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A production inventory model which considers the dependence of production rate on demand and inventory level

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Abstract. This study presents a production inventory model for deteriorating products in which the production rate at any instant depends on the demand and the inventory level. While the demand rate is assumed to decrease exponentially, shortages are allowed and excess demand is backlogged as well. Optimal expressions are obtained for the production scheduling period, maximum inventory level, unfilled order backlog and the total average cost. Some cases of the model are briefly discussed. A numerical example illustrates the practicality of the model. The sensitivity of these solutions to changes in underlying parameter values is also discussed.

1. Introduction

Although many mathematical models have been developed for controlling inventory, a method for formulating production rate policies for controlling deteriorating items has seldom been mentioned. In practice, demand and inventory level may influence production. A situation in which the demand decreases (or increases) may cause the manufacturers to decrease (or increase) their production as well. Also, the production rate may either increase or decrease with the inventory level. Thus, the effect of inventory on production rate warrants further study. In this study, we present a realistic inventory model in which the production rate depends on demand and inventory level, where demand is an exponential decreasing function of time. Such a situation frequently arises in practice. For instance, many items, e.g. electronic components, fashionable clothes and domestic goods have the rapid sales increase in the beginning; however, they may drastically decline due to either the introduction of new competitive products or changes in consumers' preferences.

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The standard EOQ model assumes a constant and known demand rate over an infinite planning horizon. Mak (1982) proposed a production lot size inventory model with a uniform demand rate over a fixed time horizon. However, most items experience a stable demand only during the saturation phase of their life cycle and for finite periods of time. Thus, the demand rate varies with time; the EOQ model must obviously be modified. Many studies have extended the EOQ model in order to accommodate time-varying demand patterns. Goswami and Chaudhuri (1991, 1992), Bhunia and Maiti (1997) as well as Bose *et al.* (1995) assumed a linear trend in demand. Hong *et al.* (1993) considered an inventory model with time-proportional demand, instantaneous replenishment and no shortage. Dave (1989) extended this work by including variable instantaneous demand, discrete opportunities for replenishment and shortages. In addition, Mandal and Phaujdar (1989), and Urban (1992) discussed an inventory level-dependent demand rate. The above investigations did not consider the effect of inventory on production rate.

Hollier and Mak (1983) developed inventory replenishment policies for deteriorating items with a declining demand. Balkhi and Benkherouf (1996) considered a production lot size inventory model with arbitrary production and demand rate which depends on the time function. Goswami and Chaudhuri (1992) developed order-level inventory models for deteriorating items in which the finite production rate is proportional to the time-dependent demand rate. Furthermore, Bhunia and Maiti (1997) assumed that the production rate is a variable. They also presented inventory models in which the production rate depends on either on-hand inventory or demand. These investigations did not allow for shortages. In this study, we further extend all of the above models to formulate an inventory model for deteriorating items by simultaneously considering the demand rate and the production rate.

This study attempts to develop a production inventory model by assuming that the deterioration rate is uniform, the finite production rate is proportional to both the demand rate and the inventory level, shortages are allowed, and the demand rate decreases exponentially. The total average cost is derived. The optimal expressions can be obtained for the production scheduling period, maximum inventory level, unfilled order backlog and total average cost. In addition, some cases are illustrated by selecting appropriate values for the model's various parameters. A numerical example demonstrates the practicality of the model. Finally, sensitivity analysis is performed along with concluding remarks provided as well.

2. Assumptions and notations

The mathematical model of the production inventory problem considered in this paper is developed on the basis of the following assumptions.

