Packing Complete Multipartite Graphs with 4-cycles

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Abstract: In this paper we completely solve the problem of finding a maximum packing of any complete multipartite graph with edge-disjoint 4-cycles, and the minimum leaves are explicitly given. © 2001 John Wiley & Sons, Inc. J Combin Designs 9: 107–127, 2001

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1. INTRODUCTION AND PRELIMINARIES

A *k-cycle packing* of a graph G is a set C of edge disjoint k-cycles in G. A k-cycle packing C of G is *maximum* if $|C| \geq |C'|$ for all other k-cycle packings C' of G. The *leave* of a k-cycle packing of G is the set of edges of G that occur in no k-cycle in C; sometimes we also refer to the subgraph induced by these edges as the leave. A *k-cycle system* of G is a k-cycle packing of G for which the leave is empty. We refer to the leave of a maximum k-cycle packing as a *minimum* leave. Also, let $K(v_1, v_2, \ldots, v_n)$ denote the complete multipartite graph with vertex set $V_1 \cup V_2 \cup \cdots \cup V_n$ and edge set E, where $|V_i| = v_i$ and E consists of all edges between vertices in V_i and V_j, $i \neq j$; there are no edges between two vertices in the same set V_i.

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TABLE I. Minimum leaves in 4-cycle packings of K_n

K_n, n mod 8:	1	2	3	4	5	6	7	0
Leave:	$\emptyset$	F	K_3	F	B, C_6 or $2K_3$	F	C_5	F

In recent years, various edge-disjoint decompositions of complete graphs and complete multipartite graphs into cycles have been investigated; see for example [6] and [2]. Moreover, maximum packings and minimum coverings of complete graphs by k-cycles for various k have also been considered; see [8], [7], and [5] for 4-, 5- and 6-cycles respectively, for instance.

The problem of partitioning the edges of a complete multipartite graph into 3-cycles has also been considered, and is proving to be an extremely difficult problem to solve. For example, one paper deals with the particular case where all parts have the same size, except possibly for one part [3]. In contrast to this, here we completely solve the problem of finding a maximum 4-cycle packing of $K(v_1, v_2, \ldots, v_n)$ (see Theorem 6.1). This generalizes the result of Cavenagh and Billington [2] which characterizes the complete multipartite graphs for which there exists a 4-cycle system.

This problem has already been solved for complete graphs; that is, when $v_1 = v_2 = \cdots = v_n = 1$. For convenience, in Table I we list the minimum leaves in this case (see [8], and also [4]). In the following, F denotes a 1-factor of the complete graph K_n when n is even, B denotes a bowtie, that is, two triangles K_3 having one common vertex, and C_i denotes a cycle of length i.

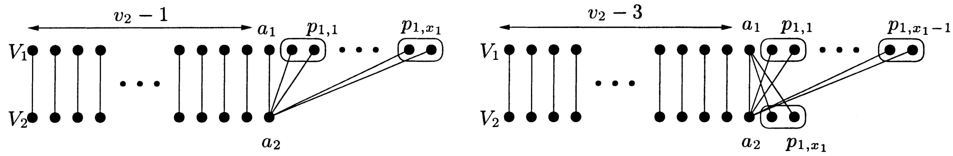
Remark 1.1. It is also possible (and will be useful in a later section) to obtain a packing of K_n with 4-cycles, having leave K_i, when $n \equiv i \pmod 8$, for $i = 1, 3, 5, 7$. (Clearly, this is not a *maximum* packing when $i = 5$ or 7, but by replacing the K_i by its maximum packing, we can obtain a maximum packing of K_n, containing a maximum packing of K_i, $i = 5$ or 7. See the inductive construction described in [4].)

One straightforward result (which is easily seen to hold for 4-cycles) follows from Sotteau [9]. This guarantees the existence of a decomposition of any complete bipartite graph into 4-cycles if and only if the two parts each have even size. We shall use this frequently in the following. This result also means that in any complete multipartite graph which has all parts of even size, there is a decomposition into 4-cycles with empty leave. We shall refer to this as the "all parts even" condition (*), and henceforth assume that at least one part has odd size.

The complement of a graph G is denoted here by $\overline{G}$. If two graphs G and H are vertex disjoint, then the *join* $G \vee H$ is formed from $G \cup H$ by joining each vertex in G to each vertex in H. For any other graph theoretic definitions, see [10].

2. THE BIPARTITE CASE

Let $K(v_1, v_2)$ be a complete bipartite graph with vertex partition $\{V_1, V_2\}$ where $|V_i| = v_i$, $i = 1, 2$. If both v_1 and v_2 are even, condition (*) ensures there exists a 4-cycle system of $K(v_1, v_2)$.



F_1 $(v_2 \equiv 1 \pmod 4)$ F_2 $(v_2 \equiv 3 \pmod 4)$

FIG. 1.

If v_1 is odd and v_2 is even, then in any minimum leave each vertex in V_2 has odd degree. Pick any vertex x_1 in V_1, and let R denote the star centered at x_1, with v_2 edges. This is our leave. What remains is the graph $K(v_1, v_2) \setminus E(R)$, which is isomorphic to $K(v_1 - 1, v_2)$, a complete bipartite graph with both parts of even size. So a 4-cycle decomposition of $K(v_1, v_2) \setminus E(R)$ follows from (*).

Now suppose that v_1 and v_2 are odd, with $v_1 \geq v_2$. In this case any minimum leave must be a spanning subgraph, with every vertex of odd degree. So, certainly the minimum number of edges in the leave is at least v_1. Thus, if $v_2 \equiv 3 \pmod 4$, any minimum leave must contain at least $v_1 + 2$ edges, in order that the number of remaining edges is 0 (mod 4). Therefore, the graph induced by the minimum leaves can be F_1 or F_2, according as $v_2 \equiv 1$ or 3 (mod 4), where F_1 and F_2 are given in Figure 1 (providing the remaining edges can be partitioned into 4-cycles).

In Figure 1, we conveniently group vertices in the component of F_1 or F_2 containing more than one edge into pairs $p_{1,i}$ as shown, together with the special pair a_1, a_2. This concept of paired vertices will also be important in Section 4 below.

Note that the number of components in both F_1 and F_2 is 1 (mod 4). In order to describe a convenient 4-cycle decomposition of $K(v_1, v_2) \setminus F_i$, $i = 1, 2$, we need the following lemma. (See also Lemma 6 of [1]; we include a brief proof below for completeness.)

Lemma 2.1. *The complete bipartite graph $K(4m + 1, 4m + 1)$ minus a perfect matching F has a decomposition into 4-cycles.*

Proof. First, when $m = 1$, a decomposition of $K_{5,5} \setminus F$ with vertex set $\{0, 1, 2, 3, 4\} \cup \{0', 1', 2', 3', 4'\}$ into 4-cycles is given cyclically by $(0, 1', 4, 3')$ (mod 5), where F is $\{\{i, i'\} \mid 0 \leq i \leq 4\}$.

Now, $K(4m + 1, 4m + 1) \setminus F$ is essentially m copies of $K_{5,5} \setminus F$ (with vertices 0 and $0'$ in each copy, and with $\{0, 0'\} \in F$), together with $m(m - 1)$ copies of $K_{4,4}$. So the result follows. □

Returning to $K(v_1, v_2) \setminus E(F_i)$, we can now apply Lemma 2.1 together with (*), and easily decompose $K(v_1, v_2) \setminus E(F_i)$ into 4-cycles. Thus we have proved:

Lemma 2.2. *The complete bipartite graph $K(v_1, v_2)$ can be decomposed into 4-cycles with leave L, where L is as follows:*

$K(v_1, v_2)$	*leave L*
$v_1 \equiv v_2 \equiv 0 \pmod 2$	$\emptyset$
$v_1 - 1 \equiv v_2 \equiv 0 \pmod 2$	*R, star with v_2 edges*
$v_1 \equiv v_2 \equiv 1 \pmod 2$, $v_1 \geq v_2 \equiv 1 \pmod 4$	F_1 (*see Figure* 1)
$v_1 \equiv v_2 \equiv 1 \pmod 2$, $v_1 \geq v_2 \equiv 3 \pmod 4$	F_2 (*see Figure* 1)

In the case $v_2 \equiv 3 \pmod 4$, *for* $1 \le i \le x_1 - 1$, *the decomposition includes the* 4*-cycle with vertex set* $p_{1,x_1} \cup p_{1,i}$.

3. AN ODD NUMBER OF PARTS, ALL OF ODD SIZE

In this case the vertices are in parts V_i, $1 \le i \le n$, with n odd and v_i odd. Let $w_i \in V_i$, for $1 \le i \le n$, and let $V_i \setminus \{w_i\}$ be denoted by Q_i. Then we may take a maximum packing as follows, with leave being exactly the same as the leave for a maximum packing of K_n with 4-cycles (see Table I).

First, on the set $\{w_i \mid 1 \le i \le n\}$, place a maximum packing of K_n with 4-cycles. Then use (*) to take a 4-cycle decomposition of the following complete bipartite graphs F_i and H_{ij}, for $1 \le i, j \le n$, $i \ne j$. The graph F_i has vertex partition $\{\{w_j \mid 1 \le j \le n, j \ne i\}, Q_i\}$, while H_{ij}, $i < j$, has vertex partition $\{Q_i, Q_j\}$.

Now each edge of the complete multipartite graph with an odd number of odd parts is used either in the leave or in a 4-cycle. Furthermore, the leave has at most six edges, so since it must be simple, the leave is a minimum leave.

We summarize this section as follows.

Lemma 3.1. *A maximum packing with* 4*-cycles of a complete multipartite graph with n parts, where all parts have odd size and where n is odd, has minimum leave exactly the same as that in a maximum packing of* K_n, *namely:* $\emptyset$; K_3; B, C_6 *or* $2K_3$; C_5, *according as* $n \equiv 1, 3, 5$ *or* $7 \pmod 8$.

