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Existence of Many Positive Nonradial Solutions for Nonlinear Elliptic Equations on an Annulus

SONG-SUN LIN*

Department of Applied Mathematics, National Chiao Tung University, Hsin-chu, Taiwan, Republic of China

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1. INTRODUCTION

In this paper we study the existence of many positive nonradial solutions of the equation

$$\Delta u + f(u) = 0 \qquad \text{in } \Omega, \tag{1.1}$$

$$u = 0 \qquad \text{on } \partial\Omega, \tag{1.2}$$

where $\Omega = \Omega_a = \{x \in \mathbb{R}^n : a < |x| < 1\}$ is an annulus in $\mathbb{R}^n$, $n \geqslant 2$, and f satisfies the following conditions:

(H-0) $f \in C^1(\mathbb{R}^1)$ and $f(u) > 0$ for u large,

(H-1) $f(0) = 0$ and $f'(0) \leqslant 0$,

(H-2) there exists $\sigma > 0$ such that $uf'(u) \geqslant (1+\sigma) f(u)$ for all $u \geqslant 0$,

(H-3) for u large,

$$f(u) \leqslant \begin{cases} Cu^p & \text{for some } p < \dfrac{n+2}{n-2} \text{ and } C > 0 \text{ if } n \geqslant 3, \\ \exp A(u) & \text{with } A(u) = o(u^2) \text{ as } u \to \infty \text{ if } n = 2. \end{cases}$$

In [3], Coffman considered Eqs. (1.1) and (1.2) when $n = 2$, $f(u) = -u + u^p$, $p = 2N+1$, N is a positive integer, and $\Omega = \Omega(r, c) = \{(x, y) \in \mathbb{R}^2 : r^2 < x^2 + y^2 < (r+c)^2\}$. He showed that for any fixed $c > 0$, the number of rotationally non-equivalent nonradial positive solutions is unbounded as $r \to \infty$. The method used in [3] was to minimize the associated Rayleigh quotients on the class of all radial functions and the class of functions which are invariant under the rotating $2\pi/k$ angles with $k \geqslant 2$. By choosing some appropriate test functions, he was able

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to show that the minimums are different as soon as r is large enough. Later, using the same idea, Li [6] extended the results to $n \geqslant 4$ and $p \in (1, (n+2)/(n-2))$. He also treated the problems when the nonlinearity was not homogeneous.

In this paper, we use a Nehari-type variational method, i.e., we consider the functionals

$$J(u) = \int_{\Omega_a} \tfrac{1}{2} |\nabla u|^2 - F(u^+),$$

and

$$M(u) = \int_{\Omega_a} |\nabla u|^2 - u^+ f(u^+)$$

on $H_0^1(\Omega_a)$, where $F(u) = \int_0^u f(t)\,dt$ and $u^+ = \max\{u, 0\}$, and manifolds,

$$M = \{u \in H_0^1(\Omega_a) \colon M(u) = 0 \text{ and } u \not\equiv 0\},$$

and

$$V_\infty = \{u \in M : u \text{ is radial}\}.$$

For any proper subgroup G of $O(n)$, denote by

$$M_G = \{u \in M : u(gx) = u(x) \text{ for all } g \in G \text{ and } x \in \Omega_a\},$$

the G-symmetric submanifold

Let

$$I_\infty = I_\infty(a) = \inf\{J(u) : u \in V_\infty\},$$

and

$$I_G = I_G(a) = \inf\{J(u) : u \in M_G\}.$$

Let Γ be a family of subgroups of $O(n)$. Then we can obtain many positive nonradial solutions if we can establish the following properties for $G \in \Gamma$:

(i) $I_G < I_\infty$ for all $G \in \Gamma$,

(ii) I_G is achieved by some $u_G \in M_G$ and u_G is a critical point of J on M_G,

(iii) u_G is a critical point of J on $H_0^1(\Omega_a)$,

(iv) $I_G \neq I_{\bar{G}}$ if $G \neq \bar{G}$ and $G, \bar{G} \in \Gamma$.

