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Families of Graphs Closed Under Taking Powers*

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Abstract. This paper gives simple proofs for "$G^k \in \mathscr{A}$ implies $G^{k+1} \in \mathscr{A}$" when $\mathscr{A}$ is the family of all interval graphs, all proper interval graphs, all cocomparability graphs, or all m-trapezoid graphs.

1. Introduction

In a graph $G = (V, E)$, the *distance* $d_G(x, y)$ between two vertices x and y is the minimum number of edges in an x-y path; $d_G(x, y) = \infty$ if there exists no x-y path. For a positive integer k, the *kth power* of a graph $G = (V, E)$ is the graph $G^k = (V, E^k)$ whose vertex set is V and edge set $E^k = \{xy : 1 \le d_G(x, y) \le k\}$.

Powers of graphs have been studied from different points of view. For instance, researchers are interested in knowing which families of graphs are closed under taking powers. Well-known families of this kind are interval graphs, proper interval graphs, strongly chordal graphs, circular-arc graphs, cocomparability graphs among others. A more general question is, for a family $\mathscr{A}$ of graphs, whether $G^k \in \mathscr{A}$ implies $G^{k+1} \in \mathscr{A}$.

The first surprising result in this line was given by Lubiw [18], who proved that powers of strongly chordal graphs are strongly chordal. Hoffman et al. [14] gave a simple proof of this result. Knowing a similar result is impossible for chordal graphs, Chang and Nemhauser [3] showed that if G and G^2 are chordal then so are all powers of G. On the other hand, Balakirishnan and Paulraja [2] proved that odd powers of chordal graphs are chordal. An even more interesting result, with an elegant proof, was given by Duchet [10] that if G^k is chordal then so is G^{k+2}.

Since then, many authors worked on the problem "whether $G^k \in \mathscr{A}$ implies $G^{k+1} \in \mathscr{A}$" for various families $\mathscr{A}$. Typical examples are strongly chordal graphs [21], interval graphs [20], proper interval graphs [20], m-trapezoid graphs [11], cocomparability graphs [11]. Most of them are proved in some clever, but slightly complicated ways. The main effort of this paper is to give simple proofs for

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interval graphs, proper interval graphs, cocomparability graphs, and m-trapezoid graphs by a "vertex ordering" methodology.

2. Powers of Graphs

The concept of intersection graphs plays an important role in graph theory. The *intersection graph* of a family $\mathscr{F}$ of sets is the graph whose vertices have a one-to-one correspondence to the sets in $\mathscr{F}$, and two distinct vertices are adjacent if and only if their corresponding sets intersect. In this definition, $\mathscr{F}$ is called an *intersection model* of its intersection graph. It is an easy exercise to show that any graph is the intersection graph of some family of sets. However, if the sets in $\mathscr{F}$ have special structures, then its intersection graph is usually well-behaved. Recently intersection graphs of the following objects have been studied extensively by many authors: intervals on the real line, boxes (balls) in the n-dimensional Euclidean space, arcs in a circle, trapezoids between two parallel lines on a plane, to name a few.

Among these, *interval graphs*, which are intersection graphs of intervals on the real line, have been most extensively studied not only because they are well-behaved, but also because they are applicable to many fields such as biology and computer science, e.g., see [24]. For studying the domination problem, Ramalingam and Pandu Rangan [19] gave that a graph G is an interval graph if and only if it has an *interval ordering*, which is an ordering of $V(G)$ into $[v_1, v_2, \ldots, v_n]$ such that

$$i < \ell < j \quad \text{and} \quad v_i v_j \in E(G) \text{ imply } v_\ell v_j \in E(G).$$

This can be seen by sorting the right endpoints of intervals correspondent to the vertices of the interval graph. Using this, we now give an alternative proof for Raychaudhuri's [20] result on interval graphs.