Remark 3.2. We may also take a packing of a complete multipartite graph with n odd parts, where n is odd, having leave as described in Remark 1.1, namely: $\emptyset$, K_3, K_5, K_7 according as $n \equiv$ 1, 3, 5 or 7 (mod 8). (Of course, this is not a minimum leave when $n \equiv$ 5 or 7 (mod 8), but this type of leave will be useful later.)

4. AN EVEN NUMBER OF PARTS, ALL OF ODD SIZE

In this section we deal with one of the two difficult cases. We begin with some preliminary results giving 4-cycle decompositions of particular graphs which arise later.

For $1 \le i \le 4$, let p_i be a set of two of non-adjacent vertices. Let $H_1(b_1, p_1; b_2, p_2; b_3, p_3; b_4, p_4)$ denote the graph with vertex set $\{b_i \mid 1 \le i \le 4\} \cup (\bigcup_{i=1}^{4} p_i)$ and edge set consisting of the eight edges joining b_i to vertices in p_i, for $1 \le i \le 4$, together with the edges of a $K_{4,4}$ with bipartition $p_1 \cup p_2$ and $p_3 \cup p_4$ (see Fig. 2). Here possibly $b_1 = b_2$, and possibly $b_3 = b_4$. This graph H_1 contains 24 edges, and the degrees of the 8 vertices in $\bigcup_{i=1}^{4} p_i$ are all odd.

Lemma 4.1. *The graph* $H_1(b_1, p_1; b_2, p_2; b_3, p_3; b_4, p_4)$ *has a* 4*-cycle packing with the leave perfectly matching between* $p_1 \cup p_2$ *and* $p_3 \cup p_4$.

Proof. Letting $p_i = \{c_{i,1}, c_{i,2}\}$, $1 \le i \le 4$, the leave is the four edges $\{c_{1,1}, c_{4,2}\}$, $\{c_{1,2}, c_{3,2}\}, \{c_{2,1}, c_{4,1}\}, \{c_{2,2}, c_{3,1}\}$, and the 4-cycles are $(b_1, c_{1,1}, c_{3,1}, c_{1,2})$, $(b_2, c_{2,1}, c_{4,2}, c_{2,2}), (b_3, c_{3,1}, c_{2,1}, c_{3,2}), (b_4, c_{4,1}, c_{1,2}, c_{4,2}), (c_{1,1}, c_{3,2}, c_{2,2}, c_{4,1})$. (see Fig. 2.).

□

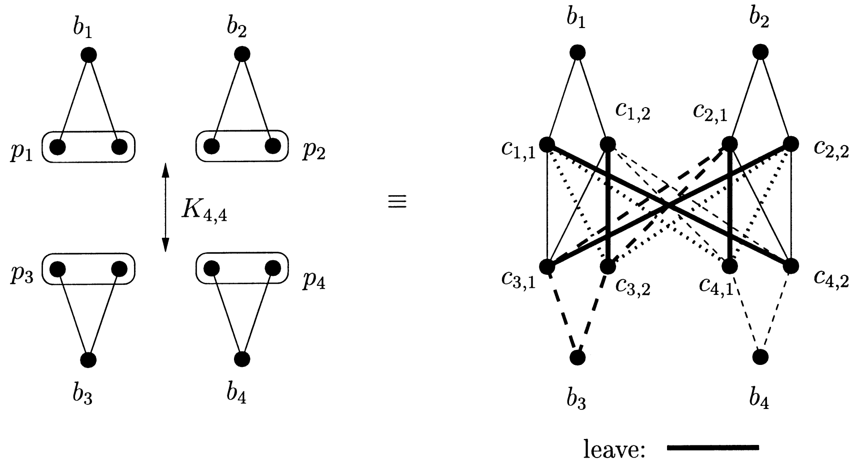


FIG. 2. $H_1(b_1, p_1; b_2, p_2; b_3, p_3; b_4, p_4)$ with its maximum 4-cycle decomposition.

We now define the graph $H_2(W_1, W_2, W_3; b_1, b_2; c_1, c_2, c_3, c_4)$ as follows (see Fig. 3). The sets W_1 and W_2 each consists of four different vertices, with a copy of $K_{4,4}$ joining them. Vertex b_i is joined by an edge to each of the four vertices in W_i, $i = 1, 2$. The set W_3 consists of eight independent vertices, paired as p_1, p_2, p_3, p_4, so that c_i is joined by edges to the two vertices in p_i, $1 \le i \le 4$. Finally, H_2 contains a copy of $K_{8,8}$ with bipartition $W_1 \cup W_2$ and W_3. Note that possibly the vertices c_1, c_2, c_3, c_4 are not all distinct (see Fig. 3).

Lemma 4.2. *The graph $H_2(W_1, W_2, W_3; b_1, b_2; c_1, c_2, c_3, c_4)$ has a 4-cycle packing with the leave being a perfect matching between $W_1 \cup W_2$ and W_3.*

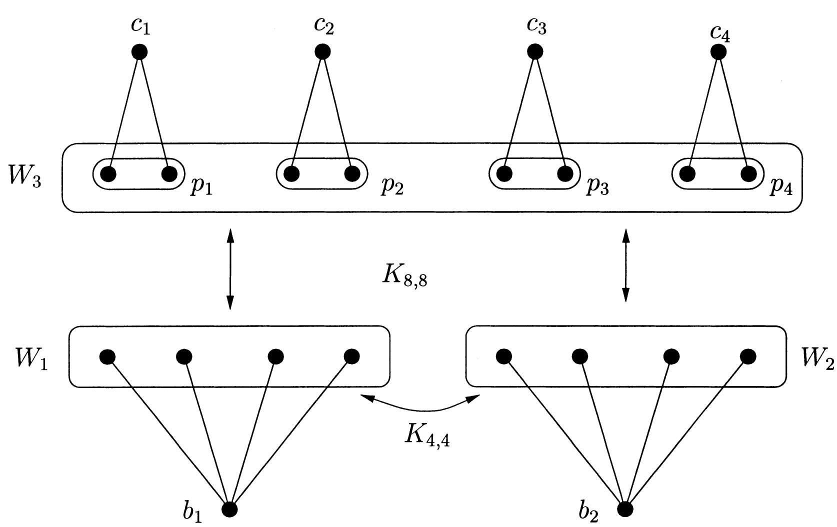


FIG. 3. The graph $H_2(W_1, W_2, W_3; b_1, b_2; c_1, c_2, c_3, c_4)$.

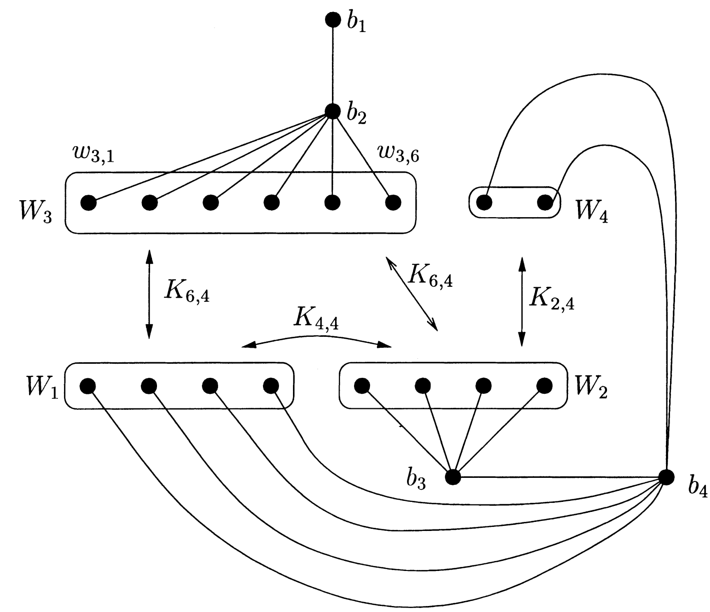


FIG. 4 The graph $H_3(W_1, W_2, W_3, W_4; b_1, b_2, b_3, b_4)$.

Proof. The graph H_2 is made up of: $H_1(c_1, p_1; c_2, p_2; b_1, p_5; b_1, p_6)$ (where $W_1 = p_5 \cup p_6$); $H_1(c_3, p_3; c_4, p_4; b_2, p_7; b_2, p_8)$ (where $W_2 = p_7 \cup p_8$); and three copies of $K_{4,4}$, one from W_1 to W_2, one from $p_1 \cup p_2$ to W_2, and one from $p_3 \cup p_4$ to W_1. Therefore, the result follows from Lemma 4.1 and the fact that $K_{4,4}$ is trivially decomposable into 4-cycles. □

The next lemma is similiar to the previous two in flavor, and is needed subsequently in one particular case. We define the graph $H_3(W_1, W_2, W_3, W_4; b_1, b_2; b_3, b_4)$ on 20 vertices as follows (see Fig. 4). The sets $W_1 = \{w_{1,i} \mid 1 \leq i \leq 4\}$ and $W_2 = \{w_{2,i} \mid 1 \leq i \leq 4\}$ each consists of four independent vertices, with a copy of $K_{4,4}$ joining them. Vertex b_4 is joined to the four vertices in W_1, vertex b_3 is joined to the five vertices in $W_2 \cup \{b_4\}$. The set W_4 consists of two independent vertices, each joined to b_4. The set W_3 consists of six independent vertices, $w_{3,i}$, $1 \leq i \leq 6$, all joined to b_2, and b_2 is joined to b_1. Finally, H_3 contains copies of $K_{6,4}$, $K_{6,4}$, and $K_{2,4}$, with bipartitions $\{W_3, W_1\}$, $\{W_3, W_2\}$, and $\{W_4, W_2\}$, respectively.