Note that (ii) is related to the compact imbedding of M_G into $L^{p+1}(\Omega_a)$, and (iii) is related to the symmetric critically principle: if u_G is a critical point of J on M_G, then it is also a critical point of J on $H_0^1(\Omega_a)$ (see, e.g., [15]).

To establish (i) and (iv), we study the nonradial instability of positive radial solutions of (1.1) and (1.2). Indeed, for $n \geqslant 2$, let u_a be a positive radial solution of (1.1) and (1.2) on Ω_a. The linearized eigenvalue problem of (1.1) and (1.2) at u_a is

$$\Delta w + f'(u_a) w = -\mu w \qquad \text{in } \Omega_a, \tag{1.3}$$

$$w = 0 \qquad \text{on } \partial\Omega_a. \tag{1.4}$$

In spherical coordinates, (1.3) and (1.4) are equivalent to

$$\varphi''(r) + \frac{n-1}{r}\varphi'(r) + \left\{ f'(u_a(r)) - \frac{\alpha_k}{r^2} \right\} \varphi(r) = -\mu_{k,l}(u_a)\,\varphi(r), \qquad a < r < 1, \tag{1.5}$$

$$\varphi(a) = 0 = \varphi(1), \tag{1.6}$$

where $\alpha_k = k(k+n-2)$, $k = 0, 1, 2, \ldots$, and $l = 1, 2, \ldots$. Note that α_k are eigenvalues of Laplacian $-\Delta$ on S^{n-1}, the unit sphere. For $k \geqslant 1$, the associated eigenspace $S_{n,k}$ of $-\Delta$ on S^{n-1} is given by $S_{n,k} = \{\psi_k : S^{n-1} \to \mathbb{R}^1 \mid \psi_k(x) = P_k(x)$ for $|x| = 1$, where $P_k(x)$ is the harmonic homogeneous polynomial of degree k on $\mathbb{R}^n\}$. The associated eigenfunctions $w_{k,l}$ of (1.3) and (1.4) are given by $w_{k,l} = \varphi_{k,l}\psi_k$. For $l = 1$, denote $\varphi_k = \varphi_{k,1}$ and then $w_k = \varphi_k \psi_k$.

u_a is said to be unstable with respect to the k-mode if $\mu_{k,1}(u_a) < 0$. In this case, it was proved in [9] that for $|t|$ small there exists $\delta(t) = o(t)$ such that

$$u_k(t) \equiv u_a + \delta(t) w_0 + t w_k \in M, \tag{1.7}$$

and

$$J(u_k(t)) < J(u_a). \tag{1.8}$$

Therefore, there is a positive nonradial solution provided that all positive radial solutions are unstable with respect to the k-mode for some $k \geqslant 1$ (a being fixed).

In this paper, by carefully studying the various kinds of symmetry of the members of $S_{n,k}$, we are able to obtain some families Γ of subgroups of $O(n)$ such that (i)–(iv) hold, which give us many non-equivalent positive nonradial solutions under the assumptions (H-0)–(H-3).

In [4], Ding showed that the imbeddings of certain symmetric submanifolds of $H^1(S^{n-1})$ into $L^p(S^{n-1})$ are compact for some p exceeding the Sobolev critical exponent $(n+2)/(n-2)$. Using the same idea, Li [6] obtained some nonradial positive solutions for p greater than $(n+2)/(n-2)$. After proving similar compact imbedding theorems, we can obtain more positive nonradial solutions in some supercritical cases.

The paper is organized as follows: In Section 2, we study the subcritical cases. In Section 3, we study the supercritical cases.

