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COMPLETE STABILITY FOR A CLASS OF CELLULAR NEURAL NETWORKS

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This work investigates a class of lattice dynamical systems originated from cellular neural networks. In the vector field of this class, each component of the state vector and the output vector is related through a sigmoidal nonlinear output function. For two types of sigmoidal output functions, Liapunov functions have been constructed in the literatures. Complete stability has been studied for these systems using LaSalle's invariant principle on the Liapunov functions. The purpose of this presentation is two folds. The first one is to construct Liapunov functions for more general sigmoidal output functions. The second is to extend the interaction parameters into a more general class, using an approach by Fiedler and Gedeon. This presentation also emphasizes the complete stability when the equilibrium is not isolated for the standard cellular neural networks.

1. Introduction

Consider the following class of cellular neural networks

$$\begin{aligned}\frac{d\mathbf{x}}{dt} &= \mathcal{F}(\mathbf{x}) := -\mathbf{x} + \mathbf{A}\mathbf{y} + \mathbf{b} \\ \mathbf{y} &= \mathbf{f}(\mathbf{x}) := (f_1(x_1),\, f_2(x_2), \ldots,\, f_n(x_n))\,.\end{aligned} \tag{1}$$

Herein, $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbf{R}^n$ is the state vector and $\mathbf{y} = (y_1, y_2, \ldots, y_n) \in \mathbf{R}^n$ is called the output vector. Each component of the state vector and the output vector is related through a sigmoidal nonlinear function f_i, namely, $y_i = f_i(x_i)$. $\mathbf{A}$ is an $n \times n$ matrix and $\mathbf{b}$ is a constant vector. In cellular neural network models, $\mathbf{A}$ is generated from the feedback operator, $\mathbf{b}$ consists of the terms from the control operator and an independent current source (bias, threshold).

Equation (1) is actually a recast or a generalization of a cellular neural network (CNN) proposed by Chua and Yang in 1988. In their model, a CNN of any dimension with finitely many cells can be recast in the form (1). To be more explicit on the equations we are interested in, let us describe this model on a two-dimensional lattice with cells sitting on a $k_1 \times k_2$ lattice $T_{\mathbf{k}} := \{(i, j) \in \mathbf{Z}^2 | 1 \le i \le k_1, 1 \le j \le k_2\}$. The circuit equation of a cell is given as

$$\frac{dx_{i,j}}{dt} = -x_{i,j} + \sum_{(k,\ell)\in N_r(i,j)} A(i, j; k, \ell) f(x_{k,\ell}) + b_{i,j}\,, \quad (i, j) \in T_{\mathbf{k}}\,, \tag{2}$$

where $N_r(i, j)$ represents the r-neighborhood of the cell at (i, j). The real numbers $A(i, j; k, \ell)$, $(i, j) \in T_{\mathbf{k}}$, $(k, \ell) \in N_r(i, j)$ constitute the template for CNN; this template describes the connection weights between cells. For $(i, j) \in \{(i, 1), (1, j), (k_1, j), (i, k_2) | 1 \le i \le k_1, 1 \le j \le k_2\}$, the $x_{i,j}$ term in (2) has to be specified according to the imposed boundary condition. For details, please see [Chua, 1998; Lin & Shih, 1999]. If we rearrange

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the coordinates $\{x_{i,j}\}$, $1 \le i \le k_1$, $1 \le j \le k_2$ into $\{x_i\}$, $1 \le i \le n$, $n = k_1 \times k_2$, it can be seen that (2) has the form (1).

Let $d(\mathbf{x}) = \sum_{i=1}^{n} x_i^2$, that is, the square of distance from $\mathbf{x}$ to the origin of $\mathbf{R}^n$. If every f_i is bounded, it can be computed that for every $\rho > 0$ large enough, the vector field (1) on the level surface $d(\mathbf{x}) = \rho$ points inward. Hence, (1) is a dissipative system if every f_i is bounded. Therefore, there exists a global attractor. For the notions of dissipative dynamical system and the global attractor, cf. [Hale *et al.*, 1984; Hale & Raugel, 1991]. One of the simplest kinds of such global attractors is the one that consists of all the equilibria and the unstable manifolds of the unstable equilibria. The heteroclinic orbits connecting the equilibria lie on these manifolds. Such a structure of global attractor leads to the notion of complete stability. By *complete stability*, we mean that every solution of a dynamical system tends to a stationary solution as time goes to positive infinity. Such a property is also called *convergence*, cf. [Fiedler & Gedeon, 1998]. In addition, a system is called *quasi-convergent* if every orbit of the system converges at least to the set of equilibria.

The complete stability for (2) with a two-sided saturated output function (see the following description) has been studied in [Chua & Yang, 1988; Lin & Shih, 1999]. The basic assumption is the symmetry condition of the circuit parameters:

$$A(i,\, j;\, k,\, \ell) = A(k,\, \ell;\, i,\, j),\ \text{for all } (i,\, j) \in T_{\mathbf{k}},\quad (k,\, \ell) \in N_r(i,\, j)\,. \tag{3}$$

With this assumption, if (2) is imposed with certain discrete-type boundary conditions, $\mathbf{A}$ is always symmetric, as (2) is reformulated into the form (1). In this presentation, we shall adopt the generalization in [Fiedler & Gedeon, 1998] and consider a larger class of interaction parameters $\mathbf{A}$. Their formulations are described as follows. Let $\mathbf{A} = [\mathbf{A}_{ik}]$ be an $n \times n$ matrix with either $\mathbf{A}_{ik} = 0$ or $\mathbf{A}_{ik}\mathbf{A}_{ki} \ne 0$ for $i,\, k = 1,\, 2, \ldots,\, n$. There corresponds an undirected graph Υ whose vertex k is joined to the vertex i by the edge e_{ik} if and only if $\mathbf{A}_{ik} \ne 0$ and $\mathbf{A}_{ki} \ne 0$. A cycle C in the graph Υ is a collection of edges $e_{i_1 i_2}, e_{i_2 i_3}, \ldots, e_{i_\ell i_1}$ with $\ell \ge 3$. The interaction parameters $\mathbf{A}$ considered in this investigation are the ones satisfying

$$\begin{aligned} &(H_1) \quad \mathbf{A}_{ik}\mathbf{A}_{ki} > 0, \quad \text{if } \mathbf{A}_{i,k} \ne 0\,, \\ &(H_2) \quad \Pi_C \mathbf{A}_{ik} = \Pi_C \mathbf{A}_{ki}, \quad \text{along every cycle } C\,, \end{aligned} \tag{4}$$

where Π denotes the product. It is straightforward to verify that symmetric $\mathbf{A}$ satisfies (H_1) and (H_2). Denote by $\mathbf{N}_n$ the set of positive integers from 1 to n.