Theorem 1. *Suppose G is a graph and k a positive integer. If G^k is an interval graph, then so is G^{k+1}.*

Proof. Let $[v_1, v_2, \ldots, v_n]$ be an interval ordering of G^k. Consider G^{k+1} and the ordering $[v_1, v_2, \ldots, v_n]$. Suppose $i < \ell < j$ and $v_i v_j \in E(G^{k+1})$, i.e., $d_G(v_i, v_j) \leq k+1$. If $d_G(v_i, v_j) \leq k$, then $v_i v_j \in E(G^k)$ and so $v_\ell v_j \in E(G^k) \subseteq E(G^{k+1})$. Now, suppose $d_G(v_i, v_j) = k+1$. Let P be a shortest v_i-v_j path in G and let v_a be the vertex adjacent to v_j on P. Then, $d_G(v_i, v_a) = k$ and $d_G(v_a, v_j) = 1$. So, $v_i v_a \in E(G^k)$ and $v_a v_j \in E(G^k)$. If $i < \ell < a$, then $v_\ell v_a \in E(G^k)$ and so $d_G(v_\ell, v_j) \leq d_G(v_\ell, v_a) + d_G(v_a, v_j) \leq k+1$. If $a < \ell < j$, then $v_\ell v_j \in E(G^k) \subseteq E(G^{k+1})$. Therefore, $v_\ell v_j \in E(G^{k+1})$ in any case. Hence, $[v_1, v_2, \ldots, v_n]$ is an interval ordering of G^{k+1} and G^{k+1} is an interval graph. $\square$

Corollary 2. *Powers of interval graphs are interval graphs.*

A *proper interval graph* is an interval graph with an interval model in which no interval is a proper subset of another interval. Ding [9] and Roberts [23] proved that a graph is a proper interval graph if and only if it has an *proper interval ordering*, which is an ordering of its vertex set into $[v_1, v_2, \ldots, v_n]$ such that $[v_1, v_2, \ldots, v_n]$ and $[v_n, v_{n-1}, \ldots, v_1]$ are interval orderings, or equivalently,

$$i < \ell < j \text{ and } v_i v_j \in E(G) \text{ imply } v_i v_\ell \in E(G) \text{ and } v_\ell v_j \in E(G).$$

Using this, we have The following simple proof for Raychaudhuri's [20] result on proper interval graphs.

Theorem 3. *Suppose G is a graph and k a positive integer. If G^k is a proper interval graph, then so is G^{k+1}.*

Proof. Let $[v_1, v_2, \ldots, v_n]$ be a proper interval ordering of G^k, i.e., both $[v_1, v_2, \ldots, v_n]$ and $[v_n, v_{n+1}, \ldots, v_1]$ are interval orderings of G^k. By the same arguments used in the proof of Theorem 1, we have that both $[v_1, v_2, \ldots, v_n]$ and $[v_n, v_{n-1}, \ldots, v_1]$ are interval orderings of G^{k+1}. Hence, G^{k+1} is a proper interval graph. □

Corollary 4. *Powers of proper interval graphs are proper interval graphs.*

A *comparability graph* is the underlying graph of an acyclic transitive digraph, which can be viewed as a poset. In other words, a graph G is comparability if it has a *transitive ordering* that is an ordering of $V(G)$ into $[v_1, v_2, ..., v_n]$ such that

$$i < \ell < j \text{ and } v_i v_\ell, v_\ell v_j \in E(G) \text{ imply } v_i v_j \in E(G).$$

A *cocomparability graph* is the complement of a comparability graph, i.e., it has a *cocomparability ordering* that is an ordering of its vertex set into $[v_1, v_2, ..., v_n]$ such that

$$i < \ell < j \text{ and } v_i v_j \in E(G) \text{ imply } v_i v_\ell \in E(G) \text{ or } v_\ell v_j \in E(G).$$

Cocomparability graphs include interval graphs and m-trapezoid graphs defined below. Flotow [11] proved the following result for cocomparability graphs by means of m-trapezoid graphs. We now give a simple and direct proof.

