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WHEN IS INDIVIDUAL TESTING OPTIMAL FOR NONADAPTIVE GROUP TESTING?*

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Abstract. The combinatorial group testing problem is, assuming the existence of up to d defectives among n items, to identify the defectives by as few tests as possible. In this paper, we study the problem for what values of n, given d, individual testing is optimal in nonadaptive group testing. Let $N(d)$ denote the largest n for fixed d for which individual testing is optimal. We will show that $N(d) = (d+1)^2$ under a prevalent constraint in practical nonadaptive algorithms and prove that $N(d) = (d+1)^2$ for $d = 1, 2, 3, 4$ without any constraint.

Key words. nonadaptive group testing, disjunct matrix, union-free matrix

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1. Introduction.

In combinatorial group testing, a prototype problem called the $(\bar{d}, n)$ problem is to assume that there are up to d defectives among n given items, and the problem is to separate the good items from the defective ones by group tests. A (group) test is administered on an arbitrary subset S of the items with two possible outcomes; a *negative* outcome means S contains no defectives and a *positive* outcome means S contains at least one defective, not knowing exactly how many or which ones. A group testing algorithm is *optimal* if it minimizes the worst-case number of tests required.

A group testing algorithm is *sequential* if the tests can be done sequentially and the outcomes of previous tests are known at the time to determine the current test. A group testing algorithm is *nonadaptive* if all tests must be specified at once. A nonadaptive algorithm can be represented by a 0-1 matrix where columns are items, rows are tests, and a 1-entry in cell (i, j) means item j is contained in test i. Note that a column can be viewed as a subset whose elements are indices of the rows incident to the column. Thus we can talk about the union of columns. Group testing has applications to blood testing, electrical and chemical testing, coding, multiaccess channel conflict resolution, etc. Recently, nonadaptive group testing has been shown to play a crucial role in the clone library screening problem.

Kautz and Singleton [8] introduced the notions of "$\bar{d}$*-separable*" and "d*-disjunct*" of 0-1 matrix $M_{t\times n}$; the former requires that no two unions of up to d columns are identical, while the latter requires that no union of d columns contains a column not in the union. They showed that both properties guarantee $M_{t\times n}$ to be a nonadaptive $(\bar{d}, n)$ algorithm, while the d-disjunct property has an extra feature of simplifying the process of identifying defectives. These two properties were also called r-union-free and r-cover-free [3, 4] in extremal set theory.

A trivial d-disjunct algorithm not using the idea of group testing would test the n items individually, which requires n tests. Thus it is of interest to know for what values of n, given d, individual testing is optimal.

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A similar question as posed in the title has been asked on sequential group testing for exactly d defectives. (Thus the state of the last item can be deduced without testing.) Hu, Hwang, and Wang [5] conjectured that individual testing is optimal if and only if $n \leq 3d$. They proved the "necessary" part, but the sufficient condition is proved only for $n \leq 2.5d$, improving an earlier sufficient condition $n \leq 2d$ of Hwang [7]. Du and Hwang [1] further improved the sufficient condition to $n \leq 2.625d$, but the conjecture remains open.

Back to the nonadaptive case, let $N(d)$ denote the largest n for fixed d for which individual testing is optimal. Bassalygo (see [2]) first gave a lower bound.

LEMMA 1.1. $N(d) \geq \binom{d+2}{2}$.

Erdös, Frankl, and Füredi [3] conjectured that

$$\lim \quad N(d)/d^2 = 1 \quad \text{(weaker version)},$$
$$N(d) \leq (d+1)^2 \quad \text{(stronger version)}$$

and stated without giving details that they can prove the stronger version for $d \leq 3$. In this paper, we will prove $N(d) \leq (d+1)^2$ for $d+1$ a prime power. Thus

$$\binom{d+2}{2} \leq N(d) \leq (d+1)^2.$$

This establishes $N(d) = O(d^2)$, as opposed to $N(d) = O(d)$ in the sequential case. We will also show that under a prevalent constraint in practical nonadaptive algorithms, $N(d) = (d+1)^2$. Finally, we prove $N(d) = (d+1)^2$ for d=1, 2, 3, 4 without any constraint.