Lemma 4.3. *The graph $H_3(W_1, W_2, W_3, W_4; b_1, b_2; b_3, b_4)$ can be decomposed into 4-cycles with the leave being a 1-factor consisting of $\{b_1, b_2\}$, $\{b_3, b_4\}$, four edges between $\{w_{3,1}, w_{3,2}, w_{3,3}, w_{3,4}\}$ and W_1, and four edges between $\{w_{3,5}, w_{3,6}\} \cup W_4$ and W_2.*

Proof. The graph H_3 is made up of: $H_1(b_2, \{w_{3,1}, w_{3,2}\}; b_2, \{w_{3,3}, w_{3,4}\}; b_4, \{w_{1,1}, w_{1,2}\}; b_4, \{w_{1,3}, w_{1,4}\})$; $H_1(b_2, \{w_{3,5}, w_{3,6}\}; b_4, W_4; b_3, \{w_{2,1}, w_{2,2}\}; b_3, \{w_{2,3}, w_{2,4}\})$; a copy of $K_{8,4}$ joining $W_1 \cup \{w_{3,1}, w_{3,2}, w_{3,3}, w_{3,4}\}$ to W_2; a copy of $K_{2,4}$ joining $\{w_{3,5}, w_{3,6}\}$ to W_1; and the two edges b_1b_2 and b_3b_4. Thus the decomposition into 4-cycles follows from Lemma 4.1 and condition (*). □

Lemma 4.4. *The graph $\overline{K}_9 \vee K_9$ has a 4-cycle packing with the leave being a perfect matching of nine edges.*

Proof. The graph K_9 can be packed with 4-cycles (with empty leave), since $9 \equiv 1 \pmod 8$. Also, $K_{9,9}$ can be decomposed into one perfect matching and a collection of 4-cycles (Lemma 2.1). So the result follows. □

Let $H_4(b_1, b_2, \ldots, b_7; b_8, \ldots, b_{11}; b_{12}, b_{13}, b_{14})$ denote the graph $\overline{K}_7 \vee (K_4 \vee \overline{K}_3)$, where $V(\overline{K}_7) = \{b_1, \ldots, b_7\}$, $V(K_4) = \{b_8, \ldots, b_{11}\}$, and $V(\overline{K}_3) = \{b_{12}, b_{13}, b_{14}\}$.

Lemma 4.5. *The graph $H_4(b_1, \ldots, b_7; b_8, \ldots, b_{11}; b_{12}, b_{13}, b_{14})$ has a 4-cycle packing, with the leave being a perfect matching of seven edges.*

Proof. Begin with a 4-cycle packing of $K_{7,7}$, with bipartition of the vertices being $\{\{b_1, \ldots, b_7\}, \{b_8, \ldots, b_{14}\}\}$, so that the leave is the set of edges $\{\{b_1, b_8\}, \{b_2, b_9\}, \{b_3, b_{10}\}, \{b_4, b_{11}\}, \{b_5, b_{12}\}, \{b_5, b_{13}\}, \{b_5, b_{14}\}, \{b_6, b_{12}\}, \{b_7, b_{12}\}\}$, and so that $(b_6, b_{13}, b_7, b_{14})$ is one of the 4-cycles (see Lemma 2.2). Remove the 4-cycle $(b_6, b_{13}, b_7, b_{14})$; the leave from $K_{7,7}$ now consists of seven copies of K_2 together with one 6-cycle $c_1 = (b_5, b_{13}, b_7, b_{12}, b_6, b_{14})$.

We also have a partition of $E(K_7 \setminus K_3)$, with $V(K_7 \setminus K_3) = \{b_8, b_9, \ldots, b_{14}\}$ and $V(K_3) = \{b_{12}, b_{13}, b_{14}\}$, which induces three 4-cycles and one 6-cycle: $\{(b_8, b_9, b_{12}, b_{10}), (b_9, b_{10}, b_{11}, b_{14}), (b_8, b_{11}, b_9, b_{13}),$ and $c_2 = (b_8, b_{12}, b_{11}, b_{13}, b_{10}, b_{14})\}$.

The edges in the two 6-cycles c_1 and c_2 together form three 4-cycles: $(b_5, b_{13}, b_{10}, b_{14})$, $(b_6, b_{14}, b_8, b_{12})$, $(b_7, b_{12}, b_{11}, b_{13})$. The only remaining leave is now $\{\{b_i, b_{i+7}\} \mid 1 \leq i \leq 7\}$. □

We are now ready to prove our main result in this section.

Theorem 4.6. *Suppose G is a complete multipartite graph with $2z$ parts $V_1, \ldots, V_{2z}$, where $|V_i| = v_i$ is odd for $1 \leq i \leq 2z$. Also, let ν be the number of vertices in the largest part, and let $\eta = \sum_{i=1}^{2z} v_i$. There exists a maximum 4-cycle packing in which the leave L satisfies either*

(A) $|L| \leq \frac{\eta}{2} + 3$, *or*
(B) $|L| \leq \nu + 3$.

(*Each leave constructed here induces one of the graphs in Fig. 5.*)

Proof. Since every vertex in G has odd degree, we first point out that any leave L will be a spanning subgraph with all vertices of odd degree; so, clearly L is a minimum leave if $|L| \leq \max\{\eta/2 + 3, \nu + 3\}$.

We pair the parts of G, V_{2i-1} with V_{2i} for $1 \leq i \leq z$; for convenience we label them $v_{2i-1} \geq v_{2i}$ for each pair of parts V_{2i-1}, V_{2i}.

For $1 \leq i \leq z$, apply Lemma 2.2 to the bipartite graph with vertex partition V_{2i-1}, V_{2i} to obtain a maximum 4-cycle packing, B_i, and let G_i be the graph containing the edges in the leave of B_i (the 4-cycles in B_i might not be part of our final set). Note that G_i contains v_{2i-1} edges if $v_{2i} \equiv 1 \pmod 4$, and $v_{2i-1} + 2$ edges if $v_{2i} \equiv 3 \pmod 4$. For $1 \leq i \leq z$, G_i consists of ε_i copies of K_2 (where necessarily $\varepsilon_i \equiv 0 \pmod 4$) together with a final component Z_i (see Fig. 1). If $v_{2i-1} = v_{2i} \equiv 1$

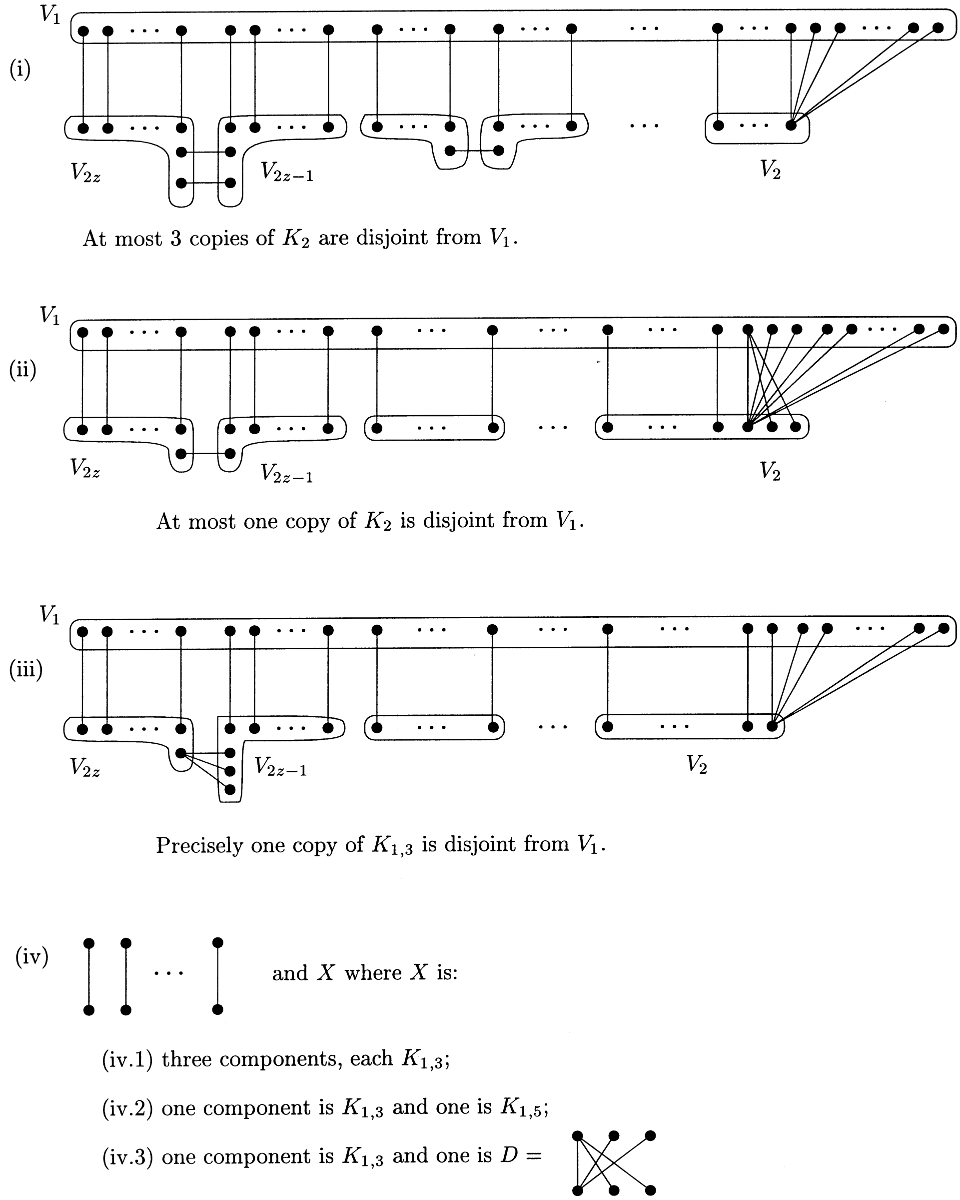


FIG. 5. Leaves: $2z$ parts, all of odd size.