2. Subcritical Cases

The existence of positive radial solutions of (1.1) and (1.2) has been studied by many authors, see [5, 7, 11, 12, 14]. In fact, conditions (H-0)–(H-2) imply that there exists at least one positive radial solution u_a of (1.1) and (1.2) for each $a \in (0, 1)$. Note that it was proved in [10] that there is no positive nonradial solution of (1.1) and (1.2) provided that a is small enough.

We first need some results concerning the nonradial instability of positive radial solutions.

Lemma 2.1. *Assume conditions* (H-0)–(H-2) *are satisfied. Then, for each* $k \geqslant 1$, *there exists an* $a_k = a_k(n, \sigma) \in (0, 1)$, *such that for any* $a \in (a_k, 1)$ *and any positive radial solution* u_a, *we have* $\mu_{k,1}(u_a) < 0$.

Proof. The lemma was proved essentially in Lemma 3.1 of [9]. Here, we give better estimates on a_k.

It is known that $\mu_{k,1} = \mu_{k,1}(u_a)$ can be characterized as

$$\mu_{k,1} = \inf \left\{ Q_k(v) \Big/ \int_a^1 r^{n-1} v^2 \, dr : v \in H_0^1(a, 1) \text{ and } v \not\equiv 0 \right\}, \tag{2.1}$$

where

$$Q_k(v) = \int_a^1 r^{n-1} \left\{ v'^2 - f'(u_a) v^2 + \frac{\alpha_k}{r^2} v^2 \right\} dr.$$

From (2.1), it is clear that $\mu_{k,1}$ is strictly increasing in k.

Since u_a is a solution of (1.1) and (1.2), we have

$$\int_{\Omega_a} |\nabla u_a|^2 = \int_{\Omega_a} u_a f(u_a). \tag{2.2}$$

By (H-2) and (2.2), we have

$$\omega_n Q_k(u_a) = \int_{\Omega_a} \left\{ u_a f(u_a) - f'(u_a)\, u_a^2 \right\} + \alpha_k \int_{\Omega_a} u_a^{2r-2}$$

$$\leqslant -\sigma \int_{\Omega_a} |\nabla u_a|^2 + \alpha_k a^{-2} \int_{\Omega_a} u_a^2,$$

where ω_n is the area of S^{n-1}.

Let $v_1(a) > 0$ be the least eigenvalue of $-\Delta$ on Ω_a with the Dirichlet boundary condition. Then, it is easy to check that $v_1(a)$ is strictly increasing in a and

$$\lim_{a \to 1} v_1(a) = \infty. \tag{2.3}$$

Using the Poincaré inequality

$$\int_{\Omega_a} |\nabla v|^2 \geqslant v_1(a) \int_{\Omega_a} v^2$$

for all $v \in H_0^1(\Omega_a)$, we have

$$\omega_n Q_k(u_a) \leqslant \{ -\sigma + \alpha_k a^{-2} v_1^{-1}(a) \} \int_{\Omega_a} |\nabla u_a|^2.$$

Now, by choosing a_k satisfying

$$a_k^2 v_1(a_k) = \alpha_k / \sigma, \tag{2.4}$$

the results follow. The proof is complete.

Next, we recall the results concerning the change of $J(u)$ along the direction of nonradial mode w_k at positive radial solution u_a.

LEMMA 2.2 *Assume conditions* (H-0)–(H-2) *are satisfied. Let* u_a *be a positive radial solution of* (1.1) *and* (1.2), w_0 *and* w_k *be associated eigenfunctions with respect to* $\mu_{0,1}$, *and* $\mu_{k,1}$, $k \geqslant 1$, *respectively, and* $\int_{\Omega_a} w_0^2 = \int_{\Omega_a} w_k^2 = 1$. *Then, there exist* $\varepsilon > 0$ *and a smooth function* $\delta : (-\varepsilon, \varepsilon) \to \mathbb{R}^1$ *with* $\delta(0) = \delta'(0) = 0$ *such that for any* $t \in (-\varepsilon, \varepsilon)$, *we have*

$$M(u_a + \delta(t) w_0 + t w_k) = 0. \tag{2.5}$$

Moreover, we have

$$J(u_a + \delta(t) w_0 + t w_k) = J(u_a) + \tfrac{1}{2}\mu_{0,1}\, \delta^2(t) + \tfrac{1}{2}\mu_{k,1} t^2 + O(t^4), \tag{2.6}$$

for $t \sim 0$.