2. A necessary condition. Let *t(d, n)* denote the minimum number of tests a nonadaptive algorithm requires, given d and n. We first make an observation.

LEMMA 2.1. *The existence of a d-disjunct matrix $M_{t\times n}$ with $t < n$ implies $N(d) \leq t$.*

Proof. *t(d, n)* is clearly nondecreasing in n. Hence, the existence of a d-disjunct $M_{t\times n}$ with $n > t$ implies $t+1$ items can be done in t tests, or, equivalently, $N(d) \leq t$. ∎

Let $\lambda_{cc'}$ denote the inner product of two columns c and c'. Define $\bar{\lambda} = \max \lambda_{cc'}$ over all pairs of columns. A 0-1 matrix is called a *weight-w matrix* if each column is a w-set. The following lemma is well known [10].

LEMMA 2.2. *A weight-w matrix is* $(\lceil w/\bar{\lambda}\rceil - 1)$*-disjunct.*

COROLLARY 2.3. *If* $\bar{\lambda} = 1$*, then a weight-w matrix is* $(w-1)$*-disjunct.*

The main result in this section is the following theorem.

THEOREM 2.4. $N(d) \leq (d+1)^2$ *for* $d+1$*, a prime power.*

Proof. By Lemma 2.1 and Corollary 2.3, it suffices to construct a weight-$(d+1)$ matrix $M_{(d+1)^2\times n}$ with $\bar{\lambda} = 1$ and $(d+1)^2 < n$.

It is well known [9] that if $d+1$ is a prime power, then there exists a set of d mutually orthogonal latin squares (MOLS). Each such square will generate $d+1$ columns (as $(d+1)$-subsets of $\{0, 1, 2, \ldots, (d+1)^2 - 1\}$) of M, one for each set of cells (i, j) having the same entry k, $0 \leq k \leq d$, and cell (i, j) is translated to the number $i(d+1) + j$. Clearly, columns generated from the same square have $\lambda_{cc'} = 0$. Due to orthogonality, columns generated from different latin squares satisfy $\lambda_{cc'} = 1$. Thus the d MOLS generate a total of $(d+1)d$ columns of M with $\bar{\lambda} = 1$. Finally, consider the $(d+1) \times (d+1)$ matrix S, where the entry in cell (i, j) is just (i, j). Clearly, the columns (rows) of S have $\lambda_{cc'} = 0$, and each row-column pair has $\lambda_{rc} = 1$.

Furthermore, the set of cells having the same entry from a latin square must be a *transversal*; i.e., they have distinct row indices and distinct column indices. Let s be a transversal, r a row, and c a column of S. Then $\lambda_{sr} = \lambda_{sc} = 1$. Thus we can add the $2(d+1)$ rows and columns of S to be columns of M and preserve the $\bar{\lambda} = 1$ property. The total number of columns in M is now

$$(d+1)d + 2(d+1) = (d+1)(d+2).$$

Note that the base set of the columns is the set $\{0, 1, \ldots, (d+1)^2 - 1\}$. By treating the base set as the set of tests, then we have constructed a $(d+1)^2 \times (d+1)(d+2)$ matrix with $\bar{\lambda} = 1$. □

3. A necessary and sufficient condition under $\bar{\lambda} = 1$.

Constructing efficient d-disjunct matrices is a difficult task, with the simplest and most prevalent method being given by Corollary 2.3, i.e., constructing matrices with $\bar{\lambda} = 1$. In this section, we study the problem given in the title of this paper under the constraint $\bar{\lambda} = 1$.

Let $M_{t\times n}$ be a 0-1 matrix. Let *G(M)* denote the graph with the rows of M as vertices and an edge between two vertices if and only if the inner product of the two corresponding rows (viewed as subsets of $\{1, 2, \ldots, n\}$) is not zero. Let $d_G(v)$ denote the degree of v in G.