(mod 4), then $Z_i \cong K_2$; otherwise, Z_i is the star R_i if $v_{2i} \equiv 1 \pmod 4$ and $v_{2i-1} > v_{2i}$, and $Z_i \cong D_i$ if $v_{2i} \equiv 3 \pmod 4$ (see Fig. 6). We now introduce the notation used for certain vertices in Figure 6. In the star R_i, pair off all but one of the vertices of degree 1 in V_{2i-1} into sets $p_{i,1}, p_{i,2}, \ldots, p_{i,x_i}$, and in D_i pair off all of the vertices of degree 1 in V_{2i-1} into sets $p_{i,1}, \ldots, p_{i,x_i-1}$ and let p_{i,x_i} be the pair of vertices of degree 1 in V_{2i} in D_i. In any case, let a_{2i-1} and a_{2i} in Z_i be the unpaired vertices in $Z_i \cap V_{2i-1}$ and $Z_i \cap V_{2i}$, respectively. Note that in each leave the two vertices in each pair $p_{i,j}$ have a

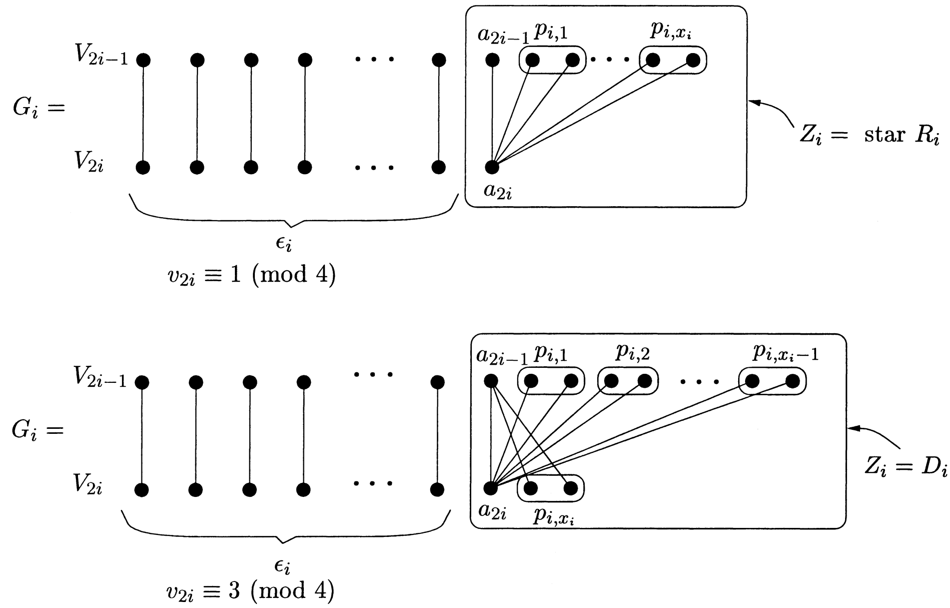


FIG. 6. In $D_i, p_{i,j} \cup p_{i,x_i}$ induces a 4-cycle in B_i for $1 \leq j < x_i$.

common neighbor. Note also that if $p_{i,j} = \{x_1, y_1\}$ and $p_{i,x_i} = \{x_2, y_2\}$ occur in *different* parts in Z_i (see D_i in Fig. 6), then by Lemma 2.2 the 4-cycle (x_1, x_2, y_1, y_2) is in B_i.

Now we concentrate on the pairs $p_{i,j}$, $1 \leq i \leq z$, $1 \leq j \leq x_i$. We partition as many of these pairs as possible into sets $S_1, S_2, \ldots, S_y$ of size 4, with the property that for $1 \leq k \leq 2z$ and $1 \leq l \leq y$, the set S_l contains at most two pairs from each part V_k. Let S denote the set of remaining pairs which do not occur in $\bigcup_{l=1}^{y} S_l$. Then S must satisfy:

(a) $|S| \leq 3$, or
(b) $|S| \geq 4$ and all pairs in S, except possibly one, belong to the same part, say V_1, in G_1.

In both cases (a) and (b), if $S_l = \{p_{i_1,m_1}, p_{i_2,m_2}, p_{i_3,m_3}, p_{i_4,m_4}\}$, then we apply Lemma 4.1 to the graph $H_1(b_1, p_{i_1,m_1}; b_2, p_{i_2,m_2}; b_3, p_{i_3,m_3}; b_4, p_{i_4,m_4})$ where $b_t = a_{2i_t-1}$ or a_{2i_t} according as p_{i_t,m_t} is in V_{2i_t} or V_{2i_t-1} for $1 \leq t \leq 4$ (so possibly $b_1 = b_2$, possibly $b_3 = b_4$, possibly $i_1 = i_2$, and possibly $i_3 = i_4$). Let $B_{1,l}$ denote the set of five 4-cycles obtained by applying Lemma 4.1 to S_l in this way, and let L_l denote the four edges in the leave. These are placed in our final set of 4-cycles, B, and our final leave L, respectively.

Now for convenience, let E_i denote the set of ϵ_i copies of K_2 in the leave $G_i \setminus Z_i$, $1 \leq i \leq z$. (Henceforth we also think of an edge such as $\{a_{2i-1}, a_{2i}\}$ as a 2-element set of vertices.) Then we partition all the vertices in V into 2-element subsets: let

$$\mathcal{P} = \{e \in E_i,\ p_{i,j},\ \{a_{2i-1}, a_{2i}\} \mid 1 \leq i \leq z, 1 \leq j \leq x_i\}.$$

In case (a), for every two distinct pairs $\{q_1, q_2\}$ and $\{q_3, q_4\}$ in $\mathscr{P}$ that are not both in S_l for $1 \leq l \leq y$, and not both contained in the same $V(G_i)$, $1 \leq i \leq z$, we take the 4-cycle (q_1, q_3, q_2, q_4) and place it in B. In case (a) we complete forming B by adding all the 4-cycles in $\bigcup_{i=1}^{z} B_i$, except those 4-cycles which have been used in taking care of the pairs in the S_l. Let $G_i' = (G_i \setminus Z_i)$, and let L_i' be formed from $E(G_i')$ by adding $\{a_{2i-1}, a_{2i}\}$ and any edges in G_i that are incident with vertices in pairs in S. Then the leave of B is $L = (\bigcup_{l=1}^{y} L_l) \cup (\bigcup_{i=1}^{z} L_i')$, and $|L| = \eta/2 + |S| \leq \eta/2 + 3$.

The minimum leave is given in Figure 5; we comment further on this at the end of the proof.

Case (b) remains. By choosing y to be maximal, we can assume (for $1 \leq l \leq y$) that each S_l contains *exactly* two pairs in V_1 in G_1. (For, if S_l contains at most one pair from V_1 then we can replace any pair in S_l that is not in V_1 with a pair from S that is in V_1.) So we can assume that V_1 is the largest part, and thus $v_1 = \nu$.

Recall that for $1 \leq i \leq z$, $E_i = E(G_i) \setminus E(Z_i)$, $\varepsilon_i = |E_i|$, and $\varepsilon_i \equiv 0 \pmod 4$. Let

$$\alpha = \min\left\{ \left\lfloor \frac{|S|}{4} \right\rfloor, \sum_{i=2}^{z} \frac{\varepsilon_i}{4} \right\}.$$

(Thus α is the minimum of the number of disjoint sets of four distinct pairs that are in S, and the number of disjoint sets of four copies of K_2 that are in $\bigcup_{i=2}^{z} G_i \setminus Z_i$.)

Now select α pairwise disjoint sets $T_1, \ldots, T_\alpha$, where each T_j contains four edges from E_i, for some i, $2 \leq i \leq z$, as well as four pairs from S.

Consider T_j. Let $e_1, \ldots, e_4$ be the four edges from E_i in T_j, so each joins a vertex in part V_{2i-1} to a vertex in V_{2i}, for some $i \geq 2$. Also, let $p_{1,j_1}, p_{1,j_2}, p_{1,j_3}, p_{k,j_4}$ be the four pairs from S in T_j. Here possibly $k = 1$ with p_{k,j_4} in V_1 or in V_2, or $k > 1$, in which case without loss of generality we say p_{k,j_4} is in V_3. So the possibilities for T_j are:

- T(i) 4 pairs in T_j all from V_1, 4 edges $e_1, \ldots, e_4$ all from G_i, for some i, $2 \leq i \leq z$.
- T(ii) 3 pairs in T_j from V_1, one pair from V_2, 4 edges $e_1, \ldots, e_4$ from G_i, for some i, $2 \leq i \leq z$.
- T(iii) 3 pairs in T_j from V_1, one pair from V_3, 4 edges $e_1, \ldots, e_4$ from G_i, for some i, $3 \leq i \leq z$.
- T(iv) 3 pairs in T_j from V_1, one pair from V_3, 4 edges $e_1, \ldots, e_4$ from G_2 (based on $V_3 \cup V_4$).

In cases T(i), T(ii), and T(iii) above, we apply Lemma 4.2 to the graph $H_2(W_1, W_2, W_3; b_1, b_2; c_1, c_2, c_3, c_4)$ where: $c_1 = c_2 = c_3 = a_2$; in cases T(i), T(ii), and T(iii), $c_4 = a_2, a_1$, and a_4, respectively; $W_1 = (\bigcup_{j=1}^{4} e_j) \cap V_{2i-1}$ and $W_2 = (\bigcup_{j=1}^{4} e_j) \cap V_{2i}$ (regarding e_j as a set of two vertices); $W_3 = \bigcup_{l=1}^{3} p_{1,j_l} \cup p_{k,j_4}$; and $b_1 = a_{2i}$ and $b_2 = a_{2i-1}$. If $k \neq 1$ then for $1 \leq l \leq 3$ add the 4-cycle between p_{1,j_l} and p_{k,j_4} to the set of 4-cycles that arises in this way from T_j, and call the resulting set $B_{2,j}$. Let $B_{2,j} \subseteq B$, and place the leave from Lemma 4.2 into L.