For the proofs, see Lemmas 6.1 and 6.2 in [9].

Next, we characterize various kinds of symmetric submanifolds of $H_0^1(\Omega_a)$ which are invariant under the group actions of certain subgroups of $O(n)$, the set of all $n \times n$ orthogonal matrices.

DEFINITION 2.3. Let G be a subgroup of $O(n)$. A function u is said to be invariant under G or G-symmetric if $u(gx) = u(x)$ for all $g \in G$ and $x \in \Omega_a$. In this case, we write $u \in G$.

DEFINITION 2.4. Two functions u and v on Ω_a are said to be equivalent if there exists a $g \in O(n)$ such that $v(x) = u(gx)$ for all $x \in \Omega_a$.

Note that if u and v are equivalent then u is a solution of (1.1) and (1.2) if and only if v is a solution of (1.1) and (1.2).

For $k \geqslant 2$, the rotational subgroup G_k is defined by

$$G_k = \left\{ g \in O(2) \colon g(x_1, x_2) = \left(x_1 \cos \frac{2\pi l}{k} + x_2 \sin \frac{2\pi l}{k}, \; -x_1 \sin \frac{2\pi l}{k} + x_2 \cos \frac{2\pi l}{k} \right), \right.$$
$$\left. (x_1, x_2) \in \mathbb{R}^2, \; l \text{ is an integer} \right\}.$$

Let

$$V_k = \{ u \in M \colon u \in G_k \times O(n-2) \},$$
$$I_k = I_k(a) = \inf\{ J(u) \colon u \in V_k \},$$

and for $2 \leqslant n - l \leqslant l$, let

$$\Sigma_l = \{ u \in M \colon u \in O(l) \times O(n-l) \},$$
$$\tilde{I}_l = \tilde{I}_l(a) = \inf\{ J(u) \colon u \in \Sigma_l \}.$$

Remark 2.5. In [6], it was proved that if $u \in O(l) \times O(n-l) \cap O(\bar{l}) \times O(n-\bar{l})$ with $l \neq \bar{l}$ and $l \neq n - \bar{l}$, then u is radial.

The following lemma shows that $S_{n,k}$ can provide eigenfunctions of $G_k \times O(n-2)$ symmetry for all $k \geqslant 1$.

LEMMA 2.6. *Let (ρ, θ) be the polar coordinates in $\mathbb{R}^2$. Then, for $n \geqslant 2$ and for each $k \geqslant 1$, choosing $\psi_k = \rho^k \cos k\theta$, we have $w_k = \varphi_k \psi_k \in G_k \times O(n-2)$.*

Proof. The associated homogeneous polynomial

$$P_k(x) = \rho^k \cos \theta$$

is of degree k and harmonic. The result follows.

Furthermore, $S_{n,k}$ can also provide more symmetric eigenfunctions when the dimension $n \geqslant 4$.

LEMMA 2.7. *For $n \geqslant 4$, k is even and $2 \leqslant n-l \leqslant l$. Then, there exist $\psi_{k,l}$ such that $w_{k,l} = \varphi_k \psi_{k,l} \in O(l) \times O(n-l)$. Moreover, for any decomposition $L = (l_1, ..., l_j)$ of n, $j \geqslant 2$, i.e., l_i satisfies* (i) *$l_i \geqslant 2$ for each i and* (ii) *$\sum_{i=1}^{j} l_i = n$, then there exists $w_{k,L} = \varphi_k \psi_{k,L} \in O(l_1) \times \cdots \times O(l_j)$.*