The sigmoidal output functions that will be investigated in this presentation contain the following three basic types:

(I) $f_i(\xi) = f(\xi) := (1/2)(|\xi + 1| - |\xi - 1|)$ for $i \in \mathbf{N}_n$.

(II) f_i is bounded, differentiable and $f_i'(\xi) > 0$ for any $\xi \in \mathbf{R}$ and $i \in \mathbf{N}_n$.

(III) f_i is continuous on $\mathbf{R}$, $f_i'(\xi) > 0$ if $\xi \in (-1,\, 1)$, $f_i(\xi) = 1$ if $\xi \ge 1$, and $f_i(\xi) = -1$ if $\xi \le -1$, for each $i \in \mathbf{N}_n$.

Typical figures for these functions are depicted in Figs. 1–3. The results in [Chua & Yang, 1988; Lin & Shih, 1999] concerned (2) with the sigmoidal function of type (I). Notice that for type (I), $f(\xi)$ is piecewise linear and it has two saturated parts $\xi \ge 1$, $\xi \le -1$ with two corners at $\xi = -1$ and

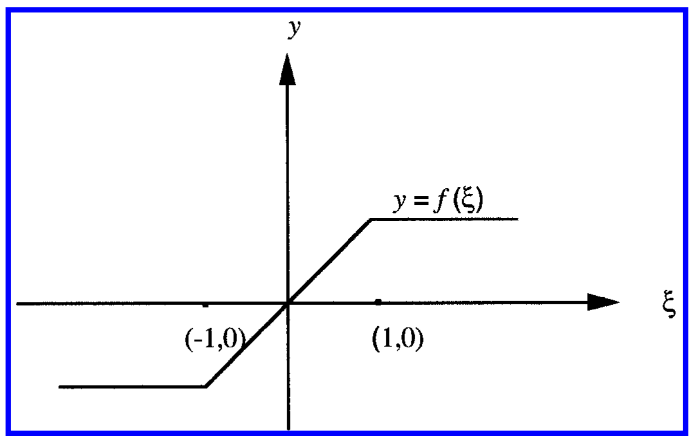

Fig. 1. Sigmoidal output function of type (I).

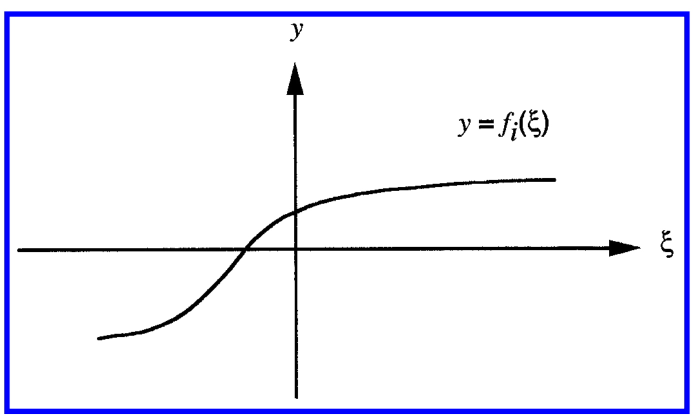

Fig. 2. Sigmoidal output functions of type (II).

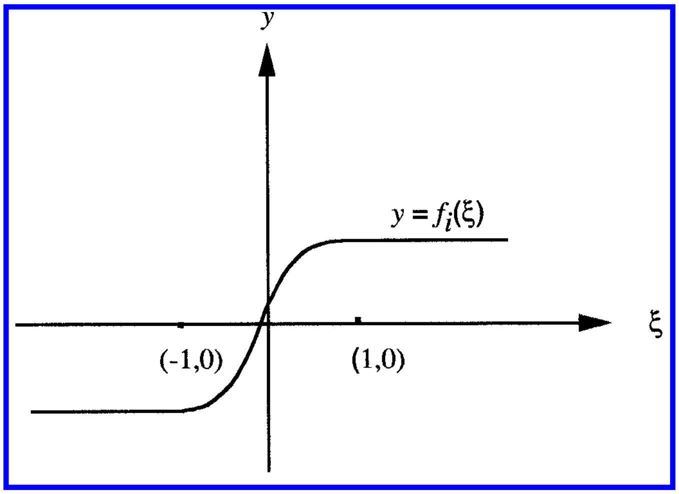

Fig. 3. Sigmoidal output functions of type (III).

$\xi = 1$. However, it is Lipschitz. On the other hand, it has been shown in [Wu & Chua, 1997] that (1) is completely stable if $\mathbf{A}$ is symmetric, every equilibrium is isolated and f_i satisfies (II). Fiedler and Gedeon [1998] studied a more general system with the sigmoidal functions of type (II).

The function of type (III) can be regarded as a generalization of type (I) or a mixture of the types (I) and (II). Note that for f_i of type (III), we do not require differentiability of f_i at -1, 1. Smooth sigmoidal functions of this type include the regularization of the type (I) function at the corners $\xi = -1, 1$.

All the aforementioned results on complete stability (convergence) and quasi-convergence are based on applying the LaSalle's invariant principle to their respective Liapunov functions. Such functions are originated from the studies of neural networks, cf. [Cohen & Grossberg, 1983]. Let us review this principle and introduce some necessary notations which will be used throughout this paper. These details can be found in [Hale, 1980].