In case T(iv) above, we apply Lemma 4.3 to the graph $H_3(W_1, W_2, W_3, W_4; b_1, b_2; b_3, b_4)$ where $W_1 = (\bigcup_{j=1}^{4} e_j) \cap V_3$, $W_2 = (\bigcup_{j=1}^{4} e_j) \cap V_4$, $W_3 = \bigcup_{l=1}^{3} p_{1,j_l}$, $W_4 = p_{2,j_4}$, and $b_i = a_i$ for $1 \leq i \leq 4$. Again, for $1 \leq l \leq 3$, add the 4-cycle between

p_{2,j_4} and p_{1,j_l} to the set of 4-cycles obtained from Lemma 4.3 to form $B_{2,j}$. Let $B_{2,j} \subseteq B$, and place the leave from Lemma 4.3 into L.

Let $S' = S \setminus \{p_{i,l} \mid p_{i,l} \in T_j \text{ for some } j, 1 \le j \le \alpha\}$, and let $A = \{\{a_{2i-1}, a_{2i}\} \mid 2 \le i \le z\}$. Choose β pairwise disjoint sets $U_1, \ldots, U_\beta$ with β as large as possible so that for $1 \le j \le \beta$

- U(i) U_j contains 4 pairs in S', each of which is a subset of V_1, and 4 edges in A, or
- U(ii) U_j contains 4 pairs in S', three of which are subsets of V_1 and one is a subset of V_2, and 2 edges in A, or
- U(iii) U_j contains 4 pairs in S', three of which are subsets of V_1 and one is a subset of V_j with $j \ge 3$ (say $j = 3$), and 2 edges in A, one of which is $\{a_3, a_4\}$.

Notice that since S contains at most one pair that is not a subset of V_1 (in case (b)), we have at most one occurrence of cases U(ii) and U(iii); in such a case we can assume that the edges in A are $\{a_3, a_4\}$ and $\{a_5, a_6\}$.

If $U_j = \{p_{1,j_l}, \{a_3, a_4\}, \{a_5, a_6\} \mid 1 \le l \le 4\}$ is formed in case U(ii), then let $p_{1,j_l} = \{p^1_{1,j_l}, p^2_{1,j_l}\}$ for $1 \le l \le 4$ with $p_{1,j_4} \subseteq V_2$. Let $B_{3,j}$ be the set of 4-cycles obtained by applying Lemma 4.5 to the graph $H_4(a_1, p^1_{1,j_1}, p^2_{1,j_1}, p^1_{1,j_2}, p^2_{1,j_2}, p^1_{1,j_3}, p^2_{1,j_3}; a_3, a_4, a_5, a_6; a_2, p^1_{1,j_4}, p^2_{1,j_4})$, and let $L_{3,j}$ be the leave.

If $U_j = \{p_{1,j_l}, p_{2,j_4}, \{a_3, a_4\}, \{a_5, a_6\} \mid 1 \le l \le 3\}$ is formed in case U(iii) then let $p_{i,j_l} = \{p^1_{i,j_l}, p^2_{i,j_l}\}$ for $1 \le j \le 4$ and $i \in \{1, 2\}$, with $p_{2,j_4} \subseteq V_3$. Let $B_{3,j}$ be the set of 4-cycles obtained by applying Lemma 4.5 to the graph $H_4(a_1, p^1_{1,j_1}, p^2_{1,j_1}, p^1_{1,j_2}, p^2_{1,j_2}, p^1_{1,j_3}, p^2_{1,j_3}; a_3, a_4, a_5, a_6; a_2, p^1_{2,j_4}, p^2_{2,j_4})$, and let $L_{3,j}$ be the leave.

Finally, if $U_j = \{p_{1,j_l}, e_l \mid 1 \le l \le 4\}$ then let $B_{3,j}$ be the set of 4-cycles obtained by applying Lemma 4.4 to the graph $\overline{K}_9 \vee K_9$ with $V(\overline{K}_9) = \{a_1\} \cup (\bigcup_{j=1}^4 p_{1,j})$ and $V(K_9) = a_2 \cup (\bigcup_{j=1}^4 e_j)$, and let $L_{3,j}$ be the leave.

For $1 \le j \le \beta$ let $B_{3,j} \subseteq B$ and let $L_{3,j} \subseteq L$.

Now let $V'_j, 1 \le j \le 2z$, be the subset of V_j containing all vertices that are *not* incident with an edge in $\bigcup_{j=1}^{\alpha} T_j$. Also, let B'_i be the set of 4-cycles in a maximum packing of the complete bipartite graph $V'_{2i-1} \cup V'_{2i}$, chosen so that the leave G'_i satisfies $G'_i \subseteq G_i$; this is possible since $|V_j| \equiv |V'_j| \pmod 4$ for $1 \le j \le 2z$. Let $B'_i \subseteq B$.

All that remains is to include those 4-cycles arising from decompositions of various complete bipartite subgraphs, all of whose parts now have even size (so condition (*) applies). First, for $2 \le i \le z$, we place in B the 4-cycles of a 4-cycle system of each of the two complete bipartite graphs, one with bipartition $V'_{2i-1} \setminus \{a_{2i-1}\}$ and $V_{2i} \setminus V'_{2i}$, the other with bipartition $V'_{2i} \setminus \{a_{2i}\}$ and $V_{2i-1} \setminus V'_{2i-1}$. Secondly, recall that V is partitioned into 2-element subsets, $\mathscr{P} = \{e \in E_i, p_{i,j}, \{a_{2i-1}, a_{2i}\} \mid 1 \le j \le x_i, 1 \le i \le z\}$. Let $\{q_1, q_2\}$, $\{q_3, q_4\}$ be any two of the 2-element subsets in $\mathscr{P}$ satisfying:

1. $\{\{q_1, q_2\}, \{q_3, q_4\}\} \not\subseteq S_l$, $1 \le l \le y$,
2. $\{\{q_1, q_2\}, \{q_3, q_4\}\} \not\subseteq T_i$, $1 \le i \le \alpha$,
3. $\{\{q_1, q_2\}, \{q_3, q_4\}\} \not\subseteq U_i$, $1 \le i \le \beta$, and
4. $\{q_1, q_2, q_3, q_4\} \not\subseteq V(G_i)$ for $1 \le i \le z$,

(again, regarding the edges in S_l, T_i, and U_i as 2-element subsets). Then we place the 4-cycle (q_1, q_3, q_2, q_4) in B.

Below we comment on the leaves described in Figure 5. Recall that since every vertex must have odd degree in the leave, it is clear that if

(a) the number of edges is less than $\frac{\eta}{2}+4$, or
(b) the number of edges is less than $\nu+4$,

then the leave is a minimum. In the case (a) above (when $|S| \leq 3$), any of the leaves (i)–(iv) can arise, while in case (b) above (when $|S| \geq 4$), one of the leaves (i)–(iii) arises. Moreover, cases (i), (ii), and (iii) of Figure 5 satisfy (B), while case (iv) satisfies (A). Thus in all cases the resulting leave is a minimum, and we have achieved our desired maximum packing.

This concludes the proof of the theorem. □

5. PARTS OF BOTH EVEN AND ODD SIZES

Here we deal with the final case, which in retrospect is probably the most difficult case, but easier to read!

First, note that if the number of parts, say *t*, of odd size is *even* (and possibly zero), then this case essentially reduces to that in Section 4 above as the following shows. Let $V_1, \ldots, V_s$ be the parts of even size, and let $V_{s+1}, \ldots, V_{s+t}$ be the parts of odd size. We use condition (*) on the bipartite graphs with bipartition $\{V_i, V_j\}$ for $1 \leq i < j \leq s$, and also on the complete bipartite graph with bipartition $\{V_1 \cup V_2 \cup \cdots \cup V_s, V_{s+1} \cup V_{s+2} \cup \cdots \cup V_{s+t}\}$. Finally, Theorem 4.6 above deals with the complete multipartite graph that remains, on the parts $V_{s+1}, \ldots, V_{s+t}$. The resulting leave is clearly a minimum since it satisfies (A) or (B), where η is now the number of vertices of odd degree, and ν is the size of the largest part.

However, if the number of odd parts is *odd*, the above simple approach needs modification. First we give some useful lemmas.

Lemma 5.1. *The graph $K(4,4,1)$ can be decomposed into five 4-cycles and four independent edges.*

Proof. Let the vertex set be partitioned as $\{\{a_1, a_2, a_3, a_4\}, \{b_1, b_2, b_3, b_4\}, \{z\}\}$. Then the decomposition is (z, a_1, b_3, a_2), (z, a_3, b_2, a_4), (z, b_1, a_4, b_3), (z, b_2, a_1, b_4), (a_2, b_1, a_3, b_4), and the edges in $\{\{a_i, b_i\} \mid 1 \leq i \leq 4\}$. □

Lemma 5.2. *The graph K_7 minus one edge can be decomposed into four 4-cycles and a path of length 4.*

Proof. We may pack K_7 with 4-cycles with minimum leave one 5-cycle (see Table I). Let the removed edge be from this 5-cycle, and the result follows. □

Lemma 5.3. *The graph K_9 minus one edge can be decomposed into eight 4-cycles and a path of length 3.*

Proof. A maximum packing of K_9 with 4-cycles has empty leave (see Table I). So removing one edge from one 4-cycle produces the required leave of a path of length 3. □

Lemma 5.4. *Let K_7 be defined on the vertex set $\{a_1, \ldots, a_7\}$. Then K_7 minus two vertex-disjoint edges $\{a_1, a_2\}$ and $\{a_3, a_4\}$ can be decomposed into four 4-cycles with leave $\{\{a_1, a_3\}, \{a_2, a_5\}, \{a_4, a_5\}\}$.*

Proof. Take the 4-cycles $(a_2, a_3, a_5, a_6), (a_1, a_4, a_7, a_5), (a_1, a_6, a_3, a_7)$, (a_2, a_4, a_6, a_7); this leaves edges $\{a_1, a_3\}, \{a_2, a_5\}$ and $\{a_4, a_5\}$. □

Lemma 5.5. *The graph K_9 minus two vertex-disjoint edges $\{a_1, a_2\}$ and $\{a_3, a_4\}$ can be decomposed into eight 4-cycles with leave $\{\{a_1, a_3\}, \{a_2, a_4\}\}$.*