(1) A single item is considered over a prescribed period of T units of time, which is subject to a constant deterioration rate.
(2) Demand rate, $D(t)$, is known and decreases exponentially. That is at time $t, t \geq 0$, $D(t) = A e^{-\lambda t}$, A is initial demand rate and λ is decreasing rate of the demand, $0 \leq \lambda \leq 1$.
(3) Production rate, $P(t)$, at any instant depends on both the demand and the inventory level. That is at time $t, t \geq 0$, $P(t) = a + bD(t) - cI(t)$, $a > 0$, $0 \leq b < 1$ and $0 \leq c < 1$,
(4) Deterioration of the units is considered only after they have been received into inventory.
(5) There is no replacement or repair of deteriorated items during a given cycle.
(6) Shortages are allowed and backlogged.

The following notations are used:

$I(t)$ inventory level at any time $t, t \geq 0$,
θ parameter of the deterioration rate function,
I_m maximum inventory level,
I_b unfilled order backlog,
C setup cost for each new cycle,
C_d the cost of a deteriorated item,
C_i inventory carrying cost per unit per month,
C_s shortage cost per unit,
T cycle time,
K the total average cost of system.

3. Mathematical modelling and analysis

Initially the stock is zero. The production inventory level starts at a time $t = 0$ and reaches I_m maximum level after t_1 time units have elapsed. Then the production is stopped, the stock level declines continuously and the inventory level becomes zero at time $t = t_2$. Now shortages start developing and accumulate to the level I_b at time $t = t_3$. At this instant of time, fresh production starts to clear the backlog by the time $t = t_4 = T$. Our purpose is to find out the optimal values of t_1, t_2, t_3, t_4, I_m and I_b that minimize K over the time horizon $[0, T]$.

The differential equations governing the stock status during the period $0 \leq t \leq T$ can be written as:

$$\frac{\mathrm{d}I(t)}{\mathrm{d}t} = a + (b - 1)A e^{-\lambda t} - (c + \theta)I(t), \quad 0 \leq t \leq t_1, \quad (1)$$

$$\frac{\mathrm{d}I(t)}{\mathrm{d}t} = -A\mathrm{e}^{-\lambda t} - \theta I(t), \quad t_1 \le t \le t_2, \qquad (2)$$

$$\frac{\mathrm{d}I(t)}{\mathrm{d}t} = -A\mathrm{e}^{-\lambda t}, \quad t_2 \le t \le t_3, \qquad (3)$$

and

$$\frac{\mathrm{d}I(t)}{\mathrm{d}t} = a + (b-1)A\mathrm{e}^{-\lambda t} - cI(t), \quad t_3 \le t \le t_4. \qquad (4)$$

Using the various boundary conditions, i.e.

$$I(t) = 0 \text{ at } t = 0, t_2 \quad \text{and } T, \qquad (5)$$

$$I(t_1) = I_{\mathrm{m}} \text{ and } -I(t_3) = I_{\mathrm{b}}, \qquad (6)$$

and after adjusting for the constant of integration, equations (1)–(4) are clearly equivalent to the following equations

$$I(t) = \frac{a}{c+\theta}\left[1 - \mathrm{e}^{-(c+\theta)t}\right] + \frac{A(1-b)}{\lambda-c-\theta}\left[\mathrm{e}^{-\lambda t} - \mathrm{e}^{-(c+\theta)t}\right],$$
$$0 \le t \le t_1, \quad (7)$$

$$I(t) = \frac{A\mathrm{e}^{-\lambda t}}{\lambda-\theta}\left[1 - \mathrm{e}^{-(\lambda-\theta)(t_2-t)}\right], \quad t_1 \le t \le t_2, \qquad (8)$$

$$I(t) = \frac{A}{\lambda}\left(\mathrm{e}^{-\lambda t} - 1\right), \quad t_2 \le t \le t_3, \qquad (9)$$

and

$$I(t) = -\frac{a}{c}\left[\mathrm{e}^{c(t_4-t)} - 1\right] - \frac{A(1-b)}{\lambda-c}\mathrm{e}^{-\lambda t}\left[\mathrm{e}^{-(\lambda-c)(t_4-t)} - 1\right],$$
$$t_3 \le t \le t_4. \quad (10)$$

