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Estimating process capability index C''_{pmk} for asymmetric tolerances: Distributional properties

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Abstract. Pearn et al. (1999) considered a capability index C''_{pmk}, a new generalization of C_{pmk}, for processes with asymmetric tolerances. In this paper, we provide a comparison between C''_{pmk} and other existing generalizations of C_{pmk} on the accuracy of measuring process performance for processes with asymmetric tolerances. We show that the new generalization C''_{pmk} is superior to other existing generalizations of C_{pmk}. Under the assumption of normality, we derive explicit forms of the cumulative distribution function and the probability density function of the estimated index $\hat{C}''_{pmk}$. We show that the cumulative distribution function and the probability density function of the estimated index $\hat{C}''_{pmk}$ can be expressed in terms of a mixture of the chi-square distribution and the normal distribution. The explicit forms of the cumulative distribution function and the probability density function considerably simplify the complexity for analyzing the statistical properties of the estimated index $\hat{C}''_{pmk}$.

Key words: Asymmetric loss function; Bias; Mean square error; Departure ratio; Normally distributed process

1 Introduction

Process capability indices (PCIs), which provide numerical measures on whether a process meets the capability requirement preset in the factory, have recently been widely used in the manufacturing industry. Examples include the manufacturing of semiconductor products (Hoskins et al. (1988)), front-end alignment for automobiles (Davis and Kaminsky (1989)), head/gimbals assembly for memory storage systems (Rado (1989)), jet-turbine engine components (Hubele et al. (1991)), flip-chips and chip-on-board (Noguera and Nielsen (1992)), piston rings for automobile engineering (Chou (1994)), cable locking terminals for automobile ignition system (Chou (1994)), speaker drivers

(Pearn and Chen (1997)), electrolytic capacitors (Pearn and Chen (1997)), and many others.

A process is said to have a symmetric tolerance if the target value T is set to be the midpoint of the specification interval (LSL, USL), i.e. $T = m = (\mathrm{USL} + \mathrm{LSL})/2$, where USL and LSL are the upper and the lower specification limits. Most research in quality assurance literature has focused on cases in which the manufacturing tolerance is symmetric. Examples include Kane (1986), Chan et al. (1988), Choi and Owen (1990), Boyles (1991), Pearn et al. (1992), Vännman (1995), Vännman and Kotz (1995), and Spiring (1997). Although cases with symmetric tolerances are common in practical situations, cases with asymmetric tolerances often occur in the manufacturing industry. In general, asymmetric tolerances simply reflect that deviations from target are less tolerable in one direction than the other (see Boyles (1994), Vännman (1997), and Wu and Tang (1998)). Asymmetric tolerances can also arise from a situation where the tolerances are symmetric to begin with, but the process follows a non-normal distribution and the data is transformed to achieve approximate normality, as shown by Chou et al. (1998) who have used Johnson curves to transform non-normal process data. Unfortunately, there has been comparatively little research published on cases with asymmetric tolerances. Exceptions are Boyles (1994), Vännman (1997), Chen (1998), Pearn and Chen (1998), Pearn et al. (1999), and Chen et al. (1999).

Three well-known basic indices are

$$C_p = \frac{\mathrm{USL} - \mathrm{LSL}}{6\sigma}, \tag{1}$$

$$C_{pk} = \frac{\min\{\mathrm{USL} - \mu, \mu - \mathrm{LSL}\}}{3\sigma}, \tag{2}$$

$$C_{pm} = \frac{\mathrm{USL} - \mathrm{LSL}}{6\sqrt{\sigma^2 + (\mu - T)^2}}, \tag{3}$$

where μ is the process mean and σ is the process standard deviation. The index C_p only considers the process variability σ thus provides no sensitivity on process departure at all. The index C_{pk} takes the process mean into consideration but it can fail to distinguish between on-target processes from off-target processes (Pearn et al. (1992)). The index C_{pm} takes the proximity of process mean from the target value into account, and is more sensitive to process departure than C_p and C_{pk}. Pearn et al. (1992) proposed an index called C_{pmk}, which combines the merits of the three basic indices C_p, C_{pk}, and C_{pm}. The index C_{pmk} has been defined as:

$$C_{pmk} = \frac{\min\{\mathrm{USL} - \mu, \mu - \mathrm{LSL}\}}{3\sqrt{\sigma^2 + (\mu - T)^2}}. \tag{4}$$

The index C_{pmk} is more sensitive to the departure of the process mean μ from the target value T than the other three indices C_p, C_{pk}, and C_{pm}. For symmetric tolerances, these indices provide reasonable measures on process potential and performance. But, for asymmetric tolerances, none of these indices provide consistent and reasonable measures on process capability.

2 Existing generalizations of C_{pmk}

Boyles (1994) presented a comprehensive study on some indices including C^{+}_{pm}, C^{*}_{pm}, C_{jkp}, S_{jkp}, C_{pmk}, and S_{pmk} for asymmetric tolerances, and compared them with each other on two performance criteria including process yield and process centering (the ability to cluster around the target). Based on the performance comparisons, Boyles (1994) recommended the index S_{pmk}, which is a generalization of C_{pmk} defined as:

$$S_{pmk} = \frac{1}{3}\Phi^{-1}\left\{\frac{1}{2}\Phi\left(\frac{\mathrm{USL}-\mu}{\sqrt{\sigma^2+(\mu-T)^2}}\right)+\frac{1}{2}\Phi\left(\frac{\mu-\mathrm{LSL}}{\sqrt{\sigma^2+(\mu-T)^2}}\right)\right\}, \tag{5}$$

where $\Phi(\cdot)$ is the cumulative function of the standard normal distribution and $\Phi^{-1}(\cdot)$ is the inverse function of $\Phi(\cdot)$.

Figures 1 and 2 display the plots of C_{pmk} and S_{pmk}, respectively, for processes with means $10 \le \mu \le 50$, and standard deviations $\sigma = 10/3$ (top) and $\sigma = 20/3$ (bottom), where $(\mathrm{LSL}, T, \mathrm{USL}) = (10, 40, 50)$ is an asymmetric tolerance. Boyles (1994) also investigated the index C_{pmk}, and commented that C_{pmk} has some advantages over S_{pmk}, particularly, the simplicity of its calculation and the easiness to work with analytically. However, for fixed process standard deviation σ, both C_{pmk} and S_{pmk} indices obtain their maximal values not at $\mu = T$, but at some value μ^* which is between the target value T and the center of the specification interval, m. The value of μ^* relative to T and m reflects the compromise established by the two indices between the process centering and the process yield.

To overcome the problem, Vännman (1997) considered another generalization of C_{pmk} to handle processes with asymmetric tolerances. Vännman's generalization has been defined as:

$$C_{pa}(u, v) = \frac{d - |\mu - m| - u|\mu - T|}{3\sqrt{\sigma^2 + v(\mu - T)^2}}, \tag{6}$$

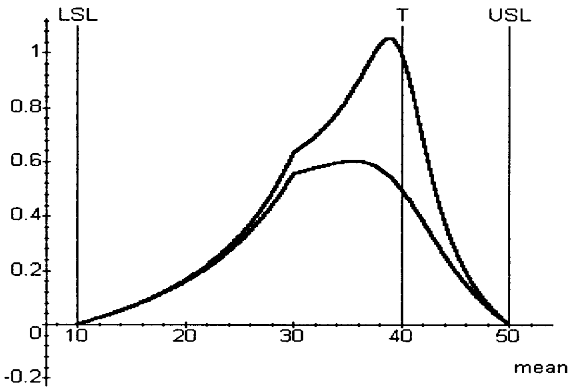

Fig. 1. Plots of C_{pmk} values for processes with $10 \le \mu \le 50$, $\sigma = 10/3$ (top), $10 \le \mu \le 50$, $\sigma = 20/3$ (bottom), and $(\mathrm{LSL}, T, \mathrm{USL}) = (10, 40, 50)$.