LEMMA 3.1. *Suppose M is weight-$(d+1)$ and $\bar{\lambda} = 1$. Then G(M) consists of n edge-disjoint K_{d+1} (complete graph of order $d+1$).*

Proof. Each column generates a K_{d+1}. The $\bar{\lambda} = 1$ property forces the K_{d+1}'s to be edge disjoint. □

Recently, W. T. Huang and Hwang observed (private communication) that a previous result of Weideman and Raghavarao [12] for the $\bar{d}$-separable matrix can be extended to the following lemma.

LEMMA 3.2. *Any d-disjunct matrix with $\bar{\lambda} = 1$ can be reduced to one where no column weight exceeds $d+1$ with the d-disjunct property preserved.*

Let $n(d, t)$ denote the largest n such that there exists a d-disjunct $M_{t\times n}$. A column in M is called *isolated* if there exists a row incident to this column only.

It is easily observed in the following lemma.

LEMMA 3.3. *Let $M_{t\times n}$ be a d-disjunct matrix containing an isolated column. Then*

$$n \le 1 + n(d, t-1).$$

Proof. Deleting the isolated column and its incident row does not affect the d-disjunct property. □

Note that to determine $N(d)$, we need to find an $M_{t\times n}$ satisfying $t < n$ and minimizing n. A matrix with no nonisolated column always satisfies $t \ge n$, and hence cannot be such a candidate, and consequently is of no interest. Therefore, we assume from now on that we consider only matrices with an isolated column.

Dyachkov and Rykov [2] proved the following lemma.

LEMMA 3.4. *Let M be a d-disjunct matrix. Then a column with weight at most d must be isolated.*

LEMMA 3.5. *Let M be a d-disjunct matrix with $\bar{\lambda} = 1$ and no column weight less than $d+1$. Then $n \le \lfloor \frac{t(t-1-r)}{(d+1)d} \rfloor$, where $t - 1 \equiv r \pmod d$.*

Proof. By Lemmas 3.2 and 3.4, we may obtain a constant weight-w matrix M^* with $w = d+1$ from M. Consider $G(M^*)$. Each vertex has a maximum possible degree of $t-1$. By Lemma 3.1, $d \mid d_G(v)$. Hence, $d_G(v) \le (t-1) - r$. Thus the total number of edges in G satisfies

$$n \times \binom{d+1}{2} = \frac{\Sigma d_G(v)}{2} \le \frac{t[(t-1)-r]}{2},$$

$$\text{or } n \le \lfloor \frac{(t(t-1-r)}{(d+1)d} \rfloor. \qquad \square$$

We are now ready to prove the main result of this section.

THEOREM 3.6. *Under* $\bar{\lambda} = 1, N(d) = (d+1)^2$ *for* $d+1$*, a prime power.*

Proof. By Theorem 2.4, it suffices to prove $N(d) \ge (d+1)^2$, which is done by proving the nonexistence of a d-disjunct matrix $M_{[(d+1)^2-1]\times(d+1)^2}$ with $\bar{\lambda} = 1$.

Let $M_{t\times n}$ be a d-disjunct matrix with $\bar{\lambda} = 1$ and $t = (d+1)^2 - 1$. By Lemma 3.2, we may assume that every column of M has weight at most $d+1$.

Case (i). No column of M has weight less than $d+1$. Since $t-1 = (d+1)^2 - 2 = (d+1)d + d - 1$, $t - 1 - r = (d+1)d$. By Lemma 3.5,

$$n \le \frac{t(d+1)d}{d(d+1)} = t.$$

Case (ii). There exists a column of M with weight at most d. Let C be the set of columns with weight at most d. By Lemma 3.4, each column $c \in C$ is isolated; i.e., there exists a row $r(c)$ incident only to c. Let M' be obtained from M by deleting c and $\{r(c) : c \in C\}$. Then M' is a $(t- \mid C \mid) \times (n- \mid C \mid)$ weight-$(d+1)$ matrix with $\bar{\lambda} = 1$, since

$$t- \mid C \mid -1 = d(d+1) + d - 1- \mid C \mid, \qquad t - 1- \mid C \mid -r \le d(d+1).$$

By Lemma 3.5,

$$(n- \mid C \mid) \le \left\lfloor \frac{(t- \mid C \mid)(d+1)d}{d(d+1)} \right\rfloor,$$

which again leads to $n \le t$. $\square$

4. $N(d)$ for small d.

