_1 is to be replaced by Ω_1 and Λ_2 by Ω_2 , and $\hat{Y}_2 \in \mathbb{R}^n \times 2$ is a pre-determined eigenmode to be embedded into the system such that (Ω_2, \hat{Y}_2) is an eigenpair of the transformed matrix polynomial. This may not be completely possible. Here we propose a strategy to choose a corresponding eigenmode $\tilde{Y}_2 = \hat{Y}_2|_{\langle Y_1, Y_2 \rangle}$, the projection of \hat{Y}_2 on to the subspace $\langle Y_1, Y_2 \rangle$ spanned by Y_1 and Y_2 , which in some sense is close to \hat{Y}_2 .

When Λ_1 is first replaced by Ω_1 , Y_2 is updated to \tilde{Y}_2 using (17). In the next step, Λ_2 is replaced by Ω_2 , but \tilde{Y}_2 remains unchanged. Since Z_1 can be chosen freely, we determine

$$\tilde{Y}_2 = Y_2 + Y_1 L = \hat{Y}_2|_{\langle Y_1, Y_2 \rangle} \tag{35}$$

Then

$$(Y_2^T Y_2)^{-1} (Y_2^T \tilde{Y}_2) = I + (Y_2^T Y_2)^{-1} (Y_2^T Y_1) L$$

thus

$$L = (Y_2^T Y_2)^{-1} (Y_2^T \hat{Y}_2 - Y_2^T Y_2)$$

Substituting L into (18) results

$$Z_1^T Y_2 = -E_1^{-1} (\Omega_1 L - L \Lambda_2) \equiv K$$

Therefore Z_1 has to be chosen to satisfy

$$Z_1^T [Y_1 \quad Y_2 \quad V_1] = [I \quad K \quad 0] \tag{36}$$

where V_1 consists of those eigenmodes one wants to keep. By computing the SVD

$$[Y_1 \quad Y_2 \quad V_1] = U \Sigma V^T$$

one obtains

$$Z_1^T = [I \quad K \quad 0] V \Sigma^+ U^T \tag{37}$$

where Σ^+ is the pseudoinverse of Σ .

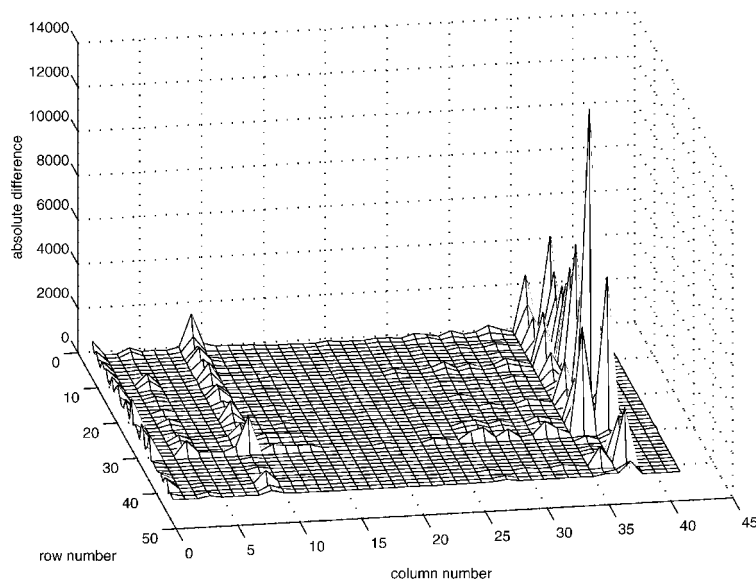


Figure 1. The absolute error $|D_1 - \tilde{D}_1|$.

4. NUMERICAL EXPERIMENT AND RESULTS

A set of pseudosimulation data was provided by The Boeing Company for testing purposes. The dimension of matrices M, D_1, D_2, K_1, K_2 are 42×42 , that is, the system matrix A is of dimension 168×168 . The numerical results are plotted in Figures 1–3 which show each entry in the absolute difference $|D_1 - \tilde{D}_1|$, $|K_1 - \tilde{K}_1|$, and $|K_2 - \tilde{K}_2|$, respectively. The figures show that relatively large errors occur in columns 35–39 as well as columns 4 and 9. These errors appear in the same locations in all three matrices. This suggests that those variables and equations corresponding to the finite approximation of the differential equation (20) are most likely in error and should be treated as optimization variables in the model updating process.