Suppose the vector field $\mathcal{F}$ in (1) is locally Lipchitzian. Let V be a scalar function defined and continuous on $\mathbf{R}^n$ and $\phi(t, \mathbf{x})$ be the flow map of (1). To determine if V decreases along the orbit of (1), we can consider

$$\dot{V}(\mathbf{x}) := \limsup_{h\to 0^+} \frac{1}{h}[V(\phi(h, \mathbf{x})) - V(\mathbf{x})]\,. \qquad (5)$$

If V is locally Lipschitz continuous, (5) is equal to

$$\limsup_{h\to 0^+} \frac{1}{h}[V(\mathbf{x} + h\mathcal{F}(\mathbf{x})) - V(\mathbf{x})]\,. \qquad (6)$$

If $\partial V/\partial \mathbf{x}$ exists and is continuous, then (5) is also equal to

$$\frac{\partial V(\mathbf{x})}{\partial \mathbf{x}}\mathcal{F}(\mathbf{x})\,. \qquad (7)$$

Suppose V is bounded in $\mathbf{R}^n$ and $\dot{V}(\mathbf{x}) \le 0$ for all $\mathbf{x} \in \mathbf{R}^n$. Let

$$\mathcal{S} = \{\mathbf{x} \in \mathbf{R}^n : \dot{V}(\mathbf{x}) = 0\} \qquad (8)$$

and let $\mathcal{I}$ be the largest invariant set of (1) in $\mathcal{S}$. LaSalle's invariant principle says that if $\phi(t, \mathbf{x})$ is bounded for $t \ge 0$, then the ω-limit set of $\phi(t, \mathbf{x})$ belongs to $\mathcal{I}$. Accordingly, if there exists a Liapunov function for (1) with one of these sigmoidal functions, then it remains to study the set $\mathcal{S}$ and the dynamics on it. For f_i of type (II), $\mathcal{S}$ consists only of equilibria. Thus, the ω-limit set for each orbit consists of equilibria only. Therefore, the quasi-convergence of the system is confirmed. The complete stability can be further concluded if every equilibrium is isolated. For f_i of type (I) or (III), $\mathcal{S}$ is a much bigger set. This is due to the saturations in f_i. To conclude the complete stability or quasi-convergence, we have to analyze the flow restricted to $\mathcal{S}$ and exclude the possibility such as existence of limit cycle or heteroclinic orbit in $\mathcal{S}$. For the case of type (I) sigmoidal function, it can be further shown that the ω-limit set of every orbit is a singleton, even if the equilibrium is not isolated. The following theorems and corollary summarize the main results in this presentation.

Proposition 1.1. *Assume that* $\mathbf{A}$ *satisfies* (H_1) *and* (H_2). *There exist Liapunov functions for* (1) *with sigmoidal output functions of types* (I), (II) *and* (III) *respectively.*

The Liapunov functions we shall construct are functions $V : \mathbf{R}^n \to \mathbf{R}$ which are nonincreasing along the orbits of (1). The dependence of these Liapunov functions on the state variable $\mathbf{x}$ are all through the respective output functions. Therefore, that the Liapunov function is strictly decreasing along all nonequilibrium orbits of (1) holds only for the sigmoidal output functions of type (II). Though this strict decrease does not hold for the sigmoidal functions of types (I) and (III), the following convergence can still be obtained through analyzing the vector field in (1).

Theorem 1.2. *Assume that* $\mathbf{A}$ *satisfies* (H_1) *and* (H_2). *Equation* (1) *with sigmoidal output* *functions of type* (II) *or type* (III) *is quasi-convergent.*

Corollary 1.3. *Assume that* **A** *satisfies* (H_1) *and* (H_2). *If the set of equilibria is finite, then* (*1*) *with sigmoidal output functions of type* (II) *or type* (III) *is completely stable* (*convergent*).

Theorem 1.4. *Assume that* **A** *satisfies* (H_1) *and* (H_2). *Equation* (*1*) *with the sigmoidal output function of type* (I) *is completely stable* (*convergent*), *even if the equilibrium is not isolated* (*thus the set of equilibria is infinite*).

We shall construct the Liapunov function as well as describe $\mathcal{S}$ for the sigmoidal functions of types (II), (III) and (I) in Secs. 2–4 respectively. Notably, type (I) function is actually contained in the class of type (III) functions. Theorems 1.2 is justified by verifying the respective proposition (Propositions 2.1 and 3.1) for each type of output function. Theorem 1.4 is confirmed from the Proposition 4.1.

2. Sigmoidal Functions of Type (II)

For the sigmoidal output functions of type (II), each f_i has an inverse f_i^{-1}, owing to $f_i'(\xi) > 0$ for all $\xi \in \mathbf{R}$. If **A** is symmetric, the Liapunov function proposed by Wu and Chua [1997] for (1) with output functions of type (II) takes the form

$$V(\mathbf{x}) = -\frac{1}{2}\langle \mathbf{y},\, \mathbf{A}\mathbf{y} + 2\mathbf{b}\rangle + \sum_{i=1}^{n} \int_{f_i(0)}^{y_i} f_i^{-1}(\xi)d\xi\,, \quad (9)$$

where $\mathbf{y} = (y_1, y_2, \ldots, y_n)$ and $y_i = f_i(x_i)$, as in (1).

For more general **A** satisfying (H_1) and (H_2), we present the following generalization. Let v_i, v_k satisfy $\mathbf{A}_{ik}/\mathbf{A}_{ki} = \exp(v_i - v_k)$ for these i, k with $i \neq k$ and $\mathbf{A}_{ik} \neq 0$. Set $v_i = 0$ if $\mathbf{A}_{ik} = 0$. Define, for $i, k \in \mathbf{N}_n$,

$$s_i := \exp(-v_i)\,, \quad p_{ik} := \frac{1}{2} s_i \mathbf{A}_{ik}\,. \quad (10)$$

Note that $p_{ik} = p_{ki}$ for $i, k \in \mathbf{N}_n$. Set $\mathbf{P} = [p_{ik}]$, the $n \times n$ matrix with entries p_{ik}, and set $\mathbf{S} = \mathrm{diag}(s_1, s_2, \ldots, s_n)$, the diagonal matrix with diagonal entries $s_1, s_2, \ldots, s_n$. The above formulation of the parameters is adopted from [Fiedler & Gedeon, 1998]. We modify the Liapunov function therein to accommodate Eq. (1) as follows.