Bassalygo (see [2]) proved the following lemma.

LEMMA 4.1. *Let* M *be a* d*-disjunct matrix and* c *a column of* M *with weight* w*. Then* $t(d,n) \ge w + t(d-1, n-1)$.

Spencer [11] proved the following lemma

LEMMA 4.2. $n(1,t) = \binom{t}{\lfloor \frac{t}{2} \rfloor}$.

THEOREM 4.3. $N(1) = 4$.

Proof. By Theorem 2.4, $N(1) \le 4$. Since $n(1,3) = 3$, $N(1) \ge 4$. Hence, $N(1) = 4$. $\square$

LEMMA 4.4. $n(2,7) = 7$.

Proof. Let $M_{7\times n}$ be a 2-disjunct matrix. If $\bar{\lambda} = 1$, then Lemma 4.4 follows from Theorem 3.6. Therefore, we may assume the existence of two columns, c and c', with $\lambda_{cc'} > 1$. Then c is either isolated or has weight at least 4.

(i) c is isolated. By Lemmas 1.1 and 3.3,

$$n \le 1 + n(2,6) = 1 + 6 = 7.$$

(ii) c has weight at least 4. Deleting c and its incident rows, the reduced matrix M' can have at most three rows and is 1-disjunct. By Lemma 4.2, M' can have at most three columns. Then $n \le 4$. ∎

THEOREM 4.5. $N(2) = 9$.

Proof. Since $d + 1 = 3$ is a prime power, by Theorem 2.4, $N(2) \le 9$. Therefore, it suffices to prove $N(2) \ge 9$, or $n(2, 8) = 8$.

Let $M_{8\times n}$ be 2-disjunct. By Theorem 3.6, we may assume the existence of two columns, c and c', with $\lambda_{cc'} > 1$.

(i) c is isolated. Then by Lemmas 3.3 and 4.4

$$n \le 1 + n(2, 7) = 8.$$

(ii) c has weight at least 4. Then by Lemmas 4.1 and 4.2

$$t(2, 8) \ge 4 + t(1, 7), \qquad \text{or} \qquad n(2, 8) \le 7. \qquad ∎$$

COROLLARY 4.6. $N(3) \ge 13$.

Proof. By Lemmas 3.4 and 4.1, $t \ge (d + 1) + t(d - 1, n - 1)$ for a d-disjunct matrix $M_{t\times n}$ (with at least one nonisolated column), since $n(2, 8) = 8$, $t(2, 12) \ge 9$. Therefore, $t \ge 4 + t(2, n - 1) \ge 4 + 9 = 13$ for $n \ge 13$. It implies that $t(3, n) = n$ for $n \le 13$, i.e., $N(3) \ge 13$. ∎

Let $I(c)$ be a collection of different b-subsets, $b \ge 2$, of $\{1, 2, 3, \dots, t\}$ denoting the intersection property of column c with other columns. For example, $I(c) = \{(1, 2), (1, 3, 4)\}$ means that there exists at least one column c_1 intersecting c at rows 1 and 2, and there exists at least one column c_2 intersecting c at rows 1, 3, 4.

Let M_1 be a $t \times n_1$ weight-4 2-disjunct matrix with no isolated column. Then $\bar{\lambda} \le 2$. We will do some deletions on M_1 to reduce weight 4 to weight 3 such that the 2-disjunct property is still preserved in $(M_1)^i$, $i \ge 0$, which is the reduced matrix after the ith deletion. Note that deleting a 1-entry of c will affect other $I(c')$ in general. Therefore, after each deletion, we need to reconsider the $I(c)$ of the reduced matrix, where c has weight 4.