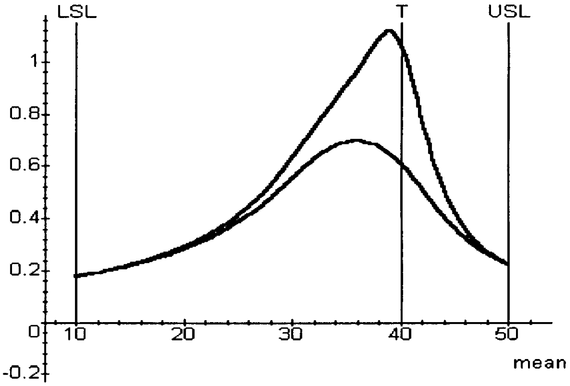

Fig. 2. Plots of S_{pmk} values for processes with $10 \le \mu \le 50$, $\sigma = 10/3$ (top), $10 \le \mu \le 50$, $\sigma = 20/3$ (bottom), and $(\text{LSL}, T, \text{USL}) = (10, 40, 50)$.

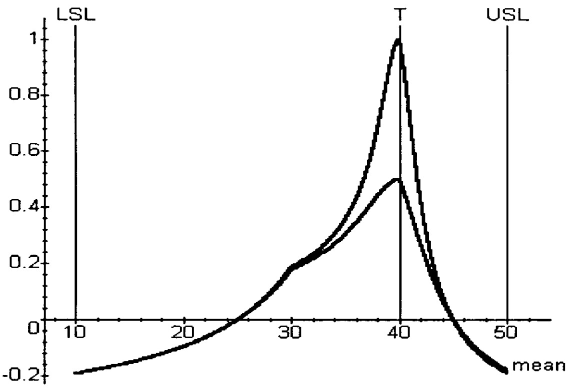

Fig. 3. Plots of $C_{pa}(1, 3)$ values for processes with $10 \le \mu \le 50$, $\sigma = 10/3$ (top), $10 \le \mu \le 50$, $\sigma = 20/3$ (bottom), and $(\text{LSL}, T, \text{USL}) = (10, 40, 50)$.

where $u, v \ge 0$ and $d = (\text{USL} + \text{LSL})/2$. We note that $C_{pa}(0, 1) = C_{pmk}$. For $u \ge 1$, the index $C_{pa}(u, v)$ decreases when mean μ shifts away from target in either direction. In fact, $C_{pa}(u, v)$ decreases faster when μ shifts away from T to the closer specification limit than that to the farther specification limit. This is an advantage since the index would respond faster to the shift towards "the wrong side" of T than towards the middle of the specification interval. Vännman (1997) showed that among many (u, v) values, $(u, v) = (0, 4)$ and $(u, v) = (1, 3)$ generate two indices which are most sensitive to process departure from the target. We note that $C_{pa}(0, 4) \ge 0$ for a process with mean value falling within the tolerance limits, i.e., for $\text{LSL} \le \mu \le \text{USL}$, which provides a clear indication about the process condition. But, $C_{pa}(1, 3) < 0$ for some processes with means falling within the tolerance limits. On the other hand, $C_{pa}(1, 3)$ obtains its maximal value when the process is on target, i.e., when $\mu = T$. But, $C_{pa}(0, 4)$ obtains its maximal value when the process is off target. Figure 3 displays the plots of $C_{pa}(1, 3)$ for processes with characteristic $10 \le \mu \le 50$,

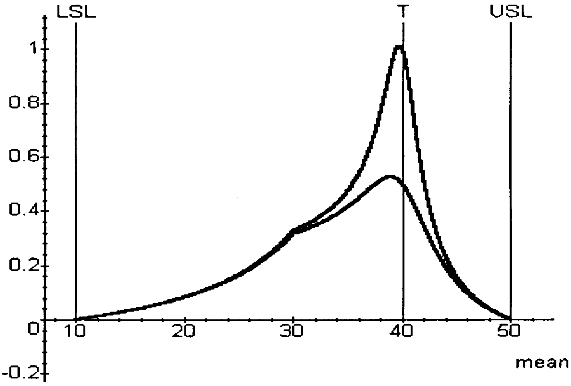

Fig. 4. Plots of $C_{pa}(0, 4)$ values for processes with $10 \le \mu \le 50$, $\sigma = 10/3$ (top), $10 \le \mu \le 50$, $\sigma = 20/3$ (bottom), and $(\text{LSL}, T, \text{USL}) = (10, 40, 50)$.

$\sigma = 10/3$ (top) and $\sigma = 20/3$ (bottom), where the manufacturing specifications $(\text{LSL}, T, \text{USL}) = (10, 40, 50)$ is an asymmetric tolerance. Figure 4 displays the plots of $C_{pa}(0, 4)$ for processes with characteristic $10 \le \mu \le 50$, $\sigma = 10/3$ (top) and $\sigma = 20/3$ (bottom), where the manufacturing specifications $(\text{LSL}, T, \text{USL}) = (10, 40, 50)$ is an asymmetric tolerance.

3 The generalization C''_{pmk}

Pearn and Chen (1998) considered a generalization of C_{pk} for processes with asymmetric tolerances. The generalization takes into account the asymmetry of the tolerance, which reflects the process capability more accurately than the original C_{pk}. Pearn et al. (1999) applied the same idea and proposed a capability index C''_{pmk}, a generalization of C_{pmk}, to handle processes with asymmetric tolerances. The generalization C''_{pmk} is defined as:

$$C''_{pmk} = \frac{d^* - A^*}{3\sqrt{\sigma^2 + A^2}}, \tag{7}$$

where $A = \max\{d(\mu - T)/D_u, d(T - \mu)/D_l\}$, $A^* = \max\{d^*(\mu - T)/D_u, d^*(T - \mu)/D_l\}$, $D_u = \text{USL} - T$, $D_l = T - \text{LSL}$, and $d^* = \min\{D_u, D_l\}$. Obviously, if $T = m$ (symmetric case), then $A = A^* = |\mu - T|$ and C''_{pmk} reduces to the original index C_{pmk}. We note that $C''_{pmk} \ge 0$ for a process with mean μ falling within the tolerance limits, as for the indices C_{pmk}, S_{pmk}, $C_{pa}(0, 4)$ and the yield-based index C_{pk}. However, according to today's modern quality improvement theories, reduction of variation from the target is just as important as meeting the manufacturing specifications. Thus, on-target would be a desired condition for a process. The factors A and A^* ensure that the new generalization C''_{pmk} obtains its maximal value at $\mu = T$ (process is on-target) regardless of whether the tolerances are symmetric $(T = m)$ or asymmetric $(T . m)$. Further, for processes E and F with $\sigma_E = \sigma_F$, $\mu_E < T, \mu_F > T$, satisfying the relationship $(\mu_F - T)/D_u = (T - \mu_E)/D_l$ (i.e., processes E and F have equal

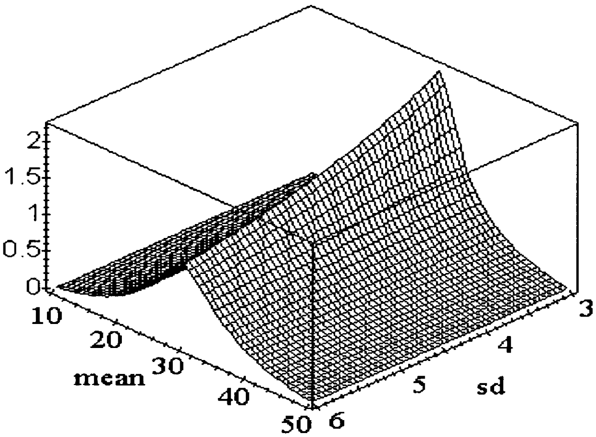

Fig. 5. Surface plot of C''_{pmk} with $10 \le \mu \le 50$ and $3 \le \sigma \le 6$ for $(\mathrm{LSL}, T, \mathrm{USL}) = (10, 30, 50)$.