Proof. Let $k = 2m \geqslant 2$, $s^2 = x_1^2 + \cdots + x_l^2$, and $t^2 = x_{l+1}^2 + \cdots + x_n^2$. Consider the homogeneous polynomials

$$P_{2m}(x) = \sum_{j=0}^{m} A_j s^{2(m-j)} t^{2j},$$

where A_j are real numbers which will be determined immediately. Then

$$\begin{aligned} \Delta P_{2m} &= \left\{ \frac{\partial^2}{\partial s^2} + \frac{l-1}{s} \frac{\partial}{\partial s} + \frac{\partial^2}{\partial t^2} + \frac{n-l-1}{t} \frac{\partial}{\partial t} \right\} P_{2m} \\ &= \sum_{j=0}^{m-1} \{ C_{j,j+1} A_j + C_{j+1,j+1} A_{j+1} \} s^{2(m-j-1)} t^{2j}, \end{aligned}$$

where the $C_{i,j}$ are positive constants. If $\Delta P_{2m} = 0$, then $A_j = C_j A_0$ with $(-1)^j C_j > 0$ for $j = 1, ..., m$. Therefore, let $\psi_{k,l} = P_k$, then $w_{k,l} = \varphi_k \psi_{k,l} \in O(l) \times O(n-l)$.

The second assertion of the lemma can be proved analogously, the detail is omitted.

After these preparations, we can now prove the following theorems.

THEOREM 2.8. *Assume conditions* (H-0)–(H-3) *are satisfied. Then there exists an increasing sequence $a_k \to 1^-$ as $k \to \infty$, such that for any $a \in (a_k, 1)$,* (1.1) *and* (1.2) *have a positive nonradial solution $u_j \in V_j$, for each $j = 1, 2, ..., k$. Furthermore, u_j are non-equivalent for $j = 1, 2, ..., k$.*

Proof. By Lemma 2.1, there exists $a_k \in (0, 1)$ such that $\mu_{k,1}(u_a) < 0$ for all positive radial solutions u_a if $a \in (a_k, 1)$.

For fixed $k \geqslant 1$ and $j = 1, 2, ..., k$, by Lemma 2.6, there exist $w_j = \varphi_j \psi_j \in G_j \times O(n-2)$, where the φ_j depend on u_a. By Lemma 2.2, $u_a + \delta(t) w_0 + t w_j \in V_j$ and

$$J(u_a + \delta(t) w_0 + t w_j) < J(u_a) \tag{2.7}$$

for $|t|$ sufficiently small, $j = 1, 2, ..., k$. Then conditions (H-0)–(H-3) imply that the minimums I_∞ and I_j are achieved by some functions $u_a \in V_\infty$ and $u_j \in V_j$, $j = 1, 2, ..., k$. By (2.7) we have

$$I_j < I_\infty, \tag{2.8}$$

for $j = 1, 2, ..., k$.

To show that all u_j are non-equivalent, we first recall the result, which was proved in [3, 6, 16], that

$$I_{jm} < I_\infty \text{ implies } I_j < I_{jm}, \tag{2.9}$$

for $j = 1, 2, \ldots$ and $m = 2, \ldots$.

Now, let $1 \leqslant i < j \leqslant k$ and l be the least common multiple of i and j. If $j = l$, then u_i and u_j are non-equivalent by (2.9). If $j < l$, then $V_i \cap V_j = V_l$. If u_i and u_j are equivalent, then we may assume $u_i = u_j = \tilde{u} \in V_l$. Therefore, by (2.8), we have

$$I_l \leqslant J(\tilde{u}) = I_i < I_\infty. \tag{2.10}$$

By (2.9) and (2.10), we obtain $I_i < I_l$, a contradiction to (2.10). Hence u_i and u_j are non-equivalent.