From equations (5) and (6), we derive

$$I_{\mathrm{m}} = \frac{a}{c+\theta}\left[1 - \mathrm{e}^{-(c+\theta)t_1}\right] + \frac{A(1-b)}{\lambda-c-\theta}\left[\mathrm{e}^{-\lambda t_1} - \mathrm{e}^{-(c+\theta)t_1}\right]$$
$$= \frac{A}{\lambda-\theta}\left[1 - \mathrm{e}^{-(\lambda-\theta)t_2}\right], \qquad (11)$$

and

$$I_{\mathrm{b}} = \frac{A}{\lambda}\left(1 - \mathrm{e}^{-\lambda t_3}\right) = \frac{a}{c}(\mathrm{e}^{ct_4} - 1) + \frac{A(1-b)}{\lambda-c}\left[\mathrm{e}^{-(\lambda-c)t_4} - 1\right]. \qquad (12)$$

Thus, t_1 and t_2 are related by the equation

$$t_2 = R(t_1) = \frac{1}{\theta-\lambda}\ln\left[1 - \frac{a(\lambda-\theta)}{A(c+\theta)}\left[1 - \mathrm{e}^{-(c+\theta)t_1}\right]\right.$$
$$\left. -\frac{(1-b)(\lambda-\theta)}{\lambda-c-\theta}\left[\mathrm{e}^{-\lambda t_1} - \mathrm{e}^{-(c+\theta)t_1}\right]\right]. \qquad (13)$$

Again t_3 and t_4 are related by the equation

$$t_3 = R(t_4) = -\frac{1}{\lambda}\ln\left[1 - \frac{a\lambda}{Ac}\left(\mathrm{e}^{ct_4} - 1\right)\right.$$
$$\left. -\frac{(1-b)\lambda}{\lambda-c}\left[\mathrm{e}^{-(\lambda-c)t_4} - 1\right]\right]. \qquad (14)$$

The deterioration cost for the period $(0, T)$ is

$$C_{\mathrm{d}}\left(\int_0^{t_1}\theta I(t)\,\mathrm{d}t + \int_{t_1}^{t_2}\theta I(t)\,\mathrm{d}t\right)$$
$$= C_{\mathrm{d}}\left\{\int_0^{t_1}\left[\frac{a\theta}{c+\theta}\left[1 - \mathrm{e}^{-(c+\theta)t}\right]\right.\right.$$
$$\left. + \frac{A\theta(1-b)}{\lambda-c-\theta}\left[\mathrm{e}^{-\lambda t} - \mathrm{e}^{-(c+\theta)t}\right]\right]\mathrm{d}t$$
$$\left. + \int_{t_1}^{t_2}\frac{A\theta}{\lambda-\theta}\left[\mathrm{e}^{-\lambda t} - \mathrm{e}^{-(\lambda-\theta)t_2}\mathrm{e}^{-\theta t}\right]\mathrm{d}t\right\}, \qquad (15)$$

The inventory carrying cost over the period $(0, T)$ is

$$C_{\mathrm{i}}\left[\int_0^{t_1}I(t)\,\mathrm{d}t + \int_{t_1}^{t_2}I(t)\,\mathrm{d}t\right]$$
$$= C_{\mathrm{i}}\int_0^{t_1}\left[\frac{a}{c+\theta}\left(1 - \mathrm{e}^{-(c+\theta)t}\right)\right.$$
$$\left. + \frac{A(1-b)}{\lambda-c-\theta}\left(\mathrm{e}^{-\lambda t} - \mathrm{e}^{-(c+\theta)t}\right)\right]\mathrm{d}t$$
$$+ C_{\mathrm{i}}\int_{t_1}^{t_2}\frac{A}{\lambda-\theta}\mathrm{e}^{-\lambda t}\left[1 - \mathrm{e}^{-(\lambda-\theta)(t_2-t)}\right]\mathrm{d}t, \qquad (16)$$