Consider $I(c_j), j \le n_1$, of $(M_1)^i, i \ge 0$, and $I(c_j) \ne \emptyset$, where c_j has weight 4. The deletion rule is as follows:

(1) If $I(c_j) \subseteq \{(x_j, y_j), (x_j, z_j), (x_j, v_j); x_j \ne y_j \ne z_j \ne v_j \in \{1, 2, 3, \dots, t\}\}$ in $(M_1)^i$, then delete the 1-entry in row x_j of c_j; hence the reduced column has weight 3 and has an inner product at most one with any other column of $(M_1)^i$. $(M_1)^i$ remains 2-disjunct by Lemma 3.2.

(2) If $I(c_j) = \{(x_j, y_j), (x_j, z_j), (y_j, z_j); x_j \ne y_j \ne z_j \in \{1, 2, 3, \dots, t\}\}$ in $(M_1)^i$, then do nothing at this moment until the last step.

Finally, we will get a reduced matrix $(M_1)^f$ with no case (1) after f deletions. If case (2) does not occur, we are done. If case (2) occurs, then there exists a $t \times n_1'$ submatrix M_1' contained in $(M_1)^f$ which has the following four properties:

(1) M_1' has at least four columns and each column of M_1' has weight 4.

(2) $\forall c_j \in C(M_1'), I(c_j)$ has the form $\{(x_j, y_j), (x_j, z_j), (y_j, z_j); x_j \ne y_j \ne z_j \in \{1, 2, 3, \dots, t\}\}$.

(3) $c_j \in M_1'$ and $|c_i \bigcap c_j| = 2 \Longrightarrow c_i \in M_1'$.

(4) M_1' doesn't contain a submatrix with fewer columns having properties (1), (2), and (3).

Let M_1'' be a $q \times n_1'$ submatrix of M_1', where the q rows of M_1'' are the collection of rows $\{x_i, y_i, z_i : c_i \in C(M'),\ I(c_i) = \{(x_i, y_i), (x_i, z_i), (y_i, z_i)\}\}$. Then we have the following lemma.

LEMMA 4.7. $n_1' \geq q$.

Proof. Suppose $c \in M_1'$ and $I(c) = \{(x,y),(x,z),(y,z)\}$. Then each row x, y, z must have at least three 1-entries in M_1''. For example, x appears once in c, once in a column intersecting c at (x,y), and once in a column intersecting c at (x,z). Suppose c has a fourth 1-entry v also in M_1''. Then $v \in \{x',y',z'\}$ for some column c' with $I(c') = \{(x',y'),(x',z'),(y',z')\}$. Therefore, row v has at least four 1-entries. Let k be the number of columns with four 1-entries in M''. Then counting the number of 1-entries by column and by row separately, we have $3n_1' + k \geq 3q + k$, or $n_1' \geq q$. ◻

LEMMA 4.8. *Suppose* $c = \{x,y,z,v\}$ *and* $I(c) = \{(x,y),(x,z),(y,z)\}$ $\forall c \in C(M_1')$. *Then* $c' \bigcap \{x,y,z\} \neq \emptyset$ *implies* $v \notin c'$ *in a 2-disjunct matrix.*

Proof. Without loss of generality, assume that c' contains x. Suppose to the contrary that c' also contains v. By the definition of $I(c)$, there exists a column c'' containing (y,z). Then $c' \bigcup c''$ contains c, contradicting the 2-disjunct. ◻

LEMMA 4.9. $n(2,9) = 12$.

Proof. The proof uses the Steiner triple system with $v = 9$ and $b = 12$ [9]. ◻

LEMMA 4.10. $n(2,10) \leq 13$.

Proof. Let $M_{10\times n}$ be a 2-disjunct matrix. If M has an isolated column, then by Lemmas 3.3 and 4.9, $n \leq 1 + n(2,9) = 1+12 = 13$. If $\bar{\lambda} = 1$ and there exists no isolated column in M, then $r \equiv 10 - 1 \equiv 1 \pmod 2$. By Lemma 3.5, $n \leq \lfloor \frac{10(10-1-1)}{3\cdot 2} \rfloor = 13$. Therefore, we may assume M has no isolated column and $\bar{\lambda} > 1$. Let column c have maximum weight of M. Then c has weight at least 4.