The proof that the u_j are solutions of (1.1) and (1.2) is rather standard. Indeed, since u_j is a minimizer of $J(u)$ over V_j, there is a Lagrange multiplier $\lambda_j \in \mathbb{R}^1$ such that

$$J'(u_j) = \lambda_j M'(u_j)$$

on V_j, see, e.g., [2]. Now, by (H-2), it is easy to verify that $\lambda_j = 0$, see, e.g., [13]. Hence, u_j is a critical point of $J(u)$ over V_j.

Finally, by using the symmetric critically principle [4, 15] or the results in [6], the u_j are positive solutions of (1.1) and (1.2). The proof is complete.

For higher dimensions, $n \geqslant 6$, we can have more positive nonradial solutions.

THEOREM 2.9. *Assume conditions* (H-0)–(H-3) *are satisfied. For* $n \geqslant 6$, k *is even*, $3 \leqslant n - l \leqslant l$, *and* $a \in (a_k, 1)$, *then there exist non-equivalent positive nonradial solutions* $u_{k,l} \in \Sigma_l$.

Proof. The proof is the same as in proving the previous theorem except using Lemma 2.7 instead of Lemma 2.6. The detail is omitted.

3. Supercritical Cases

In this section, we first prove the following compactness lemma which is motivated by Ding [4] and Li [6].

LEMMA 3.1. *For* $n \geqslant 4$ *and* $2 \leqslant n-l \leqslant l$, *the imbedding of* Σ_l *into* $L^{p+1}(\Omega_a)$ *is compact provided* $1 < p < (l+3)/(l-1)$.

Proof. Let $y = (x_1, ..., x_l)$ and $z = (x_{l+1}, ..., x_n)$, $s^2 = |y|^2$ and $t^2 = |z|^2$.

$$D = \{(s, t) \in \mathbb{R}^2 : a^2 < s^2 + t^2 < 1\},$$

$$D_1 = \left\{(s, t) \in D : t^2 > \frac{a^2}{2}\right\},$$

and

$$D_2 = D \backslash D_1.$$

In the following, the constants C_i may vary but depend only on n, l, a, and p.

Let $u \in \Sigma_l$, then there exists $v \in H_0^1(D)$ such that $u(y, z) = v(|y|, |z|)$,

$$\frac{\partial v}{\partial t}(s, t) = 0 \qquad \text{for} \quad t = 0,$$

and

$$\frac{\partial v}{\partial s}(s, t) = 0 \qquad \text{for} \quad s = 0.$$

Therefore,

$$\begin{aligned}
&\int_{\Omega_a}^{\mu} (|\nabla u|^2 + u^2)\, dy\, dz \\
&\quad = C_1 \int_D (|\nabla v|^2 + v^2)\, s^{l-1} t^{n-l-1}\, ds\, dt \\
&\quad \geqslant C_2 \left\{ \int_{D_1} (|\nabla v|^2 + v^2) s^{l-1}\, ds\, dt + \int_{D_2} (|\nabla v|^2 + v^2) t^{n-l-1}\, ds\, dt \right\}.
\end{aligned}$$

Let

$$\tilde{D}_1 = \{(y, t) \in \mathbb{R}^{l+1} : (|y|, t) \in D_1\},$$

$$\tilde{D}_2 = \{(s, z) \in \mathbb{R}^{n-l-1} : (s, |z|) \in D_2\}.$$

Define $u_i \in H^1(\tilde{D}_i)$, $i=1,2$, by $u_1(y,t)=v(|y|,t)$ and $u_2(s,z)=v(s,|z|)$. Then

$$\int_{D_1} (|\nabla v|^2+v^2)s^{l-1}\,ds\,dt=\int_{\tilde{D}_1} (|\nabla u_1|^2+u_1^2)\,dy\,dt \geqslant C_3\left\{\int_{\tilde{D}_1} |u_1|^{p+1}\right\}^{2/(p+1)},$$

for $p \leqslant (l+3)/(l-1)$. Similarly, we have

$$\int_{D_2} (|\nabla v|^2+v^2)t^{n-l-1}\,ds\,dt \geqslant C_4\left\{\int_{\tilde{D}_2} |u_2|^{p+1}\right\}^{2/(p+1)}$$

for $p \leqslant (n-l+3)/(n-l-1)$.