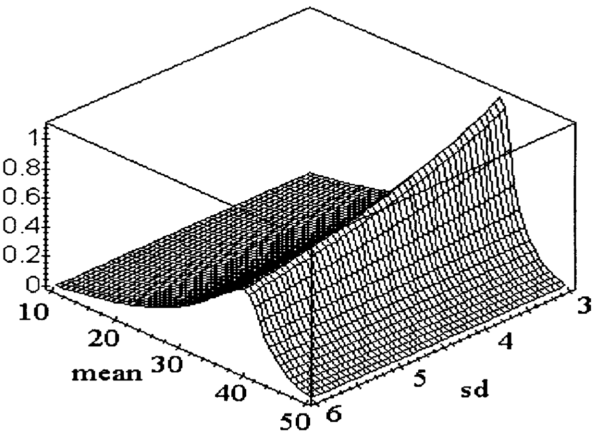

Fig. 6. Surface plot of C''_{pmk} with $10 \le \mu \le 50$ and $3 \le \sigma \le 6$ for $(\mathrm{LSL}, T, \mathrm{USL}) = (10, 40, 50)$.

departure ratio), the index values given to processes E and F are the same. In fact, the value of C''_{pmk} decreases faster when μ moves away from T to the closer specification limit, and decreases slower when μ moves away from T to the farther specification limit. We note that $C_{pa}(1, 3)$ and $C_{pa}(0, 4)$ can also differentiate those changes.

Figure 5 displays the surface plot of C''_{pmk} for $(\mathrm{LSL}, T, \mathrm{USL}) = (10, 30, 50)$, for process means $10 \le \mu \le 50$ and process standard deviations $3 \le \sigma \le 6$, we note that $C''_{pmk} = C_{pmk}$ for the symmetric case. Figure 6 displays the surface plot of C''_{pmk} for $(\mathrm{LSL}, T, \mathrm{USL}) = (10, 40, 50)$, $10 \le \mu \le 50$ and $3 \le \sigma \le 6$. Figure 7 displays the plots of C''_{pmk} for processes with $10 \le \mu \le 50$, $\sigma = 10/3$ (top) and $\sigma = 20/3$ (bottom), where the specification limits $(\mathrm{LSL}, T, \mathrm{USL}) = (10, 40, 50)$ is asymmetric. For processes with asymmetric tolerances, the corresponding loss function is also asymmetric to T. Figure 8 displays a typical

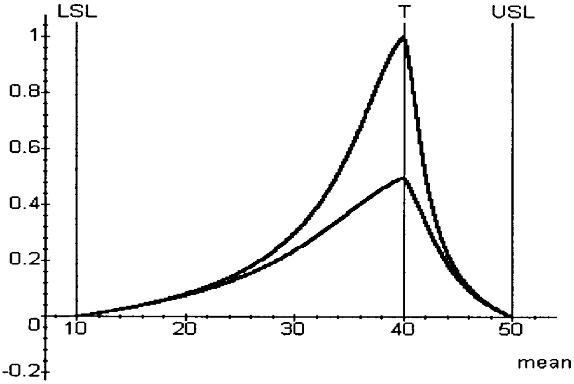

Fig. 7. Plots of C''_{pmk} values for processes with $10 \le \mu \le 50$, $\sigma = 10/3$ (top), $10 \le \mu \le 50$, $\sigma = 20/3$ (bottom), and $(\mathrm{LSL}, T, \mathrm{USL}) = (10, 40, 50)$.

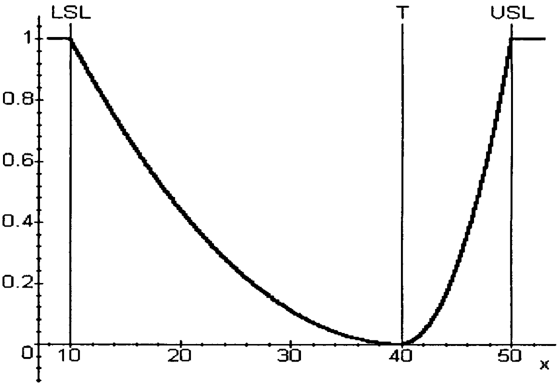

Fig. 8. An asymmetric loss function corresponding to the asymmetric tolerance $(\mathrm{LSL}, T, \mathrm{USL}) = (10, 40, 50)$.

loss function for processes with an asymmetric tolerance $(\mathrm{LSL}, T, \mathrm{USL}) = (10, 40, 50)$, which can be defined as:

$$L(x) = \begin{cases} [(T - x)/(T - \mathrm{LSL})]^2, & \mathrm{LSL} < x \le T, \\ [(x - T)/(\mathrm{USL}) - T)]^2, & T \le x < \mathrm{USL}, \\ 1, & \text{otherwise}. \end{cases} \tag{8}$$

We note that the curve for the loss function corresponds to the asymmetric tolerance. We also note that the curve in Figure 7 for the C''_{pmk} values is smooth (differentiable) uniformly over the specification interval $(10, 50)$. On the other hand, the curves for the values of C_{pmk}, $C_{pa}(1, 3)$, and $C_{pa}(0, 4)$ have reflection points or corner points (which are not differentiable) on the middle point (center) of the specification interval, as can be seen in Figure 1, Figure 3, and Figure 4.

4 Distribution of $\hat{C}''_{pmk}$

To estimate the generalization C''_{pmk}, we consider the natural estimator $\hat{C}''_{pmk}$ defined in the following. The natural estimator $\hat{C}''_{pmk}$ is obtained by replacing the process mean μ and the process variance σ^2 by their conventional estimators $\overline{X}$ and S_n^2, which may be obtained from a stable process.

$$\hat{C}''_{pmk} = \frac{d^* - \hat{A}^*}{3\sqrt{S_n^2 + \hat{A}^2}}, \tag{9}$$

where $d^* = \min\{D_u, D_l\}$, $\hat{A} = \max\{d(\overline{X} - T)/D_u, d(T - \overline{X})/D_l\}$, $\hat{A}^* = \max\{d^*(\overline{X} - T)/D_u, d^*(T - \overline{X})/D_l\}$, $\overline{X} = \sum_{i=1}^{n} X_i/n$, and $S_n^2 = \sum_{i=1}^{n}(X_i - \overline{X})^2/n$. If the manufacturing tolerance is symmetric, then $d^* = d$, $\hat{A} = \hat{A}^* = |\overline{X} - T|$, and the estimator $\hat{C}''_{pmk}$ reduces to $\hat{C}_{pmk} = (d - |\overline{X} - m|)/\{3[S_n^2 + (\overline{X} - T)^2]^{1/2}\}$, the natural estimator of C_{pmk} considered by Pearn et al. (1992) for symmetric case.