and the shortage cost can be written as

$$C_{\mathrm{s}}\left[-\int_{t_2}^{t_3}I(t)\,\mathrm{d}t - \int_{t_3}^{t_4}I(t)\,\mathrm{d}t\right]$$
$$= C_{\mathrm{s}}\left\{\int_{t_2}^{t_3}\frac{A}{\lambda}\left(1 - \mathrm{e}^{-\lambda t}\right)\mathrm{d}t + \int_{t_3}^{t_4}\left[\frac{a}{c}\left(\mathrm{e}^{c(t_4-t)} - 1\right)\right.\right.$$
$$\left.\left. + \frac{A(1-b)\mathrm{e}^{-\lambda t}}{\lambda-c}\left(\mathrm{e}^{-(\lambda-c)(t_4-t)} - 1\right)\right]\mathrm{d}t\right\}. \qquad (17)$$

Hence, the total average cost of the inventory system is

$$K = \text{setup cost} + \text{deterioration cost}$$
$$+ \text{inventory carrying cost} + \text{shortage cost}$$
$$= \frac{C}{T} + \frac{(\theta C_{\mathrm{d}} + C_{\mathrm{i}})}{T}\left\{\frac{a}{c+\theta}\left[t_1 + \frac{1}{c+\theta}\left(\mathrm{e}^{-(c+\theta)t_1} - 1\right)\right]\right.$$
$$\left. + \frac{A(1-b)}{\lambda-c-\theta}\left[\frac{1}{c+\theta}\left(\mathrm{e}^{-(c+\theta)t_1} - 1\right)\right.\right.$$

$$-\frac{1}{\lambda}\left(e^{-\lambda t_1}-1\right)\Big] + \frac{A}{\lambda-\theta}$$

$$\times\left[\frac{1}{\lambda}\left(1-e^{-\lambda t_2}\right)+\frac{1}{\theta}\left[e^{-\lambda t_2}-e^{-(\lambda-\theta)t_2}\right]\right]\Big\}$$

$$+\frac{C_s}{T}\left\{\frac{At_3}{\lambda}+\frac{A}{\lambda^2}\left(e^{-\lambda t_3}-1\right)+\frac{a}{c}\left[\frac{1}{c}\left(e^{ct_4}-1\right)-t_4\right]\right.$$

$$\left.+\frac{A(1-b)}{\lambda-c}\left[\frac{1}{\lambda}(e^{-\lambda t_4}-1)-\frac{1}{c}(e^{-\lambda t_4}-e^{-(\lambda-c)t_4})\right]\right\}.$$

$$(18)$$

4. The solution procedure

Differentiating the total average cost K given by equation (18) with respect to t_1 and t_4 and then equating to zero we find the following two equations.

$$-\frac{C}{T^2}\left[1+R'(t_1)\right]+\frac{(\theta C_d+C_i)}{T}\left\{\frac{a}{c+\theta}\left(1-e^{-(c+\theta)t_1}\right)\right.$$

$$+\frac{A(1-b)}{\lambda-c-\theta}\left[e^{-\lambda t_1}-e^{-(c+\theta)t_1}\right]$$

$$+\frac{AR'(t_1)}{\lambda-\theta}\left[e^{-\lambda R(t_1)}+\frac{\lambda-\theta}{\theta}e^{-(\lambda-\theta)R(t_1)}-\frac{\lambda}{\theta}e^{-\lambda R(t_1)}\right]\right\}$$

$$-\frac{(\theta C_d+C_i)}{T^2}\left\{\frac{a}{c+\theta}\left[t_1+\frac{1}{c+\theta}\left(e^{-(c+\theta)t_1}-1\right)\right]\right.$$

$$+\frac{A(1-b)}{\lambda-c-\theta}\left[\frac{1}{c+\theta}\left(e^{-(c+\theta)t_1}-1\right)\right.$$

$$\left.-\frac{1}{\lambda}\left(e^{-\lambda t_1}-1\right)\right]+\frac{A}{\lambda-\theta}\left[\frac{1}{\lambda}\left(1-e^{-\lambda t_2}\right)\right.$$

$$\left.\left.+\frac{1}{\theta}\left[e^{-\lambda t_2}-e^{-(\lambda-\theta)t_2}\right]\right]\right\}[1+R'(t_1)]$$

$$-\frac{C_s}{2T^2}\left\{\frac{At_3}{\lambda}+\frac{A}{\lambda^2}\left(e^{-\lambda t_3}-1\right)+\frac{a}{c}\left[\frac{1}{c}(e^{ct_4}-1)-t_4\right]\right.$$

$$\left.+\frac{A(1-b)}{\lambda-c}\left[\frac{1}{\lambda}\left(e^{-\lambda t_4}-1\right)-\frac{1}{c}\left(e^{-\lambda t_4}-e^{-(\lambda-c)t_4}\right)\right]\right\}$$

$$\times[1+R'(t_1)]=0,\qquad(19)$$

and

$$-\frac{C}{T^2}\left[1+R'(t_4)\right]-\frac{(\theta C_d+C_i)}{T^2}$$

$$\times\left\{\frac{a}{c+\theta}\left[t_1+\frac{1}{c+\theta}\left(e^{-(c+\theta)t_1}-1\right)\right]\right.$$

$$\left.+\frac{A(1-b)}{\lambda-c-\theta}\left[\frac{1}{c+\theta}\left(e^{-(c+\theta)t_1}-1\right)-\frac{1}{\lambda}\left(e^{-\lambda t_1}-1\right)\right]\right.$$

$$+\frac{A}{\lambda-\theta}\left[\frac{1}{\lambda}\left(1-e^{-\lambda t_2}\right)\right.$$

$$\left.\left.+\frac{1}{\theta}\left[e^{-\lambda t_2}-e^{-(\lambda-\theta)t_2}\right]\right]\right\}[1+R'(t_4)]$$

$$+\frac{C_s}{T}\left\{\frac{A}{\lambda}R'(t_4)(1-e^{-\lambda R(t_4)})+\frac{a}{c}(e^{ct_4}-1)\right.$$

$$\left.+A(1-b)\left[\frac{e^{-\lambda t_4}}{c}+e^{-(\lambda-c)t_4}\right]\right\}$$

$$-\frac{C_s}{2T^2}\left\{\frac{At_3}{\lambda}+\frac{A}{\lambda^2}\left(e^{-\lambda t_3}-1\right)+\frac{a}{c}\left[\frac{1}{c}\left(e^{ct_4}-1\right)-t_4\right]\right.$$

$$\left.+\frac{A(1-b)}{\lambda-c}\left[\frac{1}{\lambda}\left(e^{-\lambda t_4}-1\right)-\frac{1}{c}\left(e^{-\lambda t_4}-e^{-(\lambda-c)t_4}\right)\right]\right\}$$

$$\times[1+R'(t_4)]=0,\qquad(20)$$

the detail of $R(t_j)$ and $R'(t_j)$ are given in the Appendix.

The pair (t_1,t_4) will be an optimal solution provided the following sufficient conditions are satisfied

$$\frac{\partial^2 K}{\partial t_1^2}>0,\quad\frac{\partial^2 K}{\partial t_4^2}>0\text{ and }\frac{\partial^2 K}{\partial t_1^2}\times\frac{\partial^2 K}{\partial t_4^2}-\left(\frac{\partial^2 K}{\partial t_1\partial t_4}\right)^2>0.$$

We obtain two simultaneous non-linear equations in optimal values of t_1 and t_4 by the Newton-Raphson (NR) method using a computer. The optimal values of t_2, t_3, I_m, I_b and the minimum total average cost K can be obtained from equations (11)–(14) and (18), respectively.

5. Case illustrations

In this section, we study some important cases that follow from the problem considered in the previous sections.

5.1. *Case A. Production rate depends on demand*

$$P(t)=a+bD(t),(b\neq 0,c=0)$$

Arguing as in equation (18) above, the basic conditions for this model becomes

$$\lim_{c\to 0}K=\frac{C}{T}+\frac{(\theta C_d+C_i)}{T}\left\{\frac{a}{\theta}\left[t_1+\frac{1}{\theta}\left(e^{-\theta t_1}-1\right)\right]\right.$$

$$\left.+\frac{A(1-b)}{\lambda-\theta}\left[\frac{1}{\theta}\left(e^{-\theta t_1}-1\right)-\frac{1}{\lambda}\left(e^{-\lambda t_1}-1\right)\right]\right.$$

$$+ \frac{A}{\lambda - \theta}\left[\frac{1}{\lambda}(1 - e^{-\lambda t_2}) + \frac{1}{\theta}\left[e^{-\lambda t_2} - e^{-(\lambda-\theta)t_2}\right]\right]\Bigg\}$$

$$+ \frac{C_s A}{\lambda T}\left[t_3 + \frac{1}{\lambda}(e^{-\lambda t_3} - 1)\right]$$

$$+ \frac{a C_s t_4^2}{2T} + \frac{AC_s(1-b)}{\lambda T}\left[\frac{1}{\lambda}\left(e^{-\lambda t_4} - 1\right) + t_4 e^{-\lambda t_4}\right]. \tag{21}$$

5.2. *Case B. Production rate depends on inventory*

$$P(t) = a - cI(t), \quad (b = 0, c \neq 0)$$

In this case, equation (18) reduces to

$$K = \frac{C}{T} + \frac{(\theta C_d + C_i)}{T}\Bigg\{\frac{a}{c+\theta}\left[t_1 + \frac{1}{c+\theta}(e^{-(c+\theta)t_1} - 1)\right]$$

$$+ \frac{A}{\lambda - c - \theta}\left[\frac{1}{c+\theta}(e^{-(c+\theta)t_1} - 1)\right.$$

$$\left. - \frac{1}{\lambda}(e^{-\lambda t_1} - 1)\right] + \frac{A}{\lambda - \theta}\left[\frac{1}{\lambda}(1 - e^{-\lambda t_2})\right.$$

$$\left. + \frac{1}{\theta}\left[e^{-\lambda t_2} - e^{-(\lambda-\theta)t_2}\right]\right]\Bigg\}$$

$$+ \frac{C_s}{T}\Bigg\{\frac{A t_3}{\lambda} + \frac{A}{\lambda^2}(e^{-\lambda t_3} - 1) + \frac{a}{c}\left[\frac{1}{c}(e^{ct_4} - 1) - t_4\right]$$

$$+ \frac{A}{\lambda - c}\left[\frac{1}{\lambda}\left(e^{-\lambda t_4} - 1\right) - \frac{1}{c}\left(e^{-\lambda t_4} - e^{-(\lambda-c)t_4}\right)\right]\Bigg\}. \tag{22}$$

5.3. *Case C. Production rate is uniform*

$$P(t) = a, \quad (b = 0, c = 0)$$

Substituting the expression of this type of demand into equation (18), we can obtain

$$\lim_{b,c\to0} K = \frac{C}{T} + \frac{(\theta C_d + C_i)}{T}\Bigg\{\frac{a}{\theta}\left[t_1 + \frac{1}{\theta}(e^{-\theta t_1} - 1)\right]$$

$$+ \frac{A}{\lambda - \theta}\left[\frac{1}{\theta}(e^{-\theta t_1} - 1)\frac{1}{\lambda}(e^{-\lambda t_1} - 1)\right]$$

$$+ \frac{A}{\lambda - \theta}\left[\frac{1}{\lambda}(1 - e^{-\lambda t_2}) + \frac{1}{\theta}\left[e^{-\lambda t_2} - e^{-(\lambda-\theta)t_2}\right]\right]\Bigg\}$$

$$+ \frac{C_s A}{\lambda T}\left[t_3 + \frac{1}{\lambda}(e^{-\lambda t_3} - 1)\right]$$

$$+ \frac{a C_s t_4^2}{2T} + \frac{AC_s}{\lambda T}\left[\frac{1}{\lambda}(e^{-\lambda t_4} - 1) + t_4 e^{-\lambda t_4}\right], \tag{23}$$

where the decreasing rate of the demand $\lambda \to 0$ and $e^{\theta t} \cong 1 + \theta t + \theta^2 t^2/2$, we then obtain the same model as that given by Mak (1982). Equation (23) for the total average cost of the system is:

$$K = \frac{C}{T} + \frac{(\theta C_d + C_i)}{T}\left[\frac{(a-A)t_1^2}{2} + \frac{At_2^2}{2}\right]$$

$$+ \frac{AC_s t_3^2}{2T} + \frac{(a-A)C_s t_4^2}{2T}, \tag{24}$$

where

$$t_2 = \frac{1}{\theta}\ln\left[1 + \theta\left(\frac{a}{A} - 1\right)\left(t_1 - \frac{\theta t_1^2}{2}\right)\right],$$

and

$$t_3 = t_4\left(\frac{a}{A} - 1\right).$$

Next, if we assume $\theta = 0$, the model changes to an EOQ model. In this situation, the cost function becomes

$$K = \frac{C}{T} + \frac{C_i}{T}\left[\frac{(a-A)t_1^2}{2} + \frac{At_2^2}{2}\right] + \frac{C_s}{T}\left[\frac{At_3^2}{2} + \frac{(a-A)t_4^2}{2}\right],$$

$$= \frac{C}{T} + \frac{C_i I_m^2}{2T} \times \frac{a}{A(a-A)} + \frac{C_s I_b^2}{2T} \times \frac{a}{A(a-A)}. \tag{25}$$

where

$$t_2 = \left(\frac{a}{A} - 1\right)t_1, \quad \text{and} \quad t_3 = t_4\left(\frac{a}{A} - 1\right).$$

From equations (11) and (12), we derive

$$I_m = (a - A)t_1, \quad \text{and} \quad I_b = (a - A)t_4.$$

6. Numerical example and sensitivity analysis

A numerical example is presented in the following to illustrate the practicality of the proposed model. Assume that an inventory system's demand rate decreases negative exponentially in accordance with the equation

$$D(t) = 200e^{-\lambda t},$$

and the values of various variables are $a = 200$ units/month, $C_d = 3$/unit, $b = 0.2$, $c = 0.2$, $C_s = 10$/unit, $C = 100$ for each new cycle, $C_i = 1$/unit/month, $\lambda = 0.3$, $\theta = 0.05$. The optimum values are calculated for the model and the values are compared using different parameters b and c, as shown in table 1. For our model, the optimal production scheduling period is 0.930 month and the total average cost is \$123.938, thereby dominating the other three situations.

Now, consider a situation in which only one of the parameter values changes by a fixed proportion; the other parameters remain unchanged. The following sen-

Table 1. An illustrative example.

Cases	t_1	t_2	t_3	t_4	I_m	I_b	K
$b \neq 0, c \neq 0$	0.930	1.192	1.215	1.321	50.729	4.556	123.938
$b \neq 0, c = 0$	0.967	1.267	1.278	1.333	57.848	2.272	133.312
$b = 0, c \neq 0$	3.208	4.198	4.280	5.006	175.430	16.216	164.431
$b = 0, c = 0$	3.208	4.473	4.574	5.273	216.926	14.724	368.157

Table 2. Effects of parameters on the production inventory system.