The numerical experiment illustrates the motivation of model tuning. After the eigenvalue embedding and back-transforming are completed, a system engineer can analyse the difference between the original and the modified models to identify the problem variables and equations that were most affected by the embedding processing in order to ‘tune’ the model.

5. CONCLUSION

We have demonstrated an algebraic approach for model tuning that preserves matrix structure while allowing for the assigning of poles and limiting the changes to the eigenvectors. This approach can serve as a fast means for identifying parameters to be modified in a problem and is applicable for models in generalized co-ordinates since it identifies elements in the model matrices. It was shown to be insightful in a sample pseudotest suite provided by The Boeing Company.

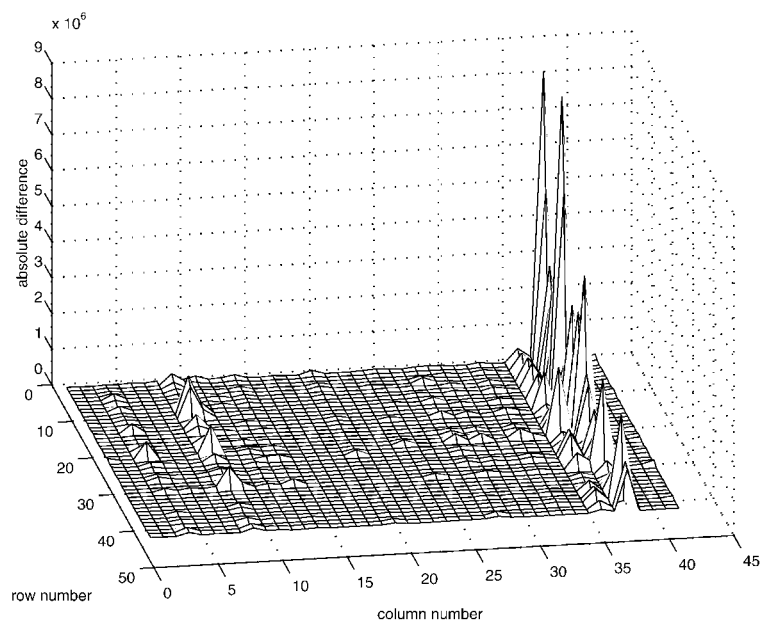


Figure 2. The absolute error $|K_1 - \tilde{K}_1|$.

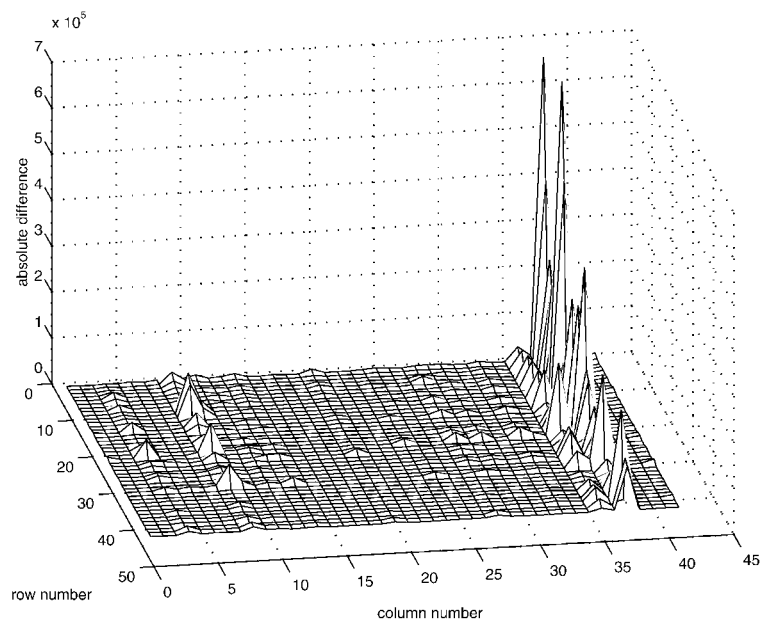


Figure 3. The absolute error $|K_2 - \tilde{K}_2|$.

Finally, we mention that although the non-equivalence transformation method is suitable for the model tuning problem we considered in this paper, to find an effective symmetric structure preserving method and to examine the impact of loss of symmetry on the physical model are still under investigation.

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