We now define $D^* = n^{1/2}(d^*/\sigma)$, $D = n^{1/2}(d/\sigma)$, $K = nS_n^2/\sigma^2$, $Z = n^{1/2}(\overline{X} - T)/\sigma$, $Y = [\max\{(d/D_u)Z, -(d/D_l)Z)\}]^2$, $\delta = n^{1/2}(\mu - T)/\sigma$, and $\lambda = \delta^2$. Then, the estimator $\hat{C}''_{pmk}$ can be rewritten as:

$$\hat{C}''_{pmk} = \frac{D^* - (d^*/d)\sqrt{Y}}{3\sqrt{K + Y}}. \tag{10}$$

Under the assumption of normality, K is distributed as χ^2_{n-1}, a chi-square distribution with $n - 1$ degrees of freedom, and Z is distributed as the normal distribution $N(\delta, 1)$ with mean δ and variance 1. Let $\Phi(\cdot)$ and $\phi(\cdot)$ be the cumulative distribution function and the probability density function of the standard normal distribution $N(0, 1)$, respectively. Then, the cumulative distribution function and the probability density function of Z can be expressed as: $F_Z(z) = \Phi(z - \delta)$ and $f_Z(z) = \phi(z - \delta)$. Hence, the cumulative distribution function of Y can be expressed as:

$$\begin{aligned} F_Y(y) &= p(Y \le y) = p(-d_1^{-1}\sqrt{y} \le Z \le d_2^{-1}\sqrt{y}) \\ &= \Phi(d_2^{-1}\sqrt{y} - \delta) - \Phi(-d_1^{-1}\sqrt{y} - \delta), \end{aligned}$$

where $d_1 = d/D_l$ and $d_2 = d/D_u$. The probability density function of Y can be expressed as:

$$f_Y(y) = \frac{d}{dx}F_Y(y) = \frac{1}{2\sqrt{y}}(d_2^{-1}\phi(d_2^{-1}\sqrt{y} - \delta) + d_1^{-1}\phi(d_1^{-1}\sqrt{y} + \delta)). \tag{11}$$

Therefore, the cumulative distribution function of $\hat{C}''_{pmk}$ can be obtained as the following (see Appendix).

$$F_{\hat{C}''_{pmk}}(x) = \begin{cases} 0, & x \le -\dfrac{u'}{3}, \\ \displaystyle\int_1^{\infty} F_K\left(\frac{(D^* - u'\sqrt{S(x)t})^2}{9x^2} - S(x)t\right) f_Y(S(x)t)S(x)\,dt, & -\dfrac{u'}{3} < x < 0, \\ 1 - \Phi(d_2^{-1}D - \delta) + \Phi(-d_1^{-1}D - \delta), & x = 0, \\ 1 - \displaystyle\int_0^{1} F_K\left(\frac{(D^* - u'\sqrt{S(x)t})^2}{9x^2} - S(x)t\right) f_Y(S(x)t)S(x)\,dt, & x > 0, \end{cases} \tag{12}$$

and the probability density function of $\hat{C}''_{pmk}$ can be derived as:

$$f_{\hat{C}''_{pmk}}(x) = \begin{cases} \displaystyle\int_1^{\infty} f_K\left(\frac{(D^* - u'\sqrt{S(x)t})^2}{9x^2} - S(x)t\right) f_Y(S(x)t) \frac{2S(x)(D^* - u'\sqrt{S(x)t})^2}{-9x^3}\,dt, & -\dfrac{u'}{3} < x < 0 \\ \displaystyle\int_0^{1} f_K\left(\frac{(D^* - u'\sqrt{S(x)t})^2}{9x^2} - S(x)t\right) f_Y(S(x)t) \frac{2S(x)(D^* - u'\sqrt{S(x)t})^2}{9x^3}\,dt, & x > 0 \end{cases} \tag{13}$$

where $D^* = n^{1/2}(d^*/\sigma)$, $u' = d^*/d$, $S(x) = [D^*/(u' + 3x)]^2$, $F_K(\cdot)$ is the cumulative distribution function of K, $f_K(\cdot)$ is the probability density function of K, and $f_Y(\cdot)$ is the probability density function of Y expressed as Eq. (11).

If the manufacturing tolerance is symmetric ($T = m$), then $D_u = D_l = d = d^*$, $D^* = n^{1/2}(d/\sigma) = D$, $u' = 1$, $d_1 = d_2 = 1$, $S(x) = [D/(1+3x)]^2$, and the cumulative distribution function of $\hat{C}''_{pmk}$ in Eq. (12) reduces to:

$$F_{\hat{C}''_{pmk}}(x) = \begin{cases} 0, & x \le -\dfrac{1}{3}, \\ \displaystyle\int_1^{\infty} F_K\left(\frac{(D - \sqrt{S(x)t})^2}{9x^2} - S(x)t\right) f_Y(S(x)t)S(x)\,dt, & -\dfrac{1}{3} < x < 0, \\ 1 - \Phi(D - \delta) + \Phi(-D - \delta), & x = 0, \\ 1 - \displaystyle\int_0^{1} F_K\left(\frac{(D - \sqrt{S(x)t})^2}{9x^2} - S(x)t\right) f_Y(S(x)t)S(x)\,dt, & x > 0, \end{cases} \tag{14}$$

and the corresponding probability density function is:

$$f_{\hat{C}''_{pmk}}(x) = \begin{cases} \displaystyle\int_1^{\infty} f_K\left(\frac{(D-\sqrt{S(x)t})^2}{9x^2} - S(x)t\right) f_Y(S(x)t) \frac{2S(x)(D-\sqrt{S(x)t})^2}{-9x^3} dt, & -\dfrac{1}{3} < x < 0, \\ \displaystyle\int_0^{1} f_K\left(\frac{(D-\sqrt{S(x)t})^2}{9x^2} - S(x)t\right) f_Y(S(x)t) \frac{2S(x)(D-\sqrt{S(x)t})^2}{9x^3} dt, & x > 0, \end{cases} \tag{15}$$

As an illustration for some of the results obtained, we plot the probability density functions of $\hat{C}''_{pmk}$ for an asymmetric case ($D_l : d : D_u = 6 : 5 : 4$) with $b = 3$, $a = -1.0, -0.5, 0.5, 1.0$, and $n = 10, 20, 50$, where $a = (\mu - T)/\sigma$ and $b = d^*/\sigma$.

Figures 9–12 display the plots of the probability density functions of $\hat{C}''_{pmk}$

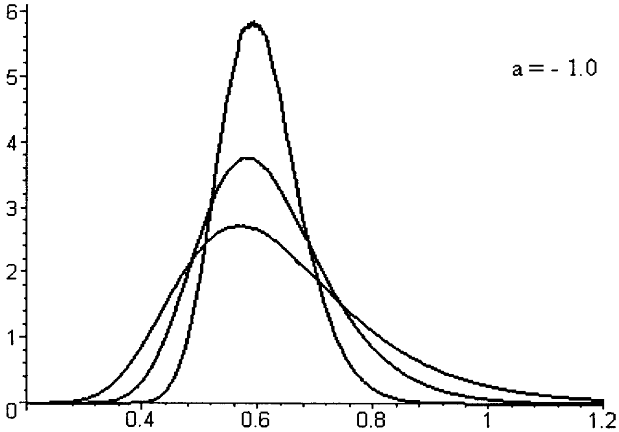

Fig. 9. The pdf of $\hat{C}''_{pmk}$ with $\sigma = d^*/3$, $a = -1.0$, and $n = 10$ (bottom), 20 (middle), and 50 (top) for the asymmetric case $D_l : d : D_u = 6 : 5 : 4$.