Theorem 5. *Suppose G is a graph and k a positive integer. If G^k is a cocomparability graph, then so is G^{k+1}.*

Proof. Let $[v_1, v_2, \ldots, v_n]$ be a cocomparability ordering of G^k. Consider G^{k+1} and the ordering $[v_1, v_2, \ldots, v_n]$. Suppose $i < \ell < j$ and $v_i v_j \in E(G^{k+1})$, i.e., $d_G(v_i, v_j) \le k + 1$. If $d_G(v_i, v_j) \le k$, then $v_i v_j \in E(G^k)$, which implies that either $v_i v_\ell \in E(G^k) \subseteq E(G^{k+1})$ or $v_\ell v_j \in E(G^k) \subseteq E(G^{k+1})$. Now, suppose $d_G(v_i, v_j) = k + 1$. Choose a vertex v_a such that $d_G(v_i, v_a) = k$ and $d_G(v_a, v_j) = 1$. Then, $v_i v_a \in E(G^k)$ and $v_a v_j \in E(G^k)$. If $i < \ell < a$, then either $d_G(v_i, v_\ell) \le k$ or $d_G(v_\ell, v_a) \le k$. For the former case, $v_i v_\ell \in E(G^k) \subseteq E(G^{k+1})$; for the latter case, $d_G(v_\ell, v_j) \le d_G(v_\ell, v_a) + d_G(v_a, v_j) \le k + 1$ and so $v_\ell v_j \in E(G^{k+1})$. If $a < \ell < j$, then either $d_G(v_a, v_\ell) \leq k$ or $d_G(v_\ell, v_j) \leq k$. For the former case, $d_G(v_\ell, v_j) \leq d_G(v_a, v_\ell) + d_G(v_a, v_j) \leq k + 1$ and so $v_\ell v_j \in E(G^{k+1})$; for the latter case, $v_\ell v_j \in E(G^k) \subseteq E(G^{k+1})$. Hence, in any case, $[v_1, v_2, \ldots, v_n]$ is a cocomparability ordering of G^{k+1} and G^{k+1} is a cocomparability graph. □

Corollary 6. *Powers of cocomparability graphs are cocomparability.*

Although, the result for cocomparability graphs can be proved without using m-trapezoid graphs, the result for m-trapezoid graphs has its own interest. As the final part of this paper, we also give a new proof for the result on m-trapezoid graphs.

Suppose $m \geq 0$ and $L_0, L_1, \ldots, L_m$ are $m + 1$ parallel lines, indexed to their ordering, in the plane. Suppose $[a_i, b_i]$ is an interval in L_i for $0 \leq i \leq m$. These intervals define an *m-trapezoid* that is the region bounded by the polygon $a_0, a_1, \ldots, a_m, b_m, b_{m-1}, \ldots, b_0, a_0$. An *m-trapezoid graph* is the intersection graph of m-trapezoids over $m + 1$ parallel lines in the plane. Without loss of generality, we may assume that all right endpoints b_i's for different m-trapezoids are distinct. Note that 0-trapezoid graphs are precisely interval graphs; 1-trapezoid graphs are the usual trapezoid graphs, which include permutation graphs; and m-trapezoid are precisely comparability graphs of posets with interval dimension at most $m + 1$ (see [11, 25]).

Lemma 7. *A graph $G = (V, E)$ is an m-trapezoid graph if and only if it has a family of m-trapezoid orderings that is a set $\{<_0, <_1, \ldots, <_m\}$ of $m + 1$ orderings of V such that the following two conditions hold for all vertices x and y.*

(T1(x, y, G)) *If x and y disagree in two orderings $<_i$ and $<_j$ (i.e., $x <_i y$ but $y <_j x$ or $x <_j y$ but $y <_i x$), then $xy \in E$.*

(T2(x, y, G)) *If x and y agree in all orderings (say, $x <_\ell y$ for all ℓ) and $xy \in E$, then there exists some i^* such that $x <_{i^*} z <_{i^*} y$ implies $zy \in E$.*

Proof. ($\Rightarrow$) Suppose G is an m-trapezoid graph whose m-trapezoid representation is over the parallel lines $L_0, L_1, \ldots, L_m$. For each vertex $v \in V$, let $[a_i^v, b_i^v]$ be the interval for the m-trapezoid of v in L_i. Define an ordering $<_i$ of V by

$$x <_i y \text{ if and only if } b_i^x < b_i^y. \tag{2.1}$$

It is straightford to check that the two conditions (T1) and (T2) hold.