Proof. Take a 4-cycle decomposition of K_9 (with empty leave) containing the 4-cycle (a_1, a_2, a_4, a_3). Then removal of the edges $\{a_1, a_2\}$ and $\{a_3, a_4\}$ yields the result. □

Lemma 5.6. *The graph K_{11} minus two vertex-disjoint edges $\{a_1, a_2\}$ and $\{a_3, a_4\}$ can be decomposed into twelve 4-cycles and leave the five edges $e_1 = \{a_1, a_3\}$, $e_2 = \{a_2, a_4\}$ and three edges that induce a copy of K_3 that is vertex-disjoint from $e_1 \cup e_2$.*

Proof. Start with a 4-cycle packing of K_{11}, which contains thirteen 4-cycles and has leave K_3 (see Table I). We choose one 4-cycle vertex-disjoint from the K_3 leave, and label it (a_1, a_2, a_4, a_3) so that removal of the two disjoint edges leaves the opposite edges, $\{a_1, a_3\}, \{a_2, a_4\}$. We can certainly find such a disjoint 4-cycle, since the vertices in the K_3 leave are each in four 4-cycles, whereas the packing of K_{11} contains thirteen 4-cycles, and $13 > 4 \times 3$. □

Lemma 5.7. *There exists a packing of $K(4, 4, 2, 1, 1, 1)$ with 4-cycles in which the leave L is a matching of five edges that saturates the ten vertices of odd degree.*

Proof. Let $v_1 = v_2 = 4$, $v_3 = 2$, and $v_4 = v_5 = v_6 = 1$. Let $V_i = \{w_{i,j} \mid 1 \le j \le v_i\}$ for $1 \le i \le 6$. Let B_1 be a set of five 4-cycles that packs $K_{5,5}$ with bipartition $\{V_1 \cup \{w_{4,1}\}, V_2 \cup \{w_{5,1}\}\}$ in which the leave is $\{\{w_{1,j}, w_{2,j}\}, \{w_{1,4}, w_{5,1}\}, \{w_{2,4}, w_{4,1}\} \mid 1 \le j \le 3\}$. (Only some of this leave is in the final leave.) Let B_2 be the set of 4-cycles in a 4-cycle system of $K_{2,6}$ with bipartition $\{V_3, \{w_{i,j} \mid 1 \le i \le 2, 1 \le j \le 3\}\}$. Let $B_3 = \{(w_{i,1}, w_{6,1}, w_{i,2}, w_{3+i,1}),\ (w_{i,3}, w_{6,1}, w_{3,i}, w_{i+3,1}),\ (w_{1,4}, w_{6,1}, w_{4,1}, w_{3,2}),\ (w_{2,4}, w_{6,1}, w_{5,1}, w_{3,1}),\ (w_{1,4}, w_{4,1}, w_{2,4}, w_{5,1}) \mid 1 \le i \le 2\}$. Then $B_1 \cup B_2 \cup B_3$ is a set of 4-cycles that packs $K(4, 4, 2, 1, 1, 1)$ with leave $L = \{\{w_{1,j}, w_{2,j}\}, \{w_{i,4}, w_{3,i}\} \mid 1 \le j \le 3, 1 \le i \le 2\}$. □

Lemma 5.8. *There exists a packing of $K(4, 4, 1, 1, 1, 1, 1)$ with 4-cycles in which the leave L is six edges: two independent edges, and four more that induce a star.*

Proof. Let the partition of the vertex set be $\{i_1 \mid 1 \le i \le 4\}$, $\{i_2 \mid 1 \le i \le 4\}$, and $\{z_i\}$ for $1 \le i \le 5$. Then on $\{3_1, 4_1, 3_2, 4_2, z_1, \ldots, z_5\}$ we place a decomposition of K_9 into 4-cycles, with one cycle being $(3_1, 4_1, 4_2, 3_2)$. The edges $\{3_1, 4_1\}$ and $\{3_2, 4_2\}$, are removed (since they do not belong to our original graph) and the edges $\{3_1, 3_2\}$, $\{4_1, 4_2\}$ become part of our leave L.

Now, on the bipartite graph $K_{4,4}$ with parts $\{1_1, 2_1, 1_2, 2_2\}$ and $\{z_2, z_3, z_4, z_5\}$, we place a 4-cycle decomposition. We also take 4-cycles $(1_1, 3_2, 2_1, 4_2)$, $(3_1, 1_2, 4_1, 2_2)$, and $(1_1, 1_2, 2_1, 2_2)$. The edges $\{x, z_1\}$ for $x = 1_1, 2_1, 1_2, 2_2$ remain; these also form part of our leave L. □

Lemma 5.9. *There exists a packing of* $K(4,4,1,1,1,1,1,1,1)$ *with* 4*-cycles in which the leave is five edges: three independent edges, and two more that induce a star.*

Proof. Let the vertex set, with partition, be $\{i_1 \mid 1 \leq i \leq 4\}$, $\{i_2 \mid 1 \leq i \leq 4\}$, $\{z_i\}$ for $1 \leq i \leq 7$. On the set $\{1_1, 1_2, z_i \mid 1 \leq i \leq 7\}$ we place a 4-cycle decomposition of K_9, ensuring that $(1_1, z_1, 1_2, z_2)$ is one of the cycles; this cycle we remove. Call the set of eight remaining cycles B_1. We place the edges $\{1_1, z_1\}$, $\{1_2, z_1\}$ into the leave L, and retain edges $\{1_1, z_2\}$, $\{1_2, z_2\}$ for later use. Next, on the bipartite graph $K_{6,6}$ with vertex partition $\{\{2_i, 3_i, 4_i \mid i = 1, 2\}, \{z_1, z_3, z_4, z_5, z_6, z_7\}\}$, we place a 4-cycle decomposition, say B_2.

The remaining edges partition into five further 4-cycles, B_3, where

$$B_3 = \{(z_2, 1_1, 3_2, 2_1),\ (z_2, 3_1, 1_2, 4_1),\ (z_2, 1_2, 2_1, 4_2),\ (z_2, 2_2, 4_1, 3_2),\ (1_1, 2_2, 3_1, 4_2)\},$$

leaving edges $\{\{x_1, x_2\} \mid 2 \leq x \leq 4\}$ to form a further part of the leave. Now $B_1 \cup B_2 \cup B_3$ is a set of twenty two 4-cycles that pack $K(4,4,1,1,1,1,1,1,1)$, with minimum leave $L = \{\{1_1, z_1\}, \{1_2, z_1\}, \{x_1, x_2\} \mid 2 \leq x \leq 4\}$. □

Theorem 5.10. *Suppose that G is a complete multipartite graph with* $s \geq 1$ *even parts and t odd parts, where t is odd. Let the even sized parts have sizes* $v_1 \geq v_2 \geq \cdots \geq v_s$*, and let* $\eta = \sum_{i=1}^{s} v_i$ *be the number of vertices of odd degree in G. There exists a maximum* 4*-cycle packing B of G with leave L in which either*

(A) $|L| \leq \frac{\eta}{2} + 3$*, or*
(B) $|L| \leq v_1 + 3$.

Proof. As observed in the proof of Theorem 4.6, if L satisfies (A) or (B) then it is a minimum leave. Let E denote the set of vertices in the even parts and O the set of vertices in the odd parts.

First we deal with $s = 1$, the case of only one even-sized part. We pack the odd parts with 4-cycles with leave as described in Remark 3.2, and then use condition $(*)$ from E to $O \setminus \{z\}$ where z is one vertex in an odd-sized part. We possibly modify this set of 4-cycles, considering four cases in turn.

(i) $t \equiv 1 \pmod 8$.
The current leave is a star that joins z to all vertices in E, so it has size v_1; hence L is minimum by (B).

(ii) $t \equiv 3 \pmod 8$.
The current leave is a star that joins z to all vertices in E, together with a K_3 on three vertices from three different odd-sized parts, so it has size $v_1 + 3$. Hence L is minimum by (B).

(iii) $t \equiv 5 \pmod 8$.
The current leave is a star centered at vertex z in O, together with (by choice) a K_5 leave on a vertex set that includes vertex z and four other vertices, each from a different odd-sized part. We apply Lemma 5.2 to a pair of vertices from the even-sized part together with the five vertices of the K_5 leave. The final leave is then a path of 4 edges starting and ending on two vertices in E,

together with a star that joins z to the remaining $v_1 - 2$ vertices in E. This leave has a total of $v_1 + 2$ edges, so it is a minimum leave, by (B).

(iv) $t \equiv 7 \pmod 8$.
We proceed exactly as in the case $t \equiv 5 \pmod 8$, but use Lemma 5.3 instead of Lemma 5.2. The final leave contains $v_1 + 1$ edges and so is a minimum leave by (B).

Next we assume that $s \geq 2$, so there are at least two even-sized parts. Form a partition $\mathscr{P}$ of the vertices in E into pairs, the two vertices in each pair belonging to the same part. Partition as many of these pairs as possible into sets $S_1, \ldots, S_y$ of size four, so that each S_l, $1 \leq l \leq y$, contains at most two pairs from each part. Let S be any of the remaining pairs. Then:

case (a) $|S| \leq 3$, and we can assume that:
(ai) no two pairs in S occur in the same part, or
(aii) S_l contains two pairs in V_1, for $1 \leq l \leq y$.

case (b) $|S| \geq 4$, and we can assume that:
(bi) all pairs in S except possibly one are in V_1, and
(bii) S_l contains two pairs in V_1, for $1 \leq l \leq y$.

To see that we can assume either (ai) or (aii) holds, suppose that W is a part containing two pairs p_1 and p_2 in S. If S_l contains two pairs in W for $1 \leq l \leq y$ then $W = V_1$ and (aii) holds. Otherwise one set, say S_1, contains at most one pair in W. Then since $|S_1| = 4$ and $|S| \leq 3$, S_1 contains a pair p_3 contained in a part which contains no pair in S. Interchange pairs p_1 and p_3 between S and S_1. If W still contains two pairs in S (so $|S| = 3$), then this process can be repeated.