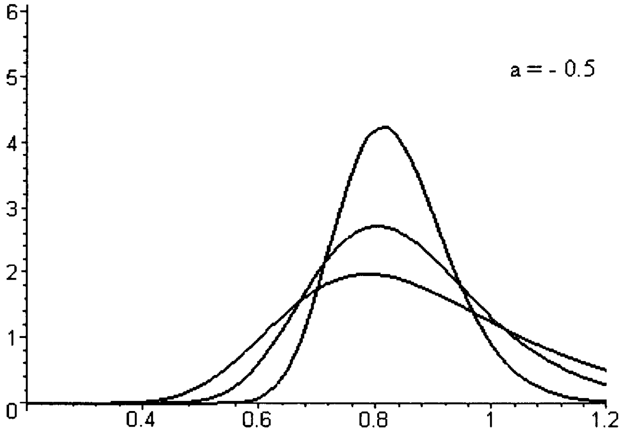

Fig. 10. The pdf of $\hat{C}''_{pmk}$ with $\sigma = d^*/3$, $a = -0.5$, and $n = 10$ (bottom), 20 (middle), and 50 (top) for the asymmetric case $D_l : d : D_u = 6 : 5 : 4$.

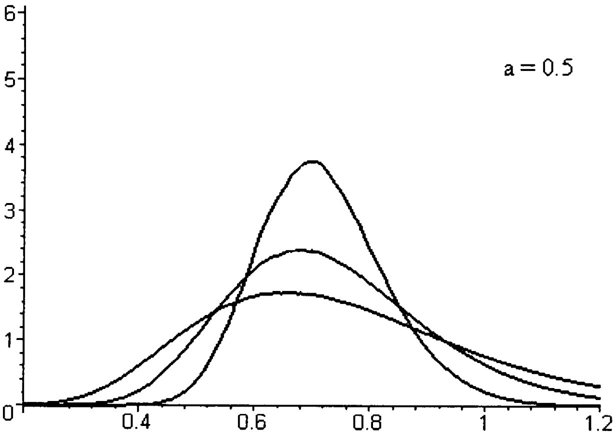

Fig. 11. The pdf of $\hat{C}''_{pmk}$ with $\sigma = d^*/3$, $a = 0.5$, and $n = 10$ (bottom), 20 (middle), and 50 (top) for the asymmetric case $D_l : d : D_u = 6 : 5 : 4$.

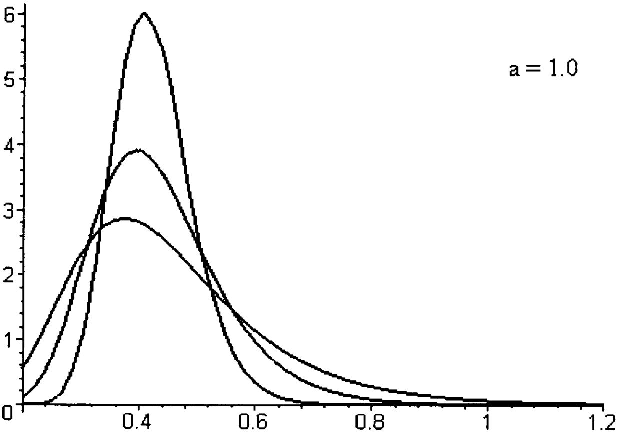

Fig. 12. The pdf of $\hat{C}''_{pmk}$ with $\sigma = d^*/3$, $a = 1.0$, and $n = 10$ (bottom), 20 (middle), and 50 (top) for the asymmetric case $D_l : d : D_u = 6 : 5 : 4$.

for the asymmetric case with $D_l : d : D_u = 6 : 5 : 4$, $\sigma = d^*/3$, $n = 10, 20$, and 50 for $a = -1.0$ ($C''_{pmk} = 0.60$), $a = -0.5$ ($C''_{pmk} = 0.82$), $a = 0.5$ ($C''_{pmk} = 0.71$), and $a = 1.0$ ($C''_{pmk} = 0.42$), respectively. From Figures 9–12 we observe that for $n = 10$ the distributions are skew and have large spread. As n increases the spread decreases and so does the skewness. We also observe that the estimated index $\hat{C}''_{pmk}$ is approximately unbiased for sample size $n > 50$.

Pearn et al. (1999) derived the r-th moment of $\hat{C}''_{pmk}$ without using the distribution of $\hat{C}''_{pmk}$. We note that the estimator $\hat{C}''_{pmk}$ is biased. The magnitude of the bias is $B(\hat{C}''_{pmk}) = E(\hat{C}''_{pmk}) - C''_{pmk}$. The mean square error can be expressed as $\mathrm{MSE}(\hat{C}''_{pmk}) = \mathrm{Var}(\hat{C}''_{pmk}) + B^2(\hat{C}''_{pmk})$, where $\mathrm{Var}(\hat{C}''_{pmk}) = E(\hat{C}''_{pmk})^2 - E^2(\hat{C}''_{pmk})$ is the variance of $\hat{C}''_{pmk}$. To investigate the behavior of the estimator $\hat{C}''_{pmk}$, the bias and the mean square error are calculated (using Maple-V computer soft-

Table 1. The values of C''_{pmk}, $B(\hat{C}''_{pmk})$ and $\mathrm{MSE}(\hat{C}''_{pmk})$ for $b = 3$, $a = -1.0(0.5)1.0$, $d_1 = 5/6$, $d_2 = 5/4$, **and** $n = 10(10)50$

n	$a = -1.0$		$a = -0.5$		$a = 0$		$a = 0.5$		$a = 1.0$	
	bias	MSE	bias	MSE	bias	MSE	bias	MSE	bias	MSE
10	0.0485	0.0361	0.0763	0.0708	0.0079	0.0801	0.0635	0.0797	0.0406	0.0340
20	0.0228	0.0144	0.0378	0.0284	−0.0130	0.0327	0.0318	0.0344	0.0191	0.0138
30	0.0149	0.0090	0.0250	0.0175	−0.0166	0.0206	0.0212	0.0219	0.0125	0.0086
40	0.0111	0.0065	0.0186	0.0126	−0.0173	0.0151	0.0158	0.0160	0.0093	0.0062
50	0.0078	0.0058	0.0148	0.0098	−0.0173	0.0119	0.0126	0.0126	0.0068	0.0052
C''_{pmk}	0.5975		0.8205		1.0000		0.7067		0.4165	