$$V(\mathbf{x}) = \sum_{i=1}^{n} s_i \int_{f_i(0)}^{y_i} f_i^{-1}(\xi)d\xi - \langle \mathbf{y},\, \mathbf{P}\mathbf{y} + \mathbf{S}\mathbf{b}\rangle\,. \quad (11)$$

Indeed,

$$\begin{aligned}
&\frac{\partial V(\mathbf{x})}{\partial \mathbf{x}} \mathcal{F}(\mathbf{x}) \\
&= \sum_{i=1}^{n} s_i f_i'(x_i) \left[x_i - b_i - \sum_{k=1}^{n} \frac{p_{ik} + p_{ki}}{s_i} y_k \right] \mathcal{F}_i(\mathbf{x}) \\
&= \sum_{i=1}^{n} s_i f_i'(x_i) \left[x_i - b_i - \sum_{k=1}^{n} \mathbf{A}_{ik} y_k \right] \mathcal{F}_i(\mathbf{x}) \\
&= -\sum_{i=1}^{n} s_i f_i'(x_i) \mathcal{F}_i(\mathbf{x})^2 \\
&\leq 0\,.
\end{aligned}$$

Notably, $\dot{V}(\mathbf{x}) = 0$ if and only if $\mathcal{F}_i(\mathbf{x}) = 0$ for all i, since $f_i'(\xi) > 0$ for all $\xi \in \mathbf{R}$. Thus, $\mathcal{S}$ consists of equilibria only. Therefore, we have the following result.

Proposition 2.1. *The* ω*-limit set for every orbit of* (*1*) *with sigmoidal functions of type* (II) *consists of equilibria only.*

This proposition confirms Theorem 1.2 for the sigmoidal functions of type (II).

3. Sigmoidal Functions of Type (III)

In this section, we study (1) with sigmoidal functions f_i of type (III). According to the configuration for the functions of type (III), the real line can be partitioned as $\mathbf{R} = (-\infty, -1] \cup (-1, 1) \cup [1, \infty)$. The phase space $\mathbf{R}^n$ for the dynamical system generated by (1) can thus be partitioned into 3^n regions. We use the following labeling and notations to describe these regions. The setting here is basically the same as in [Lin & Shih, 1999]. Let $\mathcal{A} = \{-1, 0, 1\}$. Denote by $\mathcal{A}^{\mathbf{N}_n}$ the set of all functions $\alpha : \mathbf{N}_n \to \mathcal{A}$. For each $\alpha = \{\alpha_i\} \in \mathcal{A}^{\mathbf{N}_n}$, set

$$\begin{aligned}
\Omega_\alpha := \{\mathbf{x} = \{x_i\} \in \mathbf{R}^n | x_i \geq 1 \text{ if } \alpha_i = 1;\ x_i \leq -1 \\
\text{if } \alpha_i = -1;\ |x_i| < 1 \text{ if } \alpha_i = 0\}\,. \quad (12)
\end{aligned}$$

Then,

$$\bigcup_{\alpha \in \mathcal{A}^{\mathbf{N}_n}} \Omega_\alpha = \mathbf{R}^n\,.$$

Let $\Lambda_\mathrm{e} = \{\{\alpha_i\} \in \mathcal{A}^{\mathbf{N}_n} | \alpha_i = 1 \text{ or } -1\}$, $\Lambda_\mathrm{m} = \{\{\alpha_i\} \in \mathcal{A}^{\mathbf{N}_n} | \alpha_i = 0 \text{ for some } i \in \mathbf{N}_n \text{ and } \alpha_j \neq 0 \text{ for some } j \in \mathbf{N}_n\}$. Thus, we can arrange these 3^n regions into three categories. Ω_α is called an *exterior region* if $\alpha \in \Lambda_\mathrm{e}$, a *mixed region* if $\alpha \in \Lambda_\mathrm{m}$ and an *interior region* if $\alpha_i = 0$ for all $i \in \mathbf{N}_n$. Obviously, there is only one interior region which will be denoted by Ω_0. For any two α, $\overline{\alpha} \in \mathcal{A}^{\mathbf{N}_n}$, we shall say Ω_α is *more interior* than $\Omega_{\overline{\alpha}}$, if $|\alpha_i| < |\overline{\alpha}_i|$ for some $i \in \mathbf{N}_n$ and $\alpha_j = \overline{\alpha}_j$ for the other $j \in \mathbf{N}_n$.

Consequently, the equilibria for (1) can be classified into three types, according to their locations. A stationary solution (equilibrium) $\overline{\mathbf{x}} = \{\overline{x}_i\}$ is called, *saturated* if $\overline{\mathbf{x}}$ lies in an exterior region, *mixed* if $\overline{\mathbf{x}}$ lies in a mixed region, and *interior* if $\overline{\mathbf{x}}$ lies in the interior region.

Let us describe how each of these equilibria exists. For an $\alpha \in \Lambda_\mathrm{e}$, suppose

$$\overline{x}_i = \sum_{k=1}^{n} \mathbf{A}_{ik}\alpha_k + b_i \tag{13}$$

and $\overline{x}_i \geq 1$ if $\alpha_i = 1$, $\overline{x}_i \leq -1$ if $\alpha_i = -1$. Then $\overline{\mathbf{x}} = \{\overline{x}_i\}$ is an equilibrium in the exterior region Ω_α. Notice that the equilibrium in each exterior region, if exists, is unique. Consider, for $i \in \mathbf{N}_n$,

$$x_i - \sum_{k=1}^{n} \mathbf{A}_{ik} f_k(x_k) = b_i\,. \tag{14}$$

If this system of equations is satisfied for $\mathbf{x} = \overline{\mathbf{x}}$ with $|\overline{x}_i| < 1$ for all $i \in \mathbf{N}_n$, then $\overline{\mathbf{x}} = \{\overline{x}_i\}$ is an equilibrium in the interior region Ω_0.