Since $n-l \leqslant l$, we have $(l+3)/(l-1) \leqslant (n-l+3)/(n-l-1)$ and the imbeddings of $H^1(\tilde{D}_i)$ into $L^{p+1}(\tilde{D}_i)$ are compact if $p<(l+3)/(l-1)$, for $i=1,2$.

Since $2/(p+1)<1$, we have

$$\begin{aligned}\int_{\Omega_a} (|\nabla u|^2+u^2)\,dy\,dz &\geqslant C_5\left\{\int_{\tilde{D}_1} |u_1|^{p+1}+\int_{\tilde{D}_2} |u_2|^{p+1}\right\}^{2/(p+1)}\\ &\geqslant C_6\left\{\int_D |v|^{p+1}s^{l-1}t^{n-l-1}\,ds\,dt\right\}^{2/(p+1)}\\ &= C_7\left\{\int_{\Omega_a} |u|^{p+1}\,dy\,dz\right\}^{2/(p+1)}.\end{aligned}$$

The proof is complete.

For $n \geqslant 4$, define

$$l_n=\begin{cases}\dfrac{n}{2} & \text{if } n \text{ is even,}\\[2mm] \dfrac{n+1}{2} & \text{if } n \text{ is odd,}\end{cases}$$

and $P_n^*=(l_n+3)/(l_n-1)$. It is easy to see that $P_n^*>(n+1)/(n-3)$ if $n \geqslant 6$.

We can now prove the following theorems.

THEOREM 3.2. *Let $n \geqslant 4$. Assume f satisfies* (H-0)–(H-2) *and*

(H-3)′ *for u large, $f(u) \leqslant Cu^p$, $C>0$, and for some $p<(n+1)/(n-3)$.*

Then for each $k \geqslant 2$ and $a \in (a_k, 1)$, there exist non-equivalent positive nonradial solutions $u_{j,l} \in \Sigma_l$, for $j=2, 4, ..., 2[k/2]$ and $l=l_n$.

THEOREM 3.3. *Let* $n \geqslant 4$. *Assume* f *satisfies* (H-0)–(H-2) *and* (H-3)′ *with* $p < P_n^*$. *Then for each* $k \geqslant 2$ *and* $a \in (a_k, 1)$, *there exist non-equivalent positive nonradial solutions* $u_{j,l_n} \in \Sigma_{l_n}$ *for* $j = 2, ..., 2[k/2]$.

Proof of Theorems 3.2 *and* 3.3. By Lemma 2.7, for each even $j \leqslant 2[k/2]$ and $2 \leqslant n - l \leqslant l$, there exist $w_{j,l} = \varphi_j \psi_{j,l} \in O(l) \times O(n-l)$. By Lemma 3.1, the imbeddings of Σ_l into $L^{p+1}(\Omega)$ are compact if $p < (l+3)/(l-1)$.

Since $(l+3)/(l-1)$ is decreasing in l, we have $(l+3)/(l-1) \geqslant (n+1)/(n-3)$ for $l = 2, ..., n-2$. Hence, Theorem 3.2 follows. Similarly, the imbedding of $\Sigma_{l_n} \to L^{p+1}(\Omega_a)$ is compact if $1 < p < P_n^*$, Theorem 3.3 follows. The proof is complete.

Remark 3.4. (i) The arguments used in this paper are also valid if $f(u)$ is replaced by $f(|x|, u)$ which satisfies conditions similar to (H-0)–(H-3).

(ii) It is possible to use the same techniques to bring more positive nonradial solutions in the cases $f(0) > 0$ or $f(0) = 0$ and $f'(0) > 0$, see [8] for results concerning the nonradial instability and estimations of $\mu_{k,1}$.

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