Table 2. The values of C''_{pmk}, $B(\hat{C}''_{pmk})$ and $\mathrm{MSE}(\hat{C}''_{pmk})$ for $b = 4$, $a = -1.0(0.5)1.0$, $d_1 = 5/6$, $d_2 = 5/4$, **and** $n = 10(10)50$

n	$a = -1.0$		$a = -0.5$		$a = 0$		$a = 0.5$		$a = 1.0$	
	bias	MSE	bias	MSE	bias	MSE	bias	MSE	bias	MSE
10	0.0658	0.0618	0.1039	0.1228	0.0341	0.1409	0.0842	0.1330	0.0537	0.0570
20	0.0310	0.0246	0.0509	0.0485	−0.0008	0.0566	0.0415	0.0563	0.0253	0.0230
30	0.0202	0.0153	0.0336	0.0298	−0.0086	0.0353	0.0276	0.0356	0.0165	0.0143
40	0.0150	0.0111	0.0250	0.0214	−0.0114	0.0257	0.0206	0.0260	0.0123	0.0104
50	0.0104	0.0101	0.0199	0.0167	−0.0126	0.0202	0.0164	0.0205	0.0089	0.0089
C''_{pmk}	0.8536		1.1282		1.3333		0.9893		0.6247	

Table 3. The values of C''_{pmk}, $B(\hat{C}''_{pmk})$ and $\mathrm{MSE}(\hat{C}''_{pmk})$ for $b = 5$, $a = -1.0(0.5)1.0$, $d_1 = 5/6$, $d_2 = 5/4$, **and** $n = 10(10)50$

n	$a = -1.0$		$a = -0.5$		$a = 0$		$a = 0.5$		$a = 1.0$	
	bias	MSE	bias	MSE	bias	MSE	bias	MSE	bias	MSE
10	0.0830	0.0947	0.1315	0.1895	0.0604	0.2200	0.1048	0.2003	0.0668	0.0859
20	0.0391	0.0376	0.0640	0.0742	0.0115	0.0875	0.0512	0.0838	0.0315	0.0346
30	0.0256	0.0233	0.0421	0.0455	−0.0006	0.0544	0.0339	0.0529	0.0206	0.0215
40	0.0190	0.0169	0.0314	0.0327	−0.0055	0.0395	0.0253	0.0385	0.0153	0.0156
50	0.0131	0.0157	0.0249	0.0254	−0.0079	0.0310	0.0202	0.0303	0.0109	0.0136
C''_{pmk}	1.1097		1.4359		1.6667		1.2720		0.8329	

ware) for various values of $a = (\mu - T)/\sigma$, $b = d^*/\sigma$, $d_1 = d/D_l$, $d_2 = d/D_u$, and sample size n. Tables 1, 2, and 3 display the values of C''_{pmk}, $B(\hat{C}''_{pmk})$ and $\mathrm{MSE}(\hat{C}''_{pmk})$ for $a = -1.0(0.5)1.0$, $(d_1, d_2) = (5/6, 5/4)$, and $n = 10(10)50$, with $b = 3, 4$, and 5, respectively. We note that the specification with $(d_1, d_2) = (5/6, 5/4)$ is asymmetric.

From Tables 1, 2, and 3, we observe that as the sample size n increases, both the bias and the mean square error of $\hat{C}''_{pmk}$ decrease. Figure 13 displays the plot of the bias of $\hat{C}''_{pmk}$ (vs. n) with $a = 0, 1.0$, and -1.0 (from bottom to top in the plot) for fixed $b = 3$, $d_1 = 5/6$, $d_2 = 5/4$. Figure 14 displays the plot of the

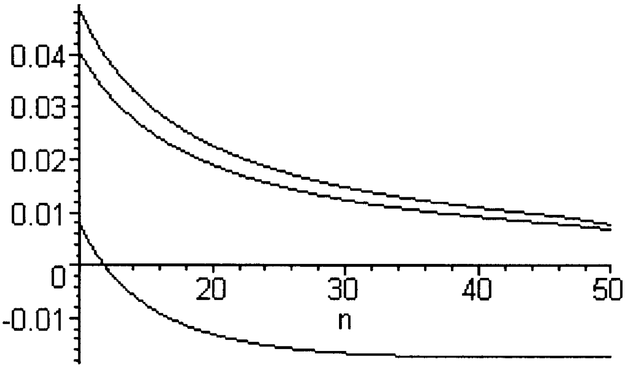

Fig. 13. Bias plot of $\hat{C}''_{pmk}$ (vs. n) for $b = 3$, $d_1 = 5/6$, $d_2 = 5/4$ with $a = 0, 1.0$, and -1.0 (from bottom to top in the plot).

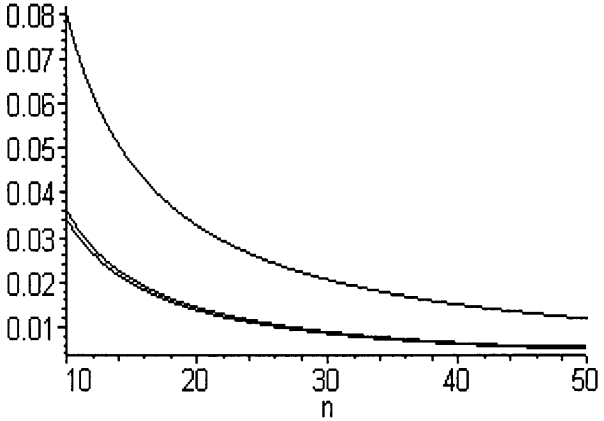

Fig. 14. MSE plot of $\hat{C}''_{pmk}$ (vs. n) for $b = 3$, $d_1 = 5/6$, $d_2 = 5/4$ with $a = 1.0, -1.0$, and 0 (from bottom to top in the plot).

MSE of $\hat{C}''_{pmk}$ (vs. n) with $a = 1.0, -1.0$, and 0 (from bottom to top in the plot) for fixed $b = 3$, $d_1 = 5/6$, $d_2 = 5/4$.

From Tables 1, 2, and 3, we also observe that as the value of b increases, both the bias and the mean square error of $\hat{C}''_{pmk}$ increase for fixed d_1, d_2, a, and n. Figure 15 displays the plot of the bias of $\hat{C}''_{pmk}$ (vs. n) with $b = 3, 4$, and 5 (from bottom to top in the plot) for fixed $a = 0.5$, $d_1 = 5/6$, $d_2 = 5/4$. Figure 16 displays the plot of the MSE of $\hat{C}''_{pmk}$ (vs. n) with $b = 3, 4$, and 5 (from bottom to top in the plot) for fixed $a = 0.5$, $d_1 = 5/6$, $d_2 = 5/4$.

We note that $\hat{C}''_{pmk}$ is a biased estimator. The results in Tables 1, 2, and 3 indicate that the bias of $\hat{C}''_{pmk}$ is positive when $\mu \neq T$. That is, C''_{pmk} is generally overestimated by $\hat{C}''_{pmk}$. On the other hand, when $\mu = T$, we have $A = A^* = 0$ and $C''_{pmk} = d^*/(3\sigma)$, the bias of $\hat{C}''_{pmk}$ tends to be negative for some cases as shown in Tables 1, 2, and 3. Thus, $\hat{C}''_{pmk}$ is smaller than C''_{pmk} and the bias is negative when $\mu = T$. This is partially contributed by the fact

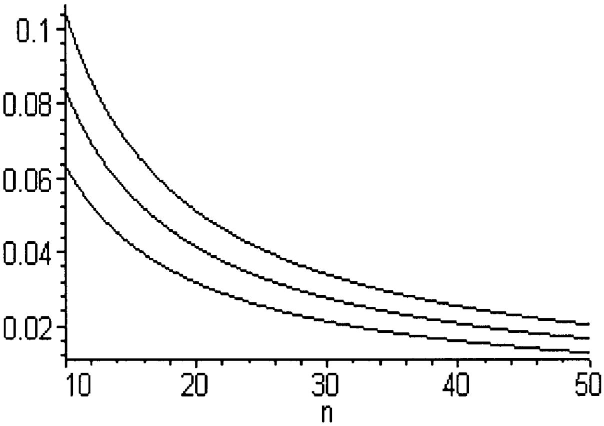

Fig. 15. Bias plot of $\hat{C}''_{pmk}$ (vs. n) for $a = 0.5$, $d_1 = 5/6$, $d_2 = 5/4$ with $b = 3, 4$, and 5 (from bottom to top in the plot).