Consider a mixed region Ω_α, $\alpha \in \Lambda_m$. Let $J_0 = \{i \in \mathbf{N}_n : \alpha_i = 0\}$ and $J_1 = \mathbf{N}_n \backslash J_0$. To find an equilibrium in Ω_α, we first solve the following system of equations for x_i, $i \in J_0$,

$$x_i - \sum_{k\in J_0} \mathbf{A}_{ik} f_k(x_k) = \sum_{k \in J_1} \mathbf{A}_{ik}\alpha_k + b_i\,. \tag{15}$$

If there exists a solution $\{x_i\}_{i\in J_0} = \{\overline{x}_i\}_{i\in J_0}$ for this system with $|\overline{x}_i| < 1$, $i \in J_0$, we substitute them into the right-hand side of (16).

$$\overline{x}_i = \sum_{k \in J_1} \mathbf{A}_{ik}\alpha_k + \sum_{k\in J_0} \mathbf{A}_{ik} f_k(\overline{x}_k) + b_i\,, \tag{16}$$

where $i \in J_1$. If $\overline{x}_i \geq 1$ for i with $\alpha_i = 1$ and $\overline{x}_i \leq -1$ for i with $\alpha_i = -1$, then $\overline{\mathbf{x}} = \{\overline{x}_i\}$ is an equilibrium of (1) in Ω_α.

The saturated equilibrium in each exterior region is unique if it exists. If a solution $\overline{\mathbf{x}}$ in (14) exists uniquely with $|\overline{x}_i| < 1$ for all $i \in \mathbf{N}_n$, then $\overline{\mathbf{x}}$ is the unique (hence isolated) equilibrium in the interior region. Moreover, the uniqueness of a mixed equilibrium in Ω_α, $\alpha \in \Lambda_m$, is determined from the uniqueness of solution in (15).

If there exists a solution $\{x_i\}_{i\in J_0} = \{\overline{x}_i\}_{i\in J_0}$ for (15) with $|\overline{x}_i| < 1$, $i \in J_0$, let I_α denote a subset in Ω_α with

$$I_\alpha = \{\mathbf{x} \in \mathbf{R}^n | x_i = \overline{x}_i,\, i \in J_0,\, x_i \geq 1 \text{ if } \alpha_i = 1,\, x_i \leq -1 \text{ if } \alpha_i = -1\}\,. \tag{17}$$

Let $\phi(t, \mathbf{x})$ be the flow map of (1). Then the flow on I_α has the following dynamic property: $\phi(t, \mathbf{x})$ with $\mathbf{x} \in I_\alpha$ does not leave I_α before it enters into the other region which is more interior than Ω_α, in forward time.

Recall the definitions of s_i, $\mathbf{P}$, $\mathbf{S}$ defined in (10). We consider the following function

$$V(\mathbf{x}) = \sum_{i=1}^{n} s_i \int_{f_i(0)}^{y_i} g_i(\xi)d\xi - \langle \mathbf{y}, \mathbf{P}\mathbf{y} + \mathbf{S}\mathbf{b}\rangle\,, \tag{18}$$

where $y_i = f_i(x_i)$, $g_i(\xi) = f_i^{-1}(\xi)$, for $\xi \in [-1, 1]$, $g_i(\xi) = 1$, if $\xi \geq 1$, $g_i(\xi) = -1$, if $\xi \leq -1$. If each f_i is differentiable on $\mathbf{R}$, then

$$\frac{d}{dx_i} \int_{f_i(0)}^{y_i} g_i(\xi)d\xi = f_i'(x_i) f(x_i)\,,$$

where f is exactly the sigmoidal function of type (I). Consequently,

$$\frac{\partial V(\mathbf{x})}{\partial \mathbf{x}} \mathcal{F}(\mathbf{x}) = -\sum_{i=1}^{n} s_i f_i'(x_i)[-f(x_i) + (\mathbf{A}\mathbf{y})_i + b_i]\mathcal{F}_i(\mathbf{x})\,. \tag{19}$$

Equation (19) is less than or equal to zero since $s_i > 0$, $f_i'(x_i) \geq 0$ for any x_i, $f_i'(x_i) = 0$ if $|x_i| \geq 1$, and $[-f(x_i) + (\mathbf{A}\mathbf{y})_i + b_i] = \mathcal{F}_i(\mathbf{x})$ if $|x_i| \leq 1$. The latter equality follows from the definition of f, that is, $f(x_i) = x_i$ if $|x_i| \leq 1$.

If some f_i is not differentiable, we elaborate on an alternative computation. Firstly,

$$\limsup_{h\to 0^+} \frac{1}{h}\left[\int_{f_i(0)}^{f_i(x_i + h\mathcal{F}_i(\mathbf{x}))} g_i(\xi)d\xi - \int_{f_i(0)}^{f_i(x_i)} g_i(\xi)d\xi\right] = f(x_i) \limsup_{h\to 0^+} Q_i(h, \mathbf{x})\,,$$

where $Q_i(h, \mathbf{x}) := (1/h)[f_i(x_i + h\mathcal{F}_i(\mathbf{x})) - f_i(x_i)]$ and, again, f is the sigmoidal function of type (I). Using the property of f_i, computations show that

$$\mathcal{F}_i(\mathbf{x}) \limsup_{h\to 0^+} Q_i(h, \mathbf{x}) \geq 0\,.$$

Moreover, $\lim_{h\to 0^+} Q_i(h, \mathbf{x}) = 0$ if $|x_i| > 1$. Note that, again, $[f(x_i) - (\mathbf{Ay})_i - b_i] = -\mathcal{F}_i(\mathbf{x})$ if $|x_i| \le 1$. With V defined in (18), it follows that

$$\begin{aligned}\limsup_{h\to 0^+} \frac{1}{h}[V(\mathbf{x} + h\mathcal{F}(\mathbf{x})) - V(\mathbf{x})] &= \limsup_{h\to 0^+} \left\{ \sum_{i=1}^{n} s_i Q_i(h, \mathbf{x})[f(x_i) - (\mathbf{Ay})_i - b_i] \right\} \\ &\le \sum_{i=1}^{n} \limsup_{h\to 0^+} \{ s_i Q_i(h, \mathbf{x})[f(x_i) - (\mathbf{Ay})_i - b_i] \} \\ &\le 0 . \end{aligned} \tag{20}$$