(i) c has weight greater than 4. Then by Lemmas 4.1 and 4.2,

$$t(2,14) \geq 5 + t(1,13) = 11, \qquad \text{or} \qquad n(2,10) \leq 13.$$

(ii) c has weight 4. Then M is 2-disjunct with $\bar{\lambda} = 2$, and each column has weight 3 or 4. Let $M = M_1 \bigcup M_2$, where M_1 is a $10 \times n_1$ weight-4 matrix, and M_2 is a $10 \times (n - n_1)$ weight-3 matrix.

We first make an observation. Because M is 2-disjunct with $\bar{\lambda} = 2$ and has maximum weight 4, it forces both M_1, M_2 to be 2-disjunct with M_1 having $\bar{\lambda} = 2, M_2$ having $\bar{\lambda} = 1$, and $\forall c_1 \in M_1, c_2 \in M_2, \lambda_{c_1 c_2} \leq 1$.

Next we want to claim that M_1 with $\bar{\lambda} = 2$ can be reduced to a matrix with $\bar{\lambda} = 1$ by deleting some 1-entries and remains 2-disjunct.

By the method we used previously, we get matrix $(M_1)^f, M_1', M_1''$ from $(M_1)^f$. If M_1', hence M_1'', does not exist, then we are done. Otherwise, by relabeling the rows, we may assume that $c_1 = \{1,2,3,4\}$. The completion of $I(c_1) = \{(1,2),(1,3),(2,3)\}$ implies the following submatrices must be in M_1' (see Figure 1):

Note that the completion of $I(c_2), I(c_3), I(c_4)$ requires at least one more row. In fact, if $q = 4$, then (b) shows the only way to complete $I(c_2), I(c_3)$, and $I(c_4)$. This case can be taken care of by deleting the circled 1's. It is easily verified that any two columns intersect at most once after the deletion. Hence, $\bar{\lambda} = 1$ in M_1'. In other words, M can be reduced to a matrix which remains 2-disjunct with $\bar{\lambda} = 1$ and has column weight 3 or 4. Hence, by Lemma 3.5, $t - 1 - r = 8$, $n \leq \lfloor \frac{10\times 8}{3\times 2} \rfloor = 13$.

If $q \geq 5$, then there are other ways to complete $I(c_2), I(c_3)$, and $I(c_4)$. By Lemma 4.7, $n_1' \geq q \geq 5$. Let c_5 be a new column in M_1''. Since M_1'' is minimal, $I(c_5)$ must intersect $\{(1,2),(1,3),(2,3)\}$. Without loss of generality, assume $(1,2) \in I(c_5)$.

Since c_5 intersects all c_1, c_2, c_3, and c_4, by Lemma 4.8, c_5 cannot intersect the fourth 1-entries of these columns. Therefore, the other two 1-entries of c_5 must take up new rows, say, rows 8 and 9 (see Figure 2(a)).

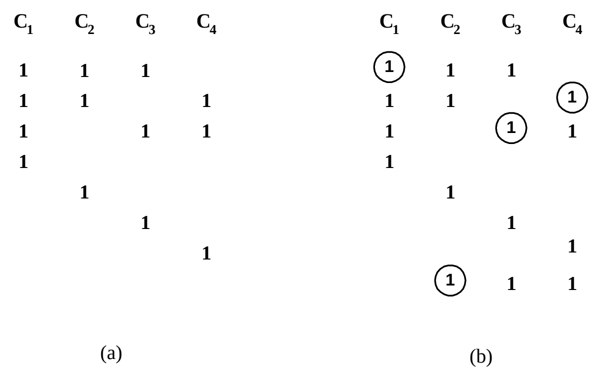

FIG. 1. *Forced submatrices in* M'_1.

c_1	c_2	c_3	c_4	c_5
1	1	1		1
1	1		1	1
1		1	1	
1				
	1			
		1		
			1	
				1
				1