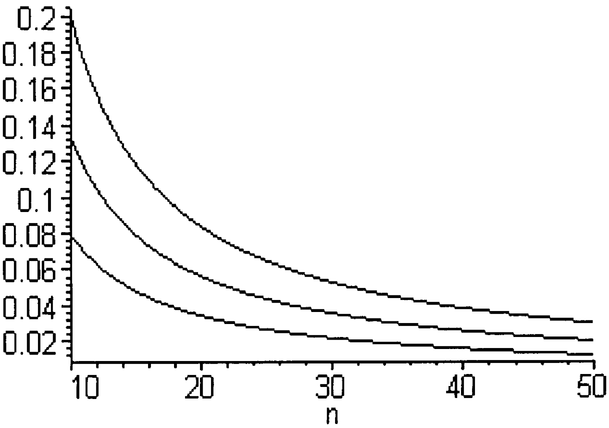

Fig. 16. MSE plot of $\hat{C}''_{pmk}$ (vs. n) for $a = 0.5$, $d_1 = 5/6$, $d_2 = 5/4$ with $b = 3, 4$, and 5 (from bottom to top in the plot).

that both $\hat{A}$ and $\hat{A}^*$ are calculated to be positive (see Eq. (9)) even when $\mu = T$ and $A = A^* = 0$. Clearly, the presence of $\hat{A}$ and $\hat{A}^*$ in Eq. (9) reduces the value of the calculated $\hat{C}''_{pmk}$. As the sample size n increases, the mean square error of $\hat{C}''_{pmk}$ decreases. Proper sample sizes for capability estimation are essential. The smaller the sample size is, the higher the value of $\hat{C}''_{pmk}$ is required to justify the true process capability.

5 Application example

The example presented in the following concerns with the capability of a process which produces electronic telecommunication amplifiers. The original data and a complete description of this process are given by Juran Institute (1990). The quality characteristic of interest is the gain (the boosting ability) of an amplifier. The design of the amplifiers had called for a gain of 10 decibels (dB) and allowed the amplifiers to be considered acceptable if the gain fell between 7.75 dB and 12.25 dB, i.e. $(\mathrm{LSL}, T, \mathrm{USL}) = (7.75, 10, 12.25)$. A sample of the gains of 120 amplifiers was taken by the quality improvement team to estimate the capability of the manufacturing process which produced the amplifiers. Chou et al. (1998) noted that the data follow a non-normal distribution. Chou et al. (1998) also used their best-fit Johnson transformation procedure to transform the non-normal data to normality. The data were then fitted by an S_B distribution. We note that it would be a mistake if someone use the original specification limits, $(\mathrm{LSL}, T, \mathrm{USL}) = (7.75, 10, 12.25)$, to evaluate the quality through the transformed data. Using the estimated transformation (see Chou et al. (1998))

$$Z = 0.96 + 0.98 \ln\left(\frac{X - 7.59}{4.68 + 7.59 - X}\right), \tag{16}$$

we have the transformed specification $(\mathrm{LSL}', T', \mathrm{USL}') = (-2.31, 1.00, 5.06)$ as well as the transformed data.

Table 4 displays the sample of the original gains of 120 amplifiers listed in Juran Institute (1990). Table 5 displays the corresponding transformed amplifier gain data, using the estimated transformation in Eq. (16). We can now apply a normal-based SPC procedure to the transformed data. We note that the

Table 4. The original amplifier gain data

8.1	10.4	8.8	9.7	7.8	9.9	11.7	8.0	9.3	9.0
8.2	8.9	10.1	9.4	9.2	7.9	9.5	10.9	7.8	8.3
9.1	8.4	9.6	11.1	7.9	8.5	8.7	7.8	10.5	8.5
11.5	8.0	7.9	8.3	8.7	10.0	9.4	9.0	9.2	10.7
9.3	9.7	8.7	8.2	8.9	8.6	9.5	9.4	8.8	8.3
8.4	9.1	10.1	7.8	8.1	8.8	8.0	9.2	8.4	7.8
7.9	8.5	9.2	8.7	10.2	7.9	9.8	8.3	9.0	9.6
9.9	10.6	8.6	9.4	8.8	8.2	10.5	9.7	9.1	8.0
8.7	9.8	8.5	8.9	9.1	8.4	8.1	9.5	8.7	9.3
8.1	10.1	9.6	8.3	8.0	9.8	9.0	8.9	8.1	9.7
8.5	8.2	9.0	10.2	9.5	8.3	8.9	9.1	10.3	8.4
8.6	9.2	8.5	9.6	9.0	10.7	8.6	10.0	8.8	8.6

Table 5. The transformed amplifier gain data

−1.1	1.4	−0.1	0.8	−2.0	0.9	2.9	−1.3	0.4	0.1
−0.9	0.0	1.1	0.5	0.3	−1.6	0.6	1.8	−2.0	−0.7
0.2	−0.6	0.7	2.0	−1.6	−0.4	−0.2	−2.0	1.4	−0.4
2.6	−1.3	−1.6	−0.7	−0.2	1.0	0.5	0.1	0.3	1.6
0.4	0.8	−0.2	−0.9	0.0	−0.3	0.6	0.5	−0.1	−0.7
−0.6	0.2	1.1	−2.0	−1.1	−0.1	−1.3	0.3	−0.6	−2.0
−1.6	−0.4	0.3	−0.2	1.2	−1.6	0.9	−0.7	0.1	0.7
0.9	1.5	−0.3	0.5	−0.1	−0.9	1.4	0.8	0.2	−1.3
−0.2	0.9	−0.4	0.0	0.2	−0.6	−1.1	0.6	−0.2	0.4
−1.1	1.1	0.7	−0.7	−1.3	0.9	0.1	0.0	−1.1	0.8
−0.4	−0.9	0.1	1.2	0.6	−0.7	0.0	0.2	1.3	−0.6
−0.3	0.3	−0.4	0.7	0.1	1.6	−0.3	1.0	−0.1	−0.3

transformed specification $(\text{LSL}', T', \text{USL}')$ is asymmetric. Therefore, we apply the proposed generalization C''_{pmk} on the transformed data. We first calculate $d = (\text{USL}' - \text{LSL}')/2 = 3.685$, $d^* = \min\{D_u, D_l\} = \min\{4.06, 3.31\} = 3.31$, $n = 120$, $\bar{Z} = (\sum_{i=1}^{n} Z_i)/n = 0.000713$, $S_n^2 = \sum_{i=1}^{n}(Z_i - \bar{Z})^2/n = 0.977$, $\hat{A} = \max\{d(\bar{Z} - T')/D_u, d(T' - \bar{Z})/D_l\} = \max\{-0.907, 1.112\} = 1.112$, and $\hat{A}^* = \max\{d^*(\bar{Z} - T')/D_u, d^*(T' - \bar{Z})/D_l\} = \max\{-0.815, 0.999\} = 0.999$. We then calculate the estimated capability index $\hat{C}''_{pmk}$ as:

$$\hat{C}''_{pmk} = \frac{d^* - \hat{A}^*}{3\sqrt{S_n^2 + \hat{A}^2}} = 0.52.$$

While all the 120 amplifiers fell within the specification limits, the low value of $\hat{C}''_{pmk}$ shows that the average quality of the amplifiers significantly deviates from the target value, which is unsatisfactory causing the communication failed. The quality improvement team could now concentrate their investigation to find out why the manufacturing line was not capable to produce amplifiers with average quality closer to the target value. Some quality improvement activities involving Taguchi's parameter designs should be initiated to identify the significant factors causing the process failing to cluster around the target value.