Therefore, V is indeed a Liapunov function. Since each term in (20) is less than or equal to zero, it follows that

$$\limsup_{h\to 0^+} \frac{1}{h}[V(\mathbf{x} + h\mathcal{F}(\mathbf{x})) - V(\mathbf{x})] = 0$$

if and only if

$$\limsup_{h\to 0^+} \{Q_i(h, \mathbf{x})[f(x_i) - (\mathbf{Ay})_i - b_i]\} = 0 ,$$

for all $i \in \mathbf{N}_n$. Thus, V remains constant along every orbit lying in $\mathcal{S}$ whose closure can be described as follows:

$$\overline{\mathcal{S}} = (\cup_{\alpha\in\Lambda_e}\Omega_\alpha) \cup (\cup I_\alpha) \cup \mathcal{E}_0 , \tag{21}$$

where $\cup_{\alpha\in\Lambda_e}\Omega_\alpha$ is the union of all exterior regions, $\mathcal{E}_0$ is the set of equilibria in the interior region, and $\cup I_\alpha$ is the union of some subsets in mixed regions, whenever they exist (see (17)). We have taken the closure of $\mathcal{S}$ in (21) since the boundary points in each region may not be in $\mathcal{S}$. By boundary points of Ω_α or I_α, we mean the points $\mathbf{x} \in \Omega_\alpha$, $\alpha \in \Lambda_e$ or $\mathbf{x} \in I_\alpha$, $\alpha \in \Lambda_m$ with $|x_i| = |\alpha_i| = 1$ for some $i \in \mathbf{N}_n$. However, all the equilibria are in $\mathcal{S}$. To verify the complete stability of (1), it suffices to investigate the dynamics in $\mathcal{S}$. Let us call each of these Ω_α, I_α and $\mathcal{E}_0$ in (21) a *component* of $\mathcal{S}$ (they are actually subsets of $\overline{\mathcal{S}}$).

If the equilibrium in each region is unique, hence isolated, then the components of $\mathcal{S}$ are disjoint. Since the Liapunov function is decreasing along an orbit off $\mathcal{S}$, the ω-limit set for an orbit of (1) has to be the maximal invariant set of some component of $\mathcal{S}$, which is an equilibrium. Thus, the maximal invariant set in $\mathcal{S}$ consists of equilibria only. For general cases, we have the following result.

Proposition 3.1. *The ω-limit set for every orbit of (1) with sigmoidal functions of type* (III) *consists of equilibrium only.*

This proposition is justified by the analysis of the vector field in (1) and its associated dynamics restricted to the components of $\mathcal{S}$. Herein, we only sketch the basic ideas. The detailed verifications resemble the ones in [Lin & Shih, 1999]. First, let us note the dynamic properties on these components. On an exterior region $\Omega_{\overline{\alpha}}$, the dynamical system is nonhomogeneous linear and uncoupled. If there is an equilibrium in $\Omega_{\overline{\alpha}}$, it is unique in $\Omega_{\overline{\alpha}}$ and it attracts every point in $\Omega_{\overline{\alpha}}$. In fact, if this equilibrium is interior in $\Omega_{\overline{\alpha}}$, then it is a sink. If there is no equilibrium in $\Omega_{\overline{\alpha}}$, then every orbit originating from $\Omega_{\overline{\alpha}}$ enters into a region which is more interior than $\Omega_{\overline{\alpha}}$. Moreover, the α-limit set for an orbit passing $\Omega_{\overline{\alpha}}$ is either empty or belongs to a more interior region. The situation is similar for $I_{\overline{\alpha}}$ in a mixed region $\Omega_{\overline{\alpha}}$, $\overline{\alpha} \in \Lambda_m$. If there is an equilibrium in $I_{\overline{\alpha}}$, then it attracts every point in $I_{\overline{\alpha}}$. If there does not exist an equilibrium in $I_{\overline{\alpha}}$, then every orbit originating from $I_{\overline{\alpha}}$ enters into a region Ω_α which is more interior than $\Omega_{\overline{\alpha}}$. Furthermore, the α-limit set for an orbit passing $I_{\overline{\alpha}}$ is either empty or belongs to a more interior region. Note that every point in $\mathcal{E}_0$ is an equilibrium. Consider an arbitrary orbit $\gamma : \phi(t, \mathbf{x})$ of (1). $\omega(\gamma)$, the ω-limit set of γ is bounded, invariant and contained in $\mathcal{S}$. Let $\mathbf{x}^* \in \omega(\gamma)$. If $\mathbf{x}^*$ is not an equilibrium, then $\phi(t, \mathbf{x}^*)$ does not exist as $t \to -\infty$, according to the above discussions on the dynamics in each component of $\mathcal{S}$. This contradicts to the property of $\omega(\gamma)$. Thus, $\mathbf{x}^*$ has to be an equilibrium.

4. Sigmoidal Function of Type (I)

We consider (1) with sigmoidal function f of type (I) in this section. The isolated condition for each equilibrium can be completely characterized by the interaction parameters $\mathbf{A}$ (i.e. independent of f), if type (I) output function f is considered in (1). In addition, since type (I) function belongs to the set of type (III) functions, Proposition 3.1 confirms the quasi-convergence for the system. Herein, it will be demonstrated that the complete stability can be further concluded even if the equilibria exist as a continuum. We shall outline these results and lay emphasis on the local structure of the phase space near a continuum of equilibria.