(a)

c_1	c_2	c_3	c_4	c_5	c_6
1	1	1		1	1
1	1		1	1	
1		1	1		
1					
	1				
		1			
			1		1
				1	1
				1	

(b)

FIG. 2. *Larger forced submatrices.*

Inspecting the relation between c_5 and the other c_i, we notice its first two 1-entries are symmetric, and so are its last two. Therefore, we may assume $(1, 8) \in I(c_5)$. To complete $I(c_5)$, there must exist c_6 containing rows 1 and 8. Since c_6 intersects all other columns except c_4, among the existing rows, it can contain only row 7. Therefore, the fourth 1-entry of c_6 must be a new row, say, row 10. However, we still need at least one more row to complete $I(c_2), I(c_3)$, and $I(c_4)$. Hence, the total number of rows exceed 10. ∎

LEMMA 4.11. $n(3, 13) = 13$.

Proof. Let $M_{13\times n}$ be a 3-disjunct matrix. If $\bar{\lambda} = 1$, then Lemma 4.11 follows from Theorem 3.6. Therefore, we may assume the existence of two columns c and c' with $\lambda_{cc'} > 1$. Then c is either isolated or has weight at least 5.

(i) c is isolated. By Lemma 3.3 and Corollary 4.6,

$$n \leq 1 + n(3, 12) = 1 + 12 = 13.$$

(ii) c has weight at least 5. Deleting c and its incident rows, the reduced matrix M' can have at most 8 rows and is 2-disjunct. By $n(2, 8) = 8$, then $n \leq 9$. ∎

LEMMA 4.12. $n(3, 14) = 14$.

Proof. Let $M_{14\times n}$ be a 3-disjunct matrix. If $\bar{\lambda} = 1$, then Lemma 4.12 follows from Theorem 3.6. Therefore, we may assume the existence of two columns c and c' with $\lambda_{cc'} > 1$. Then c either is isolated or has weight at least 5.

(i) c is isolated. By Lemmas 3.3 and 4.11,

$$n \leq 1 + n(3, 13) = 1 + 13 = 14.$$

(ii) c has weight at least 5. Deleting c and the incident rows, the reduced matrix M' can have at most 9 rows and is 2-disjunct. By Lemma 4.9,

$$n \leq 1 + n(2, 9) = 13. \qquad \square$$

THEOREM 4.13. $N(3) = 16$.

Proof. Since $d + 1 = 4$ is a prime power, by Theorem 2.4, $N(3) \leq 16$. Therefore, it suffices to prove $N(3) \geq 16$ or $n(3, 15) = 15$. Let $M_{15\times n}$ be a 3-disjunct matrix. By Theorem 3.6, we may assume the existence of two columns c and c' with $\lambda_{cc'} > 1$.

(i) c is isolated. Then by Lemmas 3.3 and 4.12

$$n \leq 1 + n(3, 14) = 15.$$

(ii) c has weight at least 5. Then by Lemmas 4.1 and 4.10

$$n \leq 1 + n(2, 10) \leq 1 + 13 = 14. \qquad \square$$

With similar but slightly more complicated arguments, we can also prove $N(4) = 25$ [6].

5. Conclusion.

In this paper, we studied the problem of when individual testing is optimal for nonadaptive group testing. We showed that $N(d) \leq (d + 1)^2$ is a necessary condition by constructing the d-disjunct matrix $M_{[(d+1)^2-1]\times(d+1)^2}$, where $d + 1$ is a prime power. Besides, we showed that $N(d) \geq (d + 1)^2$ is a sufficient condition under $\bar{\lambda} = 1$. Hence, under $\bar{\lambda} = 1$ and $d+1$ a prime power, $N(d) = (d+1)^2$ is a necessary and sufficient condition of the problem we studied. We also prove that $N(d) = (d+1)^2$ for $d = 1, 2, 3, 4$ without any constraint, giving further support to the conjecture $N(d) = (d+1)^2$ for $d+1$, a prime power. However, for $d+1$, not a prime power, we still know little about $N(d)$.

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