6 Conclusions

In this paper, we first reviewed the existing generalizations of C_{pmk}, including S_{pmk} and $C_{pa}(u, v)$ which are proposed by Boyles (1994) and Vännman (1997), respectively, for processes with asymmetric tolerances. We investigated the new generalization C''_{pmk} proposed by Pearn et al. (1999) and compared with S_{pmk} and $C_{pa}(u, v)$. We demonstrated that the new generalization C''_{pmk} is superior to S_{pmk} and $C_{pa}(u, v)$. For processes with normal distributions, we obtained the cumulative distribution function and the probability density function of the estimated index $\hat{C}''_{pmk}$. We showed that the cumulative distribution function and the probability density function of the estimated index $\hat{C}''_{pmk}$ can be expressed in terms of a mixture of the chi-square distribution and the normal distribution, which are considerably more explicit and simpler than those presented in Pearn et al. (1999). We also analyzed the bias and the MSE of the estimated index $\hat{C}''_{pmk}$ for normally distributed processes.

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Appendix

Under the assumption of normality, the cumulative distribution function and the probability density function of $\hat{C}''_{pmk}$ can be derived as follows. We first consider the case with $x > 0$. Using the technique of conditioning $\hat{C}''_{pmk}$ on Y in Eq. (10), we may obtain

$$
\begin{aligned}
F_{\hat{C}''_{pmk}}(x) &= p\left(\frac{D^* - u'\sqrt{Y}}{3\sqrt{K+Y}} \le x\right) \\
&= 1 - p\left(\sqrt{K+Y} < \frac{D^* - u'\sqrt{Y}}{3x}\right) \\
&= 1 - \int_0^\infty p\left(\sqrt{K+Y} < \frac{D^* - u'\sqrt{Y}}{3x} \,|\, Y = y\right) f_Y(y)\, dy \\
&= 1 - \int_0^\infty p\left(\sqrt{K+y} < \frac{D^* - u'\sqrt{y}}{3x}\right) f_Y(y)\, dy,
\end{aligned}
$$

where $u' = d^*/d$. Noting that $D^*/u' = n^{1/2}(d/\sigma) = D$, since

$$p\left(\sqrt{K+y} < \frac{D^* - u'\sqrt{y}}{3x}\right) = 0,$$

for $x > 0$ and $y > D^2$, then

$$F_{\hat{C}''_{pmk}}(x) = 1 - \int_0^{D^2} p\left(\sqrt{K+y} < \frac{D^* - u'\sqrt{y}}{3x}\right) f_Y(y)\, dy. \tag{A1}$$

Further, since

$$p\left(K < \frac{(D^* - u'\sqrt{y})^2}{9x^2} - y\right) = 0,$$

for $[D^*/(u'+3x)]^2 < y \le D^2$, then by rearranging Eq. (A1) we obtain

$$\begin{aligned} F_{\hat{C}''_{pmk}}(x) &= 1 - \int_0^{D^2} p\left(K < \frac{(D^* - u'\sqrt{y})^2}{9x^2} - y\right) f_Y(y)\, dy \\ &= 1 - \int_0^{[D^*/(u'+3x)]^2} p\left(K < \frac{(D^* - u'\sqrt{y})^2}{9x^2} - y\right) f_Y(y)\, dy. \end{aligned} \tag{A2}$$

Changing the variable with $t = y/S(x)$ in the above integral, where $S(x) = [D^*/(u'+3x)]^2$, we have $y = S(x)t$ and $dy = S(x)\,dt$. Hence,

$$F_{\hat{C}''_{pmk}}(x) = 1 - \int_0^1 F_K\left(\frac{(D^* - u'\sqrt{S(x)t})^2}{9x^2} - S(x)t\right) f_Y(S(x)t)S(x)\, dt, \quad \text{for } x > 0. \tag{A3}$$

For the case with $x < 0$, we have

$$\begin{aligned} F_{\hat{C}''_{pmk}}(x) &= p\left(\frac{D^* - u'\sqrt{Y}}{3\sqrt{K+Y}} \le x\right) \\ &= p\left(\sqrt{K+Y} \le \frac{D^* - u'\sqrt{Y}}{3x}\right) \\ &= \int_0^{\infty} p\left(\sqrt{K+Y} \le \frac{D^* - u'\sqrt{Y}}{3x} \,|\, Y = y\right) f_Y(y)\, dy \\ &= \int_0^{\infty} p\left(\sqrt{K+y} \le \frac{D^* - u'\sqrt{y}}{3x}\right) f_Y(y)\, dy. \end{aligned}$$

Since

$$p\left(\sqrt{K+y} \le \frac{D^* - u'\sqrt{y}}{3x}\right) = 0,$$

for $x < 0$ and $y < D^2$, then

$$F_{\hat{C}''_{pmk}}(x) = \int_{D^2}^{\infty} p\left(\sqrt{K+y} \le \frac{D^* - u'\sqrt{y}}{3x}\right) f_Y(y)\, dy. \tag{A4}$$

Further, since

$$p\left(K \le \frac{(D^* - u'\sqrt{y})^2}{9x^2} - y\right) = 0,$$

for $x \le -u'/3$, $y \le D^2$, and for $-u'/3 < x < 0$, $D^2 < y \le [D^*/(u' + 3x)]^2$, then by rearranging Eq. (A4) we obtain

$$\begin{aligned} F_{\hat{C}''_{pmk}}(x) &= \int_{D^2}^{\infty} p\left(K \le \frac{(D^* - u'\sqrt{y})^2}{9x^2} - y\right) f_Y(y)\, dy \\ &= \int_{[D^*/(u'+3x)]^2}^{\infty} F_K\left(\frac{(D^* - u'\sqrt{y})^2}{9x^2} - y\right) f_Y(y)\, dy, \end{aligned} \tag{A5}$$

for $-u'/3 < x < 0$. Changing the variable with $t = y/S(x)$ in the above integral, where $S(x) = [D^*/(u' + 3x)]^2$, we have $y = S(x)t$ and $dy = S(x)\, dt$. Hence,

$$F_{\hat{C}''_{pmk}}(x) = \int_{1}^{\infty} F_K\left(\frac{(D^* - u'\sqrt{S(x)t})^2}{9x^2} - S(x)t\right) f_Y(S(x)t)S(x)\, dt, \tag{A6}$$

for $-u'/3 < x < 0$.

Noting that $D^*/u' = D$, the result for the case with $x = 0$ is trivial. Combining Eqs. (A3) and (A6), we obtain Eq. (12) for the cumulative distribution function of $\hat{C}''_{pmk}$. Taking the derivative of the cumulative distribution function of $\hat{C}''_{pmk}$ with Leibniz's rule, we obtain the probability density function of $\hat{C}''_{pmk}$ in Eq. (13).