To see that (bi) and (bii) can be assumed we argue as follows. Clearly, the maximality of y forces all but at most one pair in S to occur together in one part, say V_i. If there exists a set S_l, $1 \leq l \leq y$, say S_1, that contains at most one pair in V_i, then replace one pair p_1 in S_1 that is not in V_i with a pair p_2 in S that is in V_i to form S'_1. Let $S' = (S \cup \{p_1\}) \setminus \{p_2\}$. By the maximality of y, p_1 is the only pair in S' that is not in V_i (for otherwise we can form another set S_{l+1} from the pairs in S'). Also, all sets S'_1 and S_l for $2 \leq l \leq y$ contain exactly 2 pairs in V_i; for otherwise the above step could be repeated, producing a second pair in S'' not in V_i which together with p_1 and two of the $|S| - 2 \geq 2$ pairs in S'' that are in V_i form a set S_{l+1}, again contradicting the maximality of y. Therefore, (bii) holds, and so clearly V_i is the largest part, so $i = 1$ and (bi) holds.

Let $z \in O$. For each l, $1 \leq l \leq y$, let $S_l = \{p_{l,i} \mid 1 \leq i \leq 4\}$ where possibly $p_{l,2j-1}$ and $p_{l,2j}$ are subsets of the same part for $j = 1, 2$. Let $p_{l,i} = \{w^1_{l,i}, w^2_{l,i}\}$. Let B_l be formed from: the 4-cycles in a packing of $K(4,4,1)$ (see Lemma 5.1) with partition $\{p_{l,1} \cup p_{l,2}, p_{l,3} \cup p_{l,4}, \{z\}\}$; and for $j = 1, 2$, if $p_{l,2j-1}$, and $p_{l,2j}$ are subsets of *different* parts, include also the 4-cycle $(w^1_{l,2j-1}, w^1_{l,2j}, w^2_{l,2j-1}, w^2_{l,2j})$. Now the 4-cycles in B_l cover all edges joining vertices in $\bigcup_{i=1}^4 p_{l,i} \cup \{z\}$ that are in different parts, except for the leave L_l that consists of a 1-factor joining vertices in $p_{l,1} \cup p_{l,2}$ to vertices in $p_{l,3} \cup p_{l,4}$. Note that in case (b), by (bii) we have that each edge in $\bigcup_{l=1}^y L_l$ is incident with a vertex in V_1.

Next, let B'_2 be a set of 4-cycles that form a packing of $K(v_{s+1}, \ldots, v_{s+t})$ with leave K_γ defined on the vertex set $\{z = z_1, z_2, \ldots, z_\gamma\}$, where $\gamma \equiv t \pmod 8$ with

$\gamma \in \{1, 3, 5, 7\}$ (see Remark 3.2). Then we pair off the vertices in $O \setminus \{z\}$ so that, in particular, $\{z_2, z_3\}, \ldots, \{z_{\gamma-1}, z_\gamma\}$ are pairs, and for each pair from $O \setminus \{z\}$, together with each pair in $\mathscr{P}$ (the partition of E into pairs), we take the induced 4-cycle, and place these 4-cycles in B'_1. Also, for each $\{p_1, p_2\} \subseteq \mathscr{P}$ such that

(1) p_1 and p_2 are in different parts, and
(2) for $1 \le l \le y$, $\{p_1, p_2\} \not\subseteq S_l$,

place the 4-cycle induced by $p_1 \cup p_2$ in B'_3. (If $t \not\equiv 1 \pmod 8$, note that not all of these 4-cycles will be in our final maximum packing.)

Now consider the cases $t \equiv 1, 3, 5$, and 7 (mod 8) in turn. Let

$$B = \left(\bigcup_{l=1}^{y} B_l\right) \cup \left(\bigcup_{j=1}^{3} B'_j\right)$$

and

$$L = \left(\bigcup_{l=1}^{y} L_l\right) \cup E(R),$$

where R is a star that joins z to each vertex in each pair of S.

(i) $t \equiv 1 \pmod 8$.
The cycles in B form a maximum packing with leave L which is a minimum leave because in case (a), $|L| = |E|/2 + |S| \le \eta/2 + 3$ (see (A)), and in case (b), $|L| = v_1$ or $v_1 + 2$, according as all pairs in S are in V_1, or one pair is not in V_1 (see (B)).

For $t \equiv i \pmod 8$ when $i = 3, 5$, or 7, we now start with B, that has leave $L \cup L'_i$ where L'_i is a copy of K_i defined on the vertex set $\{z = z_1, z_2, \ldots, z_\gamma\}$. We shall show how to modify B to obtain a maximum packing.

(ii) $t \equiv 3 \pmod 8$.
If $|S| = 0$ then $|L'| = \frac{\eta}{2} + 3$, where $L' = L \cup L'_3$, and thus L' is a minimum leave by (A). Therefore we can assume that $|S| \ge 1$.

If S contains two pairs p_1 and p_2 from different parts, then remove the 4-cycle induced by $p_1 \cup p_2$ from B'_3, and remove the two 4-cycles joining $p_1 \cup p_2$ to $\{z_2, z_3\}$ from B'_1; then let B'_4 be the set of 4-cycles formed by applying Lemma 5.4 to $K_7 \setminus \{p_1, p_2\}$ (regarding p_i as an edge here), on the vertex set $p_1 \cup p_2 \cup \{z_i \mid 1 \le i \le 3\}$. This results in the modified leave L' with

$$|L'| = |L| - 1 = \begin{cases} \dfrac{\eta}{2} + |S| - 1 & \text{in case (a) (so } 2 \le |S| \le 3) \\ v_1 + 1 & \text{in case (b).} \end{cases}$$

(See Figure 7.)

If all pairs in S belong to one part, say V_i (in case (b), $i = 1$), then either there exists a set, say S_1, containing at most one pair in V_i (by (bii), this only arises in case (a), and so by (ai) and (aii), $|S| = 1$), or else each S_l contains two pairs in V_i, for $1 \le l \le y$ (so $i = 1$). Let $S_1 = \{p_1, p_2, p_3, p_4\}$.

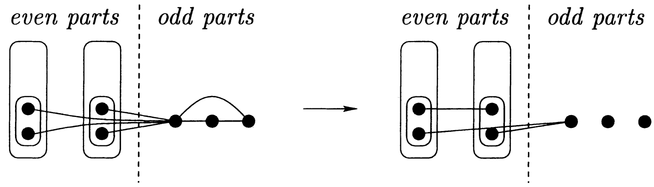


FIG. 7.

In the former case, if S_1 contains a pair in V_i, let it be p_1. If S_1 contains one (or two) pairs of pairs that are both subsets of the same part, then we can suppose that p_3 and p_4 are in the same part (or p_3 and p_4 occur in one part, and p_1 and p_2 occur in one part). Let $p \in S$. Remove the 4-cycles in B_1 from B. Remove from B_1' the 4-cycles induced by $p \cup \{z_2, z_3\}$ and by $p_i \cup \{z_2, z_3\}$ for $1 \leq i \leq 4$. Let B_4' be the set of 4-cycles formed by applying Lemma 5.7 to $K(4,4,2,1,1,1)$ with vertex parts defined as follows. If $p_1 \in V_i$ then the parts are $p \cup p_1$, $p_3 \cup p_4$, p_2, $\{z_1\}$, $\{z_2\}$, $\{z_3\}$, in which case if p_3 and p_4 are in different parts we add to B_4' the 4-cycle formed by the edges joining p_3 to p_4. If $p_1 \notin V_i$ then the parts are $p_1 \cup p_2$, $p_3 \cup p_4$, p, $\{z_1\}, \{z_2\}, \{z_3\}$, in which case for $i = 1, 2$, if p_{2i-1} and p_{2i} are in different parts then we also add to B_4' the 4-cycle formed by the edges joining p_{2i-1} to p_{2i}. The modified leave L' satisfies

$$|L'| = \frac{\eta}{2} + |S| - 1 = \frac{\eta}{2}, \tag{1}$$

so L' is a minimum leave by (A). (See Figure 8.)

In the latter case, every edge in L meets V_1, so $|L \cup L_3'| = v_1 + 3$ and thus $L' = L \cup L_3'$ is already a minimum leave by (B).

(iii) $t \equiv 5 \pmod 8$.

If $|S| = 0$ then clearly $y \geq 1$, so let $S_1 = \{p_1, p_2, p_3, p_4\}$, where we can assume for $i = 1, 2$ and $j = 3, 4$ that p_i and p_j occur in different parts. Replace the 4-cycles in B_1'

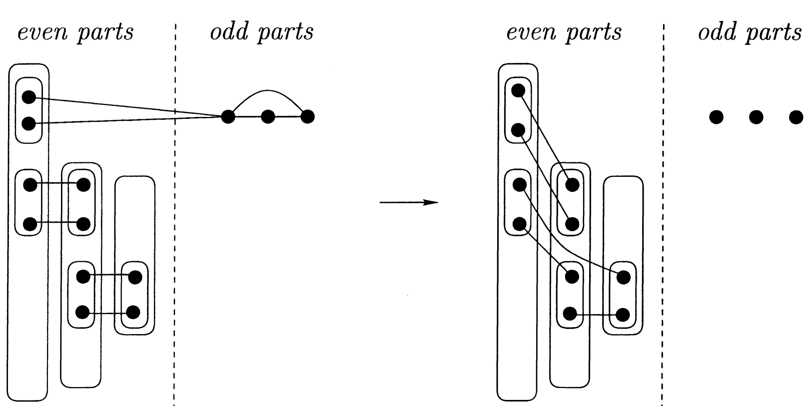


FIG. 8.

joining p_i, $1 \le i \le 4$, to $\{z_j \mid 2 \le j \le 5\}$ and the 4-cycles in B_1 with the set B'_4 of 4-cycles defined as follows. Let B'_4 contain the 4-cycles formed by applying Lemma 5.8 to $K(4,4,1,1,1,1,1)$ with parts $p_1 \cup p_2$, $p_3 \cup p_4$ and $\{z_i\}$ for $1 \le i \le 5$, together with the 4-cycles that join p_{2i-1} to p_{2i}, $i = 1,2$, whenever p_{2i-1} and p_{2i} are in different even-sized parts. The resulting leave L' satisfies $|L'| = \frac{\eta}{2} + 2$, so is a minimum leave by (A). Therefore, we can assume that $|S| \ge 1$.