($\Leftarrow$) Conversely, suppose G has a family of m-trapezoid orderings $\{<_0, <_1, \ldots, <_m\}$. Construct $m + 1$ parallel lines $L_0, L_1, \ldots, L_m$. For any vertex $v \in V$ and any line L_i, choose b_i^v such that (2.1) holds. Define

$$a_i^v = \min(\{b_i^v\} \cup \{b_i^x : xv \in E,\ x <_i v,\ \text{and } zv \in E \text{ whenever } x <_i z <_i v\}).$$

Then the $|V|$ m-trapezoids defined by the intervals $[a_i^v, b_i^v]$ determine an m-trapezoid graph, which can be verified to be the graph G. □

Theorem 8. *Suppose G is a graph and k a positive integer. If* G^k *is an m-trapezoid graph, then so is* G^{k+1}.

Proof. Let $\{<_0, <_1, \ldots, <_m\}$ be a family of m-trapezoid orderings of G^k. Consider G^{k+1} and the family $\{<_0, <_1, \ldots, <_m\}$. Since (T1$(x, y, G^k)$) holds for all x and y and $E(G^k) \subseteq E(G^{k+1})$, (T1$(x, y, G^{k+1})$) holds for all x and y. For (T2(x, y, G^{k+1})), suppose $x <_\ell y$ for all ℓ and $xy \in E(G^{k+1})$. Choose a vertex w such that $d_G(x, w) \leq k$ and $d_G(w, y) = 1$.

Note that either $w <_i x$ for some i or $x <_\ell w$ for all ℓ. For the former case, choose $i^* = i$. For the later case, (T2(x, w, G^k)) holds for some $<_{i_1^*}$ and we choose $i^* = i_1^*$. In either case, consider any z with $x <_{i^*} z <_{i^*} y$. If z and y disagree in two orderings, then (T1(z, y, G^k)) implies $zy \in E(G^k) \subseteq E(G^{k+1})$. So we may assume that z and y agree in all orderings. If z and w disagree in two orderings, then (T1(z, w, G^k)) implies $zw \in E(G^k)$ and so $zy \in E(G^{k+1})$. So we may assume that z and w agree in all orderings.

Case 1. $w <_i x$ for some i and $i^* = i$.

As $w <_i x <_i z <_i y$, we have $w <_\ell z <_\ell y$ for all ℓ. Since $wy \in E(G) \subseteq E(G^k)$, (T2$(w, y, G^k)$) holds for some i_2^*. In this case, $zy \in E(G^k) \subseteq E(G^{k+1})$.

Case 2. $x <_\ell w$ for all ℓ and $i^* = i_1^*$.

In this case, $x <_{i_1^*} z <_{i_1^*} w$ or $z = w$ or $w <_{i_1^*} z <_{i_1^*} y$. For the case of $x <_{i_1^*} z <_{i_1^*} w$, (T2$(x, w, G^k)$) implies $zw \in E(G^k)$ and so $zy \in E(G^{k+1})$. For the case of $z = w$, $zy = wy \in E(G) \subseteq E(G^{k+1})$. For the case of $w <_{i_1^*} z <_{i_1^*} y$, we have $w <_\ell z <_\ell y$ for all ℓ and so (T2(w, y, G^k)) holds for some i_2^*, which implies $zy \in E(G^k) \subseteq E(G^{k+1})$.

In any case, (T2(x, y, G^{k+1})) holds. Hence $\{<_0, <_1, \ldots, <_m\}$ is a family of m-trapezoid orderings for G^{k+1} and G^{k+1} is an m-trapezoid graph. □

Corollary 9. *Powers of trapezoid graphs are trapezoid graphs.*

In fact, Theorems 1 and 5 also follow from Theorem 8.

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