Firstly, we describe the existence for the three types of equilibria. The existence of a saturated equilibrium is exactly the same as (13). An interior equilibrium of (1) $\overline{\mathbf{x}}$ exists if $\mathbf{x} = \overline{\mathbf{x}}$ satisfies

$$x_i - \sum_{k=1}^{n} \mathbf{A}_{ik} x_k = b_i \,, \tag{22}$$

with $|\overline{x}_i| < 1$ for all $i \in \mathbf{N}_n$. The existence for a mixed equilibrium is as described in Sec. 3 except that (15) is changed to

$$x_i - \sum_{k\in J_0} \mathbf{A}_{ik} x_k = \sum_{k\in J_1} \mathbf{A}_{ik}\alpha_k + b_i \,, \tag{23}$$

as well as (16) is replaced by

$$\overline{x}_i = \sum_{k\in J_1} \mathbf{A}_{ik}\alpha_k + \sum_{k\in J_0} \mathbf{A}_{ik}\overline{x}_k + b_i \,. \tag{24}$$

We represent the system of linear equations in (22) by

$$M_0\mathbf{x} = \mathbf{b}\,, \tag{25}$$

and the one in (23) by

$$M_\alpha \mathbf{w} = \mathbf{c}_\alpha \,. \tag{26}$$

Note that M_α is formed from $\mathbf{A}_{ik}$, $i, k \in J_0$; $\mathbf{c}_\alpha$ depends on $\mathbf{A}_{ik}$, α_k, b_i for $i \in J_0$, $k \in J_1$; $\mathbf{w} = \{x_i\}_{i\in J_0}$. Due to the above arguments, the equilibrium in a mixed region or the interior region can appear as a continuum. However, for almost all parameters, the equilibrium in each region Ω_α is unique and isolated, if it exists. These parameters can be described as

$$\{\mathbf{A}_{ik},\ i,\, k \in \mathbf{N}_n |\, \det(M_0) \neq 0 \text{ and } \det(M_\alpha) \neq 0 \text{ for every } \alpha \in \Lambda_m\}\,,$$

where M_0 and M_α are given in (25) and (26) respectively. The parameters $\{\mathbf{A}_{ik},\ i,\, k \in \mathbf{N}_n\}$ in this set are called *regular parameters*.

The Liapunov function in (18) now takes the following form for the sigmoidal function of type (I).

$$\begin{aligned} V(\mathbf{x}) &= \left\langle \mathbf{y},\, \frac{1}{2}\mathbf{S}\mathbf{y} - (\mathbf{P}\mathbf{y} + \mathbf{S}\mathbf{b}) \right\rangle \\ &= \sum_{i=1}^{n} \left\{ s_i \left(\frac{1}{2} y_i^2 - b_i y_i \right) - \sum_{k=1}^{n} p_{ik} y_i y_k \right\}. \end{aligned}$$

If the parameters are regular, then the components of $\mathcal{S}$ are disjoint and (1) is completely stable. If the parameters are not regular, then some of these components are no longer disjoint. Nevertheless, (1) is quasi-convergent according to Proposition 3.1. The convergence of dynamics can further be concluded by showing that the ω-limit set of every orbit consists of a single equilibrium. Herein, we would like to emphasize this property and present its details.

Proposition 4.1. *The ω-limit set for every orbit of (1) with the sigmoidal function of type* (I) *consists of a single equilibrium.*

Proof. We shall only prove the case that $\mathbf{A}$ is symmetric. A transformation as discussed in the next section converts the matrices $\mathbf{A}$ satisfying (H_1) and (H_2) into symmetric ones. It will be verified that, if the parameters are not regular, there is exactly one equilibrium in the ω-limit set for each orbit of (1). Assume that there is a connected set of equilibria (a continuum of equilibria), denoted by $\mathcal{E}_\alpha$, in a mixed region Ω_α, $\alpha \in \Lambda_m$. This occurs exactly when M_α has zero eigenvalue and there exists $\overline{\mathbf{x}}$ satisfying (24) and (25) (i.e. (23)). If this is the case and the kernel of M_α has dimension k, then $\mathcal{E}_\alpha$ is the intersection of Ω_α and an k-dimensional hyperplane in $\mathbf{R}^n$. More precisely,

$$\mathcal{E}_\alpha = \left\{ \overline{\mathbf{x}} | \overline{\mathbf{x}} = \sum_{j=1}^{k} r_j \mathbf{v}_j + \mathbf{c} \right\} \cap \Omega_\alpha \,, \tag{27}$$

where $r_j \in \mathbf{R}$, $\mathbf{c}$ is some constant vector in $\mathbf{R}^n$, and each $\mathbf{v}_j$ is an $\mathbf{R}^n$ vector with $\mathbf{w} = \mathbf{v}_j|_{J_0}$ satisfying $M_\alpha \mathbf{w} = \mathbf{0}$ for $j = 1, 2, \ldots, k$. Here, $\mathbf{v}_j|_{J_0}$ represents the restriction of $\mathbf{R}^n$ vector $\mathbf{v}_j$ to its J_0 components. Equation (27) follows from the solution structures of the linear system (25). Notably, by continuity of solutions satisfying (24) and (26), the boundary point of the closure (relative to k-dim. subspace topology) for this intersection is still an equilibrium of (1), which belongs to Ω_α or the regions neighboring Ω_α. Now, let us analyze the local structures of the phase space near an equilibrium $\overline{\mathbf{x}}$ in the interior of $\mathcal{E}_\alpha$ (k-dim. subspace topology). Assume that $|\overline{x}_i| \neq 1$ for all $i \in \mathbf{N}_n$. Notice that the vector field $\mathcal{F}$ is smooth off $\{\mathbf{x} : |x_i| = 1$ for some $i \in \mathbf{N}_n\}$. Thus, we can consider the linearization of $\mathcal{F}$ at $\overline{\mathbf{x}}$. Renaming the coordinates, $D\mathcal{F}(\overline{\mathbf{x}})$ has a matrix representation of the following form:

$$D\mathcal{F}(\overline{\mathbf{x}}) = \begin{bmatrix} -I & C \\ 0 & M_\alpha \end{bmatrix}, \tag{28}$$