If S contains two pairs p_1 and p_2 from different parts, then remove the 4-cycle induced by $p_1 \cup p_2$ from B'_3 and the four 4-cycles joining $p_1 \cup p_2$ to $\{z_2, z_3, z_4, z_5\}$ from B'_1. Let B'_4 be the set of 4-cycles formed by applying Lemma 5.5 to $K_9 \setminus \{p_1, p_2\}$ (regarding p_i as an edge here) on the vertex set $p_1 \cup p_2 \cup \{z_i \mid 1 \le i \le 5\}$. This results in the modified leave

$$|L'| = |L| - 2 = \begin{cases} \dfrac{\eta}{2} + |S| - 2 & \text{in case (a) (so } |S| = 2 \text{ or } 3), \\ v_1 & \text{in case (b).} \end{cases}$$

(See Figure 9.)

Now if all pairs in S belong to one part, say V_i, then let $p \in S$ and remove the two 4-cycles joining p to $\{z_2, z_3, z_4, z_5\}$ from B'_1. Then apply Lemma 5.2 to $K_7 - \{p\}$ with vertex set $p \cup \{z_1, z_2, \ldots, z_5\}$. If $|S| = 1$, then $|L'| = \frac{\eta}{2} + 3$, so L' is minimum by (A). If $|S| \ge 2$, and all pairs in S belong to V_i, then since (ai) is not satisfied, each S_l contains two pairs in V_i for $1 \le l \le y$ (so $i = 1$). Therefore, every edge in L meets V_1, so we have final leave L' satisfying $|L'| = v_1 + 2$.

(iv) $t \equiv 7 \pmod 8$.
If $|S| = 0$ then proceed exactly as in the case where $t \equiv 5 \pmod 8$ and $|S| = 0$, except that Lemma 5.9 is used instead of Lemma 5.8. Then the resulting modified leave L' satisfies $|L'| = \frac{\eta}{2} + 1$, so is a minimum leave by (A). Therefore, we can assume that $|S| \ge 1$.

If $|S| = 2$ or $|S| \ge 4$ and S contains two pairs p_1 and p_2 from two different parts, then remove: the 4-cycle induced by $p_1 \cup p_2$ from B'_3, and the six 4-cycles joining $p_1 \cup p_2$ to $\{z_i \mid 2 \le i \le 7\}$ from B'_1. Then apply Lemma 5.6 to $K_{11} - \{p_1, p_2\}$ on the vertex set $p_1 \cup p_2 \cup \{z_i \mid 1 \le i \le 7\}$ to obtain a set of 4-cycles B'_4. This results in the modified leave L' with:

$$|L'| = \begin{cases} \dfrac{\eta}{2} + 3 & \text{if } |S| = 2, \text{ and} \\ v_1 + 3 & \text{if } |S| \ge 4. \end{cases}$$

(See Figure 10.)

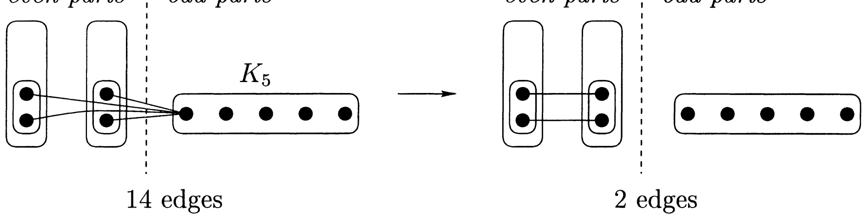


FIG. 9.

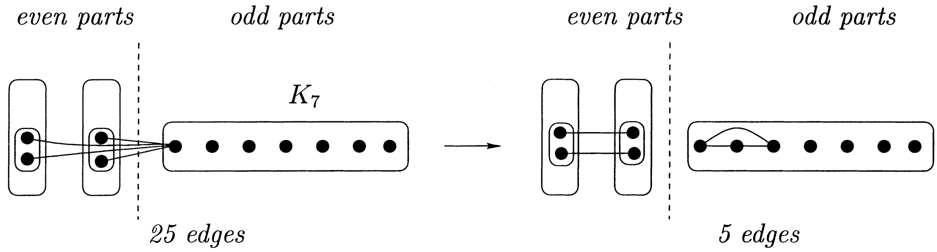


FIG. 10.

If $|S| \neq 3$, and all pairs occur in one part (so if $|S| \leq 2$, then in fact $|S| = 1$ by (ai)), apply Lemma 5.3 to $K_9 - p$ with vertex set $\{z_i \mid 1 \leq i \leq 7\} \cup p$, where $p \in S$. The resulting leave L' satisfies

$$|L'| = \begin{cases} \dfrac{\eta}{2} + |S| + 1 = \dfrac{\eta}{2} + 2, & \text{if } |S| \leq 2, \\ v_1 + 1, & \text{if } |S| \geq 4. \end{cases}$$

So L' is a minimum leave. (See Figure 11.)

Finally, suppose $|S| = 3$. Let $S = \{p_1, p_2, p_3\}$.

(1) If all three of these pairs in S belong to different parts, then remove the 4-cycles in B'_3 joining p_i to p_j for $1 \leq i < j \leq 3$, and remove from B'_1 the nine 4-cycles that join $p_1 \cup p_2 \cup p_3$ to $\{z_i \mid 2 \leq i \leq 7\}$. Let B'_4 be a packing of K_{13} on the vertex set $\{p_1, p_2, p_3, z_i \mid 1 \leq i \leq 7\}$, with leave being the 6-cycle $(p_1^1, p_2^1, p_2^2, p_3^1, p_3^2, p_1^2)$, where pair $p_i = \{p_i^1, p_i^2\}$. This is equivalent to a 4-cycle packing of $K(2,2,2,1,1,1,1,1,1,1)$ with three disjoint edges in the leave, between vertices in the parts of size 2. Thus the final leave L' satisfies $|L'| = \eta/2$.

(2) If exactly two pairs, p_1 and p_3, belong to one part, W_1 and p_2 to a third part, then (ai) is not satisfied. Therefore, each S_l, $1 \leq l \leq y$, contains two pairs from W_1, so $W_1 = V_1$. Remove the 4-cycle in B'_3 joining p_1 to p_2, and remove from B'_1 the six 4-cycles that join $p_1 \cup p_2$ to $\{z_i \mid 2 \leq i \leq 6\}$. Let B'_4 be the set of

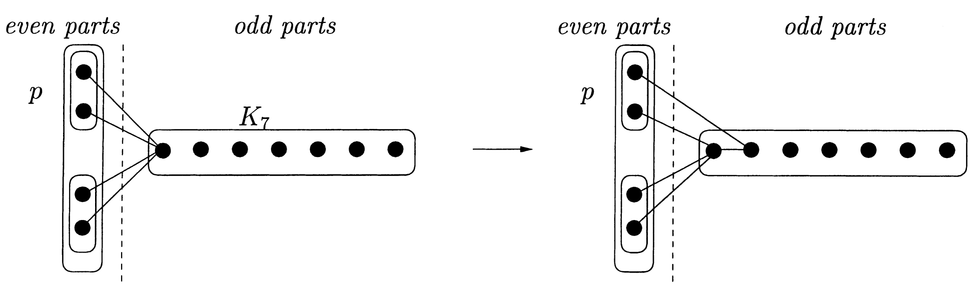


FIG. 11.

4-cycles formed by applying Lemma 5.6 to $K_{11} \setminus \{p_1, p_2\}$, on the vertex set $\{z_i \mid 1 \leq i \leq 7\} \cup p_1 \cup p_2$. This packing of $K_{11} \setminus \{p_1, p_2\}$ has the leave of two disjoint edges between p_1 and p_2 and a K_3 on the vertex set $\{z_1, z_2, z_3\}$, so the final leave L' satisfies $|L'| = v_1 + 3$.

(3) Finally, we can assume that $|S| = 3$ and all three pairs in S occur in the same part, say W. Since (ai) is not satisfied, each S_l contains two pairs from W, for $1 \leq l \leq y$, so $W = V_1$. Then using Lemma 5.3 on $K_9 \setminus \{p_1\}$ yields $|L'| = v_1 + 1$. □

6. CONCLUDING REMARKS

We now have the following result.

Theorem 6.1. *Let G be a complete multipartite graph with η vertices of odd degree and ν vertices in the largest part containing vertices of odd degree (if such a part exists). Then there exists a maximum* 4*-cycle packing of G with leave L satisfying*

(i) $\max\{\eta/2, \nu\} \leq |L| \leq \max\{\eta/2, \nu\} + 3$, *if G does not have n parts all of odd size with $n \equiv 5$ or* 7 (mod 8); *and*

(ii) $|L| = 6$ *or* 5 *if G has n parts, all of odd size, with $n \equiv 5$ or* 7 (mod 8), *respectively.*

Remark: Note that the size of the leave is completely determined by the inequalities.

Proof. Clearly, $|L| \geq \eta/2$ and $|L| \geq \nu$. Also, for any other 4-cycle packing of G with leave L', 4 must divide $|L'| - |L|$. Therefore, the result follows from the 4-cycle packing of G with leave L that is constructed in one of Lemmas 2.2 and 3.1 and Theorems 4.6 and 5.10. □

We remark that of course many different minimum leaves are possible in most cases. For instance, in cases where a component of the leave as described above is a star with center at a vertex in an odd part, this could be split into several smaller stars having centers at different vertices in odd sized parts.

The related problem of packing a λ-fold complete multipartite graph will be the subject of a subsequent paper, for reasons of length.

ACKNOWLEDGMENT

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