Changing parameters			t_1	t_2	t_3	t_4	I_m	I_b	K
The proposed model	θ	0.045	0.927	1.189	1.211	1.315	50.630	4.464	123.845
$b \neq 0, c \neq 0$		0.050	0.930	1.192	1.215	1.321	50.729	4.556	123.938
		0.055	0.933	1.195	1.218	1.325	50.826	4.602	124.106
	λ	0.27	0.929	1.181	1.192	1.244	49.056	2.149	127.992
		0.30	0.930	1.192	1.215	1.321	50.729	4.556	123.938
		0.33	0.932	1.204	1.239	1.395	52.447	6.983	121.764
	b	0.18	0.854	1.075	1.100	1.225	43.049	4.943	122.421
		0.20	0.930	1.192	1.215	1.321	50.729	4.556	123.938
		0.22	0.991	1.291	1.312	1.400	57.826	4.088	126.951
	c	0.18	0.933	1.198	1.220	1.321	51.336	4.322	124.421
		0.20	0.930	1.192	1.215	1.321	50.729	4.556	123.938
		0.22	0.927	1.186	1.210	1.320	50.131	4.746	123.544

sitivity measures are then calculated for 10% changes in the parameters on either side. Table 2 summarizes these results. Based on the sensitivity analysis, we can infer the following.

(1) An increase in parameters b, θ, λ and a decrease in parameter c cause increases in the optimal production scheduling period t_1 and the maximum inventory level I_m.
(2) An increase in parameters c, θ, λ and a decrease in parameter b cause an increase in the unfilled order backlog I_b.
(3) An increase in parameters b, θ and a decrease in parameters c, λ cause an increase in the total average inventory cost K.
(4) The total average inventory cost K increases with an increase in parameters b, θ and a decrease in parameters c, λ. Of course, the change in K is small even for a significant change in the value of λ.
(5) The optimal production scheduling period t_1 is more sensitive to the parameter b than others.

It is the intention of this result to provide managers with simple guidelines for determining the production scheduling period. Furthermore, because different parameters are involved in this problem, the decision-maker is faced with the difficult task of accurately estimating these parameters. The above study is robust in

the sense that the decision-maker is provided with a range of parameter values, instead of specific ones, for which one mode of production is preferred.

7. Conclusion

Two aspects of deriving inventory models have received increasing interest: the demand rate and the production rate. In this study, we present a production inventory model for deteriorating items by considering that the production rate depends on demand and inventory level. The demand rate is assumed to decrease exponentially. The developed model is solved by using the Newton–Raphson method. Results presented herein provide a valuable reference for decision-makers in planning production and controlling inventory. A numerical example demonstrates that applying the proposed model can minimize the total average inventory cost and result in a relatively small production scheduling period. In addition, sensitivity analysis is performed to examine the effect of the parameters. According to those results, the proposed model is more sensitive with respect to the parameter b than others. The analysis provided in this study can be very useful for managers in deciding whether they should have a continuous production policy. A future study should incorporate more realistic assumptions into the proposed model, e.g. variable deterioration rate and stochastic nature of demand.

Acknowledgements

Appendix

Taking the derivative of equations (13) and (14), we obtain

$$R'(t_1) = \frac{(\beta - \alpha)(c + \theta)e^{-(c+\theta)t_1} + \lambda\beta e^{-\lambda t_1}}{(\theta - \lambda)\{1 - \alpha[1 - e^{-(c+\theta)t_1}] - \beta[e^{-\lambda t_1} - e^{-(c+\theta)t_1}]\}},$$

(A1)

where

$$\alpha = \frac{a(\lambda - c)}{A + c + \theta}, \text{ and } \beta = \frac{(1 - b)(\lambda - \theta)}{\lambda - c - \theta}.$$

and

$$R'(t_4) = \frac{mce^{ct_4} - n(\lambda - c)e^{-(\lambda-c)t_4}}{\lambda\{1 - m(e^{ct_4} - 1) - n[e^{-(\lambda-c)t_4} - 1]\}}.$$

(A2)

where

$$m = \frac{a\lambda}{Ac}, \quad \text{and} \quad n = \frac{(1 - b)\lambda}{\lambda - c}.$$

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