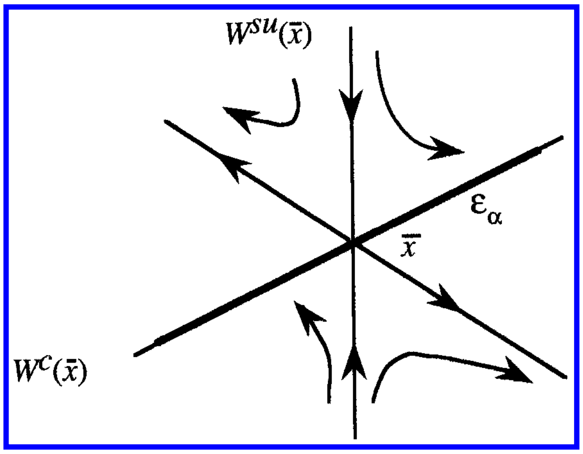

Fig. 4. Local phase space decomposition around a continuum of equilibria for (1) with sigmoidal output function of type (I).

where I is the identity matrix of size $\ell \times \ell$ with $\ell = \mathrm{card}(J_1)$. Note that M_α is symmetric, since $\mathbf{A}$ is symmetric. From (27) and (28), it can be verified that for $\mathbf{v}$ with $\mathbf{v}|_{J_0} \in \mathrm{kernel}(M_\alpha)$ and for small δ, $\overline{\mathbf{x}} + \delta\mathbf{v} \in W^c(\overline{\mathbf{x}})$ if and only if $\overline{\mathbf{x}} + \delta\mathbf{v} \in \mathcal{E}_\alpha$. Let $W^c(\overline{\mathbf{x}})$ (resp. E^c) and $W^{su}(\overline{\mathbf{x}})$ (resp. E^{su}) be the center and stable–unstable manifolds (resp. subspaces) for $\overline{\mathbf{x}}$ respectively. The phase space decomposition at $\overline{\mathbf{x}}$ then has a local structure: there is a neighborhood U of $\overline{\mathbf{x}}$ in $\mathbf{R}^n$ such that $\mathbf{R}^n = E^c \oplus E^{su}$ and $\mathcal{E}_\alpha = (\overline{\mathbf{x}} + E^c) \cap U = W^c(\overline{\mathbf{x}}) \cap U$, $(\overline{\mathbf{x}} + E^{su}) \cap U = W^{su}(\overline{\mathbf{x}}) \cap U$, see Fig. 4. This local structure of phase space holds for every point in the interior of $\mathcal{E}_\alpha$. If $\mathcal{E}_\alpha$ is contained in $\{\mathbf{x} : |x_i| = 1$ for some $i \in \mathbf{N}_n\}$, the linearization at $\overline{\mathbf{x}}$ is not available. However, by combining a sequence of similar phase space analysis, each time in a sector of $\mathbf{R}^n$, similar local phase space structure along $\mathcal{E}_\alpha$ can still be obtained. Analogous arguments can verify the case for a continuum of equilibrium in the interior region. Also note that the equilibrium in an exterior region is unique, if it exists. Therefore, in a neighborhood of every interior point $\overline{\mathbf{x}}$ of a connected set of equilibrium, the phase flows do not wander around $\overline{\mathbf{x}}$. Furthermore, if the phase flow approaches $\overline{\mathbf{x}}$, it approaches in the directions of $W^s(\overline{\mathbf{x}})$. That there is exactly one equilibrium in the ω-limit set for every orbit of (1) can thus be seen from the scenario of local phase space around $\mathcal{E}_\alpha$. Such a scenario has been analyzed in various ordinary and partial differential equations, see [Hale, 1992] and the references therein. This completes the proof for this proposition. ■

5. Conclusions

Recently, a study in [Shih & Weng, 2000] has indicated that the matrices satisfying (H_1) and (H_2), called *cycle-symmetric*, can be characterized as matrices which are similar to symmetric matrices by real diagonal matrices. Restated, if $\mathbf{A}$ satisfies (H_1) and (H_2), then there exists an invertible diagonal matrix $\mathbf{Q}$ such that $\mathbf{QAQ}^{-1}$ is a symmetric matrix. With this characterization, all the results in this presentation can also be obtained by transforming (1) to a similar system, but with symmetric interaction parameters. Let us elaborate on this formulation. Let $\mathbf{A} = [\mathbf{A}_{ik}]$ be a cycle-symmetric matrix and let $\mathbf{Q}$ be an invertible diagonal matrix such that $\mathbf{QAQ}^{-1} = \mathbf{B}$ with $\mathbf{B} = [b_{ik}]$, a symmetric matrix. Denote the diagonal entries of $\mathbf{Q}$ by $q_1, q_2, \ldots, q_n$ where every q_i is nonzero. Set $\mathbf{u} = \mathbf{Qx}$, that is, $u_i = q_i x_i$ for each i. Equations (1) in new variables becomes

$$\frac{d\mathbf{u}}{dt} = -\mathbf{u} + \mathbf{B}\tilde{\mathbf{f}}(\mathbf{u}) + \tilde{\mathbf{b}}\,,$$

where $\tilde{\mathbf{f}}(\mathbf{u}) = (\tilde{f}_1(u_1), \tilde{f}_2(u_2), \ldots, \tilde{f}_n(u_n))$, $\tilde{f}_i(u_i) = q_i f_i(q_i^{-1} u_i)$, and $\tilde{\mathbf{b}} = \mathbf{Qb}$. Notably, if f_i is of type (II), then $\tilde{f}_i$ is also of type (II). If f_i is of type (I) or type (III), then $\tilde{f}_i$ is a rescaling of f_i with analogous sigmoidal configuration. Similar partitioning of phase space as in Secs. 3 and 4 can be performed to obtain Propositions 3.1 and 4.1.

We remark that in [Wu & Chua, 1997], a transformation was proposed to extend the complete stability to nonsymmetric $\mathbf{A}$ in (1). They considered matrices $\mathbf{A}$ with the property that there exist diagonal matrices $\mathbf{D} = \mathrm{diag}(d_1, d_2, \ldots, d_n)$ and $\mathbf{T} = \mathrm{diag}(\tau_1, \tau_2, \ldots, \tau_n)$ with $d_i\tau_i > 0$ so that $\mathbf{DAT}$ is symmetric. Computations show that such matrices $\mathbf{A}$ are exactly the ones satisfying (H_1) and